# Statistical methods

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**Theorem 2 (CLT, Lévy).** *Consider i.i.d.* $X_1, \ldots, X_n$ *with* $\langle x \rangle = \langle x_i \rangle$ *and* $Var(x) = Var(x_i) < \infty$. *Then,*

$$\lim_{n\to\infty} \bar{x}_n \sim N\left( \langle x \rangle, \sqrt{\frac{Var(x)}{n}} \right), \quad \bar{x}_n \equiv \frac{1}{n} \sum_{i=1}^{n} x_i \ .$$

$$s_n^2 \equiv \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$$

**Proposition.** Consider i.i.d. $\{X_1, \ldots, X_n\}$, and suppose that $\langle x \rangle, \langle x^2 \rangle, \langle x^3 \rangle, \langle x^4 \rangle$ all exist and are finite. Then,

$$\langle s_n^2 \rangle = Var(x) \, .$$

# III. Frequency Interpretation of Probability Distributions:

"In order to make the theory operational, we must introduce a concept of probability that links the mathematics to an external world of measurable phenomena." (A. Stuart, J. K. Ord (1994), § 8.8, p. 290.)

"The most striking achievement of the physical sciences is prediction." (G. Pólya (1954), Chap. XIV, § 4, p. 64.)

"The pure mathematician can do what he pleases, but the applied mathematician must be at least partially sane." (M. Kline (1980). *Mathematics: The Loss of Certainty,* Chap. XIII, p. 285.)

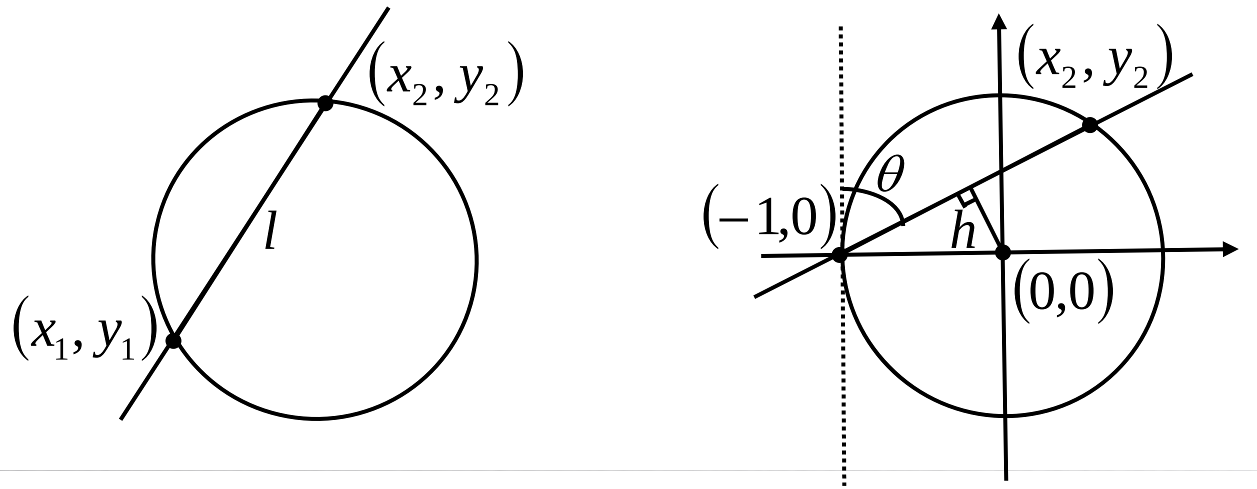
# III. Frequency Interpretation of Probability Distributions:

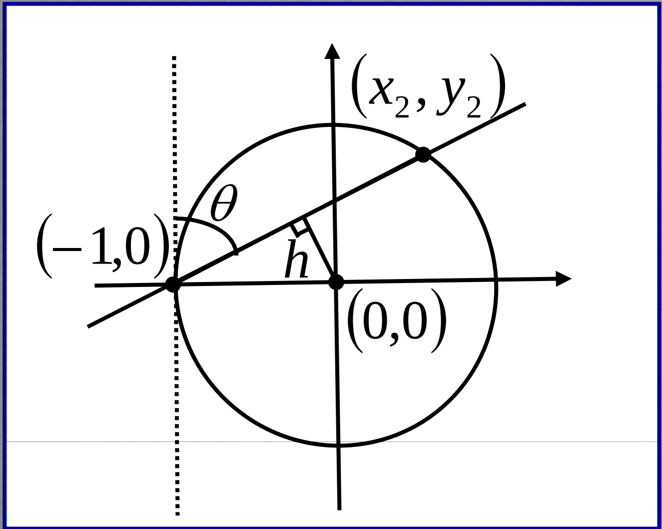
1. Example,
2. Binary random sequence,
3. Random sequence of real numbers,
4. Monte Carlo methods.

# 1. Example 1. (Bertrand's paradox).

A straw is tossed at random so that the line determined by the straw intersects the unit circle. What is the expected length $\langle l \rangle$ of the chord thus defined?

J.L. Bertrand (1889), *Calcul des Probilités*, pp. 4-5.
J.B. Paris (1994), *The Uncertain's Reasoner Companion*, Chap. 6, pp. 71-72.
E.T. Jaynes (2003), *Probability Theory,* § 12.4.4, pp. 386-394.






$$a)\quad f(h)=\begin{cases}1\,;\,0\le h\le 1\\ 0\,;\,\text{otherwise}\end{cases}\quad\Rightarrow\langle l\rangle=\frac{\pi}{2}\approx 1.57\,;$$

$$b)\quad f(\theta)=\begin{cases}\dfrac{2}{\pi}\,;\,0\le\theta\le\dfrac{\pi}{2}\\ 0\,;\,\text{otherwise}\end{cases}\quad\Rightarrow\langle l\rangle=\frac{4}{\pi}\approx 1.27\,;$$

$$c)\quad f(x_2)=\begin{cases}\dfrac{1}{2}\,;\,-1\le x_2\le 1\\ 0\,;\,\text{otherwise}\end{cases}\quad\Rightarrow\langle l\rangle=\frac{4}{3}\approx 1.33\,.$$

## 2. Binary random sequences.

Consider an **infinite** binary sequence $1,0,1,1,0,1,0,0,0,1,0,1,1,0,1,...$ with equal relative frequencies of appearance of 1's and 0's,

$$\nu_1 = \nu_0 = \frac{1}{2};$$

or more precisely,

$$\lim_{n\to\infty} P\left(\left|\frac{n_1}{n} - \frac{1}{2}\right| < \varepsilon\right) = 1.$$

We say that $\nu_1=\nu_0=1/2$ is true almost everywhere with respect to the Bernoulli measure $Bn(1/2)$ on the space of infinite binary sequences, called Cantor space ($Bn(1/2)$ on the Cantor space is isomorphic to the Lebesgue measure on the interval $[0,1]$).

For a $Bn(1/2)$ *-typical* binary sequence we would further expect that

$$\nu_{1,1} = \nu_{1,0} = \nu_{0,1} = \nu_{0,0} = \frac{1}{4},$$

$$\nu_{1,1,1} = \nu_{0,1,1} = \nu_{1,0,1} = \nu_{1,1,0} = \nu_{0,0,1} = \nu_{0,1,0} = \nu_{1,0,0,} = \nu_{0,0,0} = \frac{1}{8},$$

$$\vdots$$

holds $Bn(1/2)$-almost everywhere.

That is, from a $Bn(1/2)$ -typical binary sequence we would naively expect to satisfy all properties true $Bn(1/2)$-almost everywhere. Unfortunately, such a definition is vacuous.

**Definition 1 ($Bn(1/2)$- random binary sequence).** *An infinite binary sequence is called (*Martin-Löf*)* $Bn(1/2)$- random *iff it is not rejected by the Martin-Löf test (i.e., if it satisfies a (special) countable sequence of properties true* $Bn(1/2)$-almost everywhere).

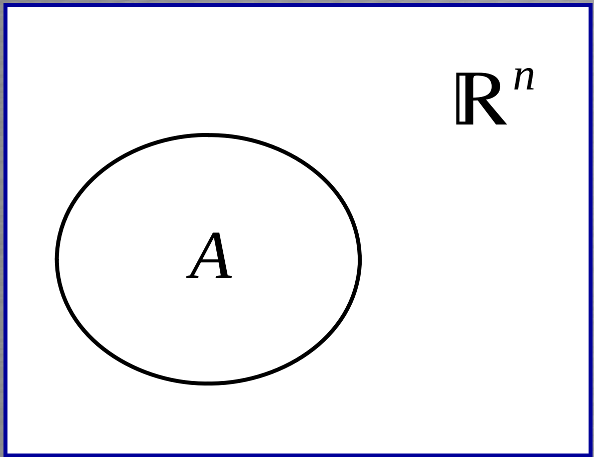
P. Martin-Löf (1966), Inform. Control **9**, 602-619.

The limiting frequencies $\nu_1$ and $\nu_0$ need not be the same, e.g., $\nu_1=2/3$ and $\nu_0=1/3$.

**Definition 2 ($Bn(\nu_1)$- random binary sequence).** *An infinite binary sequence is called (*Martin-Löf*)* $Bn(\nu_1)$- random *iff it is not rejected by the Martin-Löf test (i.e., if it satisfies a countable sequence of properties true* $Bn(\nu_1)$-almost everywhere).

**Remark 1.** *No finite binary sequence is random.*

# 3. Real random sequences.




Given a probability space $(\mathbb{R}^n, \mathcal{B}^n, \Pr_{\mathbf{X}})$, a set $A \in \mathcal{B}^n$ and an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$, $(\mathbf{x}_i \in \mathbb{R}^n)$ give rise to a binary sequence $b_1, b_2, \ldots$, where

$$b_i = \begin{cases} 1\,;\ \mathbf{x}_i \in A \\ 0\,;\ \text{otherwise} \end{cases}.$$

**Definition 3 ($\Pr_{\mathbf{X}}$-random sequence).** *Given a probability space* $(\mathbb{R}^n, \mathcal{B}^n, \Pr_{\mathbf{X}})$, *an infinite sequence* $\mathbf{x}_1, \mathbf{x}_2, \ldots$, $(\mathbf{x}_i \in \mathbb{R}^n)$ *is* $\Pr_{\mathbf{X}}$-random *iff for every* $A \in \mathcal{B}^n$ *the corresponding binary sequence* $b_1, b_2, \ldots$ *is* $Bn[\Pr_{\mathbf{X}}(A)]$-*random.*

In this way, the probability distribution $\Pr_{\mathbf{X}}$ on $\mathcal{B}^n$ coincides with the (frequency) distribution of the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$, which is characteristic of the frequency interpretation of probability.

**Remark 2.** Every finite sequence is **non**-random. Consequently, the randomness of QM cannot be verified, it can only be postulated.

**Remark 3.** Every (possibly infinite) sequence that results from an algorithm is **non**-random. Consequently, none of the numbers from random number generators, based on algorithms, is truly random. Rather, they are pseudo-random numbers.

There are random number generators based on QM processes such as, for example, radioactive decays. The numbers from these generators may be (parts of) truly random sequences.

# 4. Monte Carlo methods.

**Basis:** Generator of (pseudo-) random numbers, uniformly distributed on an interval, often $[0,1]$.

MC integration:

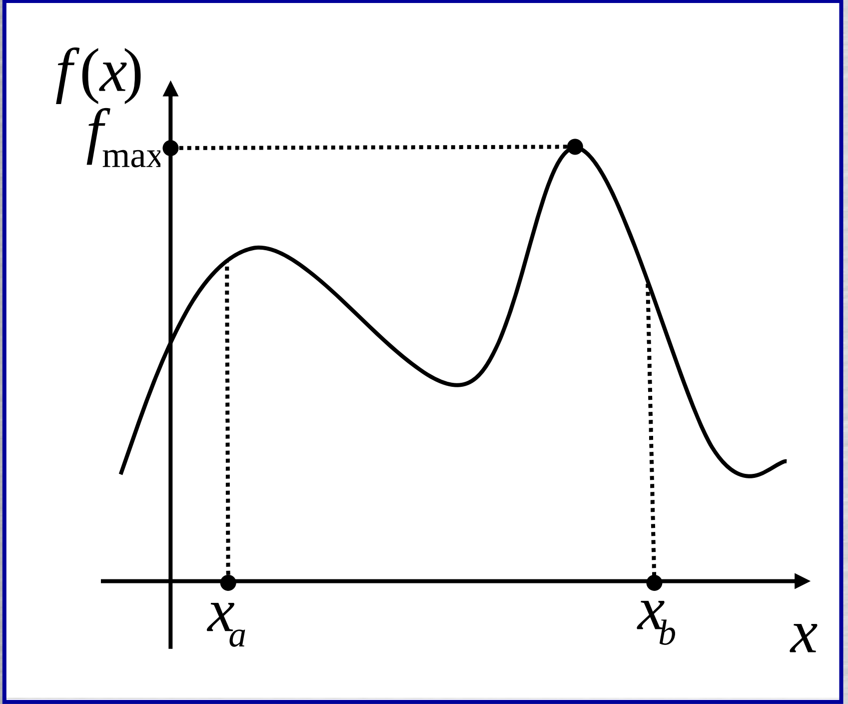
1. $x_i = x_a + \mathrm{rndm}_i \times (x_b - x_a)$

2. $y_i = \mathrm{rndm}'_i \times f_{\max}$

3. $y_i \le f(x_i) \Rightarrow N_{\mathrm{acc}} = N_{\mathrm{acc}} + 1$

– – – – – – – – – – – – – – – – –

$$\int_{x_a}^{x_b} f(x)\,dx = \frac{N_{\mathrm{acc}}}{N_{\mathrm{gen}}} \times (x_b - x_a) \times f_{\max}$$



## (Pseudo-) Random numbers for arbitrary $f_X(x)$:

$$V_X = [x_a, x_b]$$

1. $x_i = x_a + \text{rndm}_i \times (x_b - x_a)$
2. $y_i = \text{rndm}'_i \times f_{\max}$
3. $y_i \le f_X(x_i) \Rightarrow \text{accept } x_i$

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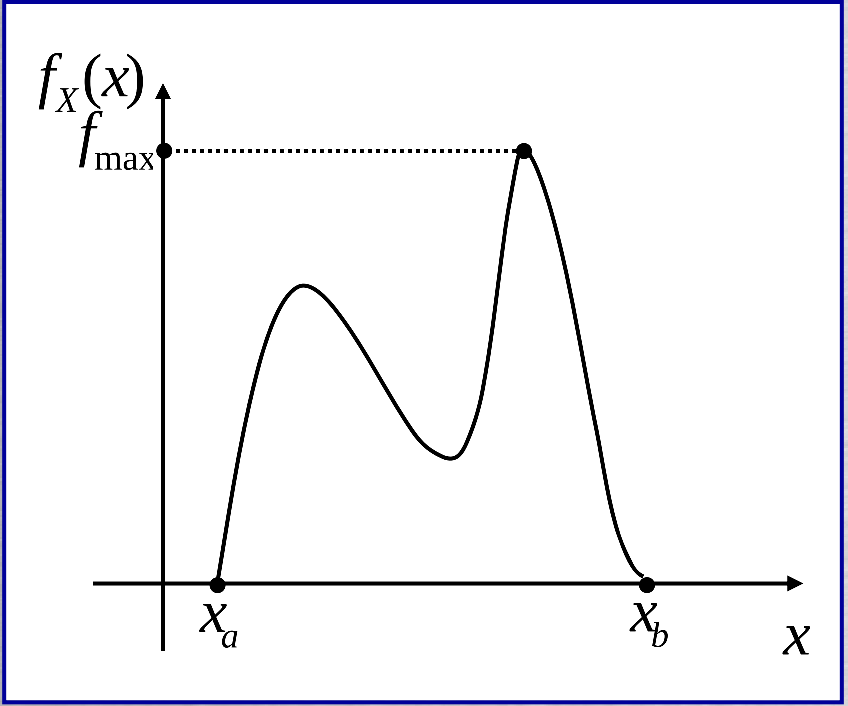
accepted $\{x_i\} \sim f_X(x)$



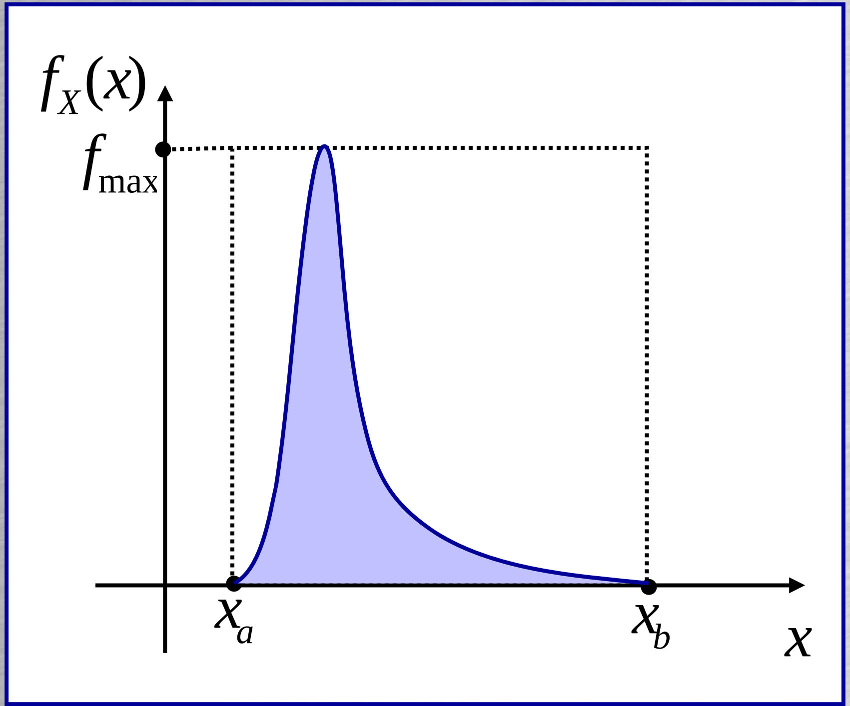
Low efficiencies may represent a serious problem:

$$V_X = [x_a, x_b]$$

1. $x_i = x_a + \mathrm{rndm}_i \times (x_b - x_a)$
2. $y_i = \mathrm{rndm}'_i \times f_{\mathrm{max}}$
3. $y_i \le f_X(x_i) \Rightarrow$ accept $x_i$

$$S_{\mathrm{rec}} = [x_a, x_b] \times f_{\mathrm{max}}$$

$$\frac{N_{\mathrm{acc}}}{N_{\mathrm{gen}}} = \frac{S_{\mathrm{shad}}}{S_{\mathrm{rec}}}$$



Solution if $F_X(x)$ simple (analytic) expression (e.g., for Exponential distr.):

$$y \equiv F_X(x) \Rightarrow f_Y(y) = \begin{cases} 1\,;\, 0 \le y \le 1 \\ 0\,;\, \text{otherwise} \end{cases};$$

$1.\ y_i = \mathrm{rndm}_i$ (100% efficiency)

$2.\ x_i = F_X^{-1}(\mathrm{y_i})$

Solutions for Normal distributions:

a) sum of $n$ uniform i.i.d. variables,

b) 2D Normal distribution……..: $\Rightarrow$

$$\begin{cases} f_{X,Y}(x,y) = f_Y(y)\, f_Y(y) \Rightarrow f_{R,\Phi}(r,\phi) = f_R(r)\, f_\Phi(\phi)\,; \\ f(\phi) = \begin{cases} \dfrac{1}{2\pi};0 \le \phi \le 2\pi \\ 0\ \ ;\text{otherwise} \end{cases},\ f_R(r) = r\exp\{-r^2/2\}\,; \\ \Rightarrow F_R(r) = 1 - \exp\{-r^2/2\}, r \ge 0\,; \Rightarrow z \equiv F_R(r) \\ \Rightarrow f_{Z,\Phi}(z,\phi) = f_Z(z)\, f_\Phi(\phi)\,;\, f_Z(z) = \begin{cases} 1;0 \le z \le 1 \\ 0;\text{otherwise} \end{cases}. \end{cases}$$

# IV. (Classical) Confidence Intervals:

1. Motivation,
2. Construction,
3. Intervals based on likelihood-ratio ordering,
4. Intervals for constrained parameters,
5. Confidence intervals for discrete distributions,
6. On the shortest confidence intervals.

# 1. Motivation.

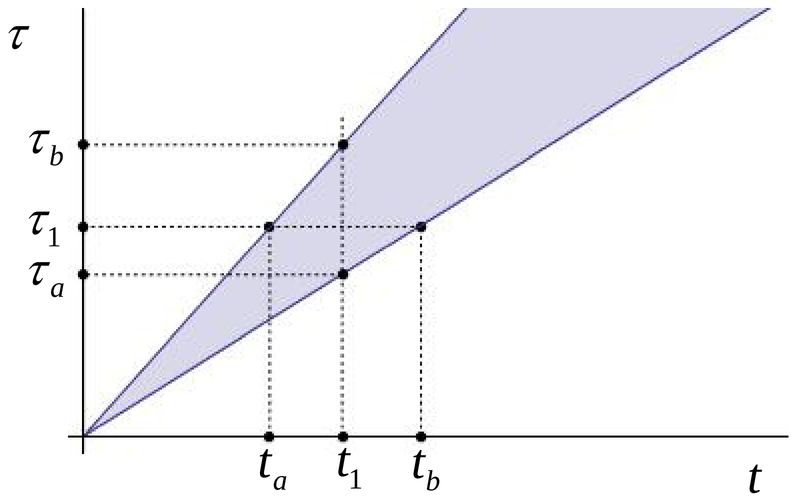
**Example 1 (Prolog):** given $t_1$, can we say anything about $\tau$ ?

The parameter $\tau$ may take on every value in a continuum
$\mathbb{R}^+ \Longrightarrow$ a measure of a single point in the continuum is 0.

For verifiable predictions we must turn to interval estimations.

J. Neyman (1937). Phil. Trans. R. Soc., **A 236**, 333-380.

## 2. Construction.




1) $\alpha \in [0, 1-\gamma]$

2) $\tau_1$

3) $t_a : F_I(t_a \mid \tau_1) = \alpha$

4) $t_b : F_I(t_b \mid \tau_1) = \alpha + \gamma$

$\Rightarrow \Pr_I(t_a < t \le t_b \mid \tau)$

$= F_I(t_b \mid \tau_1) - F_I(t_a \mid \tau_1)$

$= \gamma$

5) $\tau \in (0, \infty)$

6) $t_1$; $\tau_1$ true value

7) $\tau_1 \in (\tau_a, \tau_b) \Leftrightarrow t_1 \in (t_a, t_b)$

**Remark 4.** $$\left.\begin{aligned} F_I(t_1 \mid \tau_b) &= \alpha \\ F_I(t_1 \mid \tau_a) &= \alpha + \gamma \end{aligned}\right\} \Rightarrow \gamma = F_I(t_1 \mid \tau_a) - F_I(t_1 \mid \tau_b).$$

**Remark 5.** $\alpha = \alpha(\tau)$, as long as $t_a(\tau)$ and $t_b(\tau)$ strictly monotone in $\tau$.

# 3. Confidence intervals based on likelihood-ratio ordering.

G.J. Feldman, R.D. Cousins (1998), Phys. Rev. **D 57**, 3873-3889.

May be regarded as a definition of $\alpha(\tau)$ (of $\alpha(\theta)$).

Given $\theta : R(x,\theta) \equiv \dfrac{f_I(x\,|\,\theta)}{f_I(\hat{x}\,|\,\theta)}$ ; $x = \hat{x} : f_I(x\,|\,\theta) = \text{max.}$,

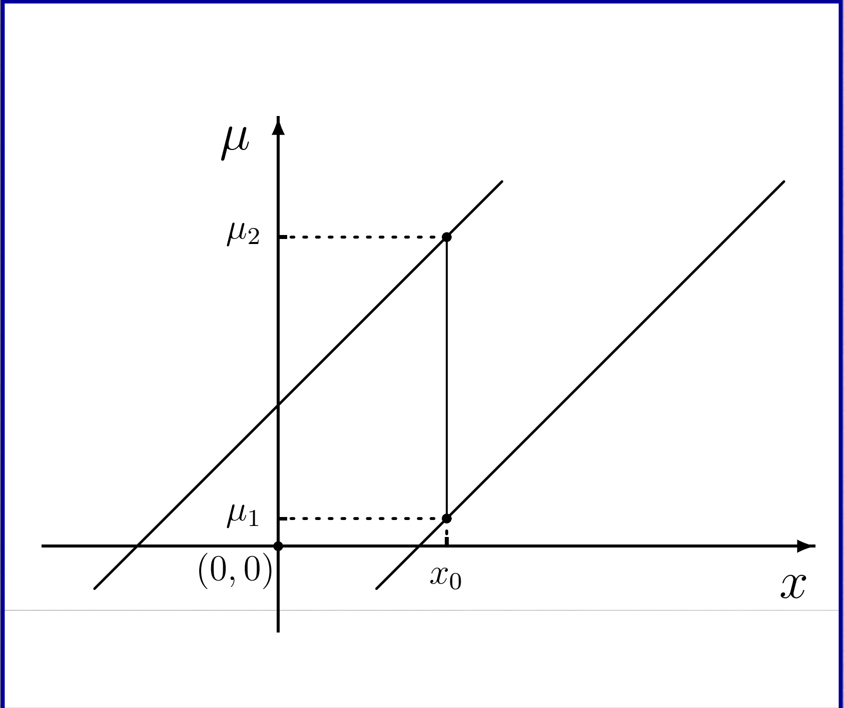
$$A = (x_a, x_b) = \{x \in V_X : R(x,\theta) \geq R_0 \text{ and } \Pr_X(A\,|\,\theta) = \gamma\}.$$

The rest of the procedure is identical to the one on the previous slide.

**Remark 6.** These confidence intervals are equivariant under ono - to - one reparametrizations $y \equiv s(x)$ and $\nu \equiv \bar{s}(\theta)$:

$$(\nu_a[s(x_1)], \nu_b[s(x_1)]) = (\bar{s}[\theta_a(x_1)], \bar{s}[\theta_b(x_1)]).$$

## **Example 2.** Confidence intervals for $x \sim N(\mu, \sigma = 1)$:




$\gamma = 0.9$

a) $\alpha = 0.05 = \text{const.}$,

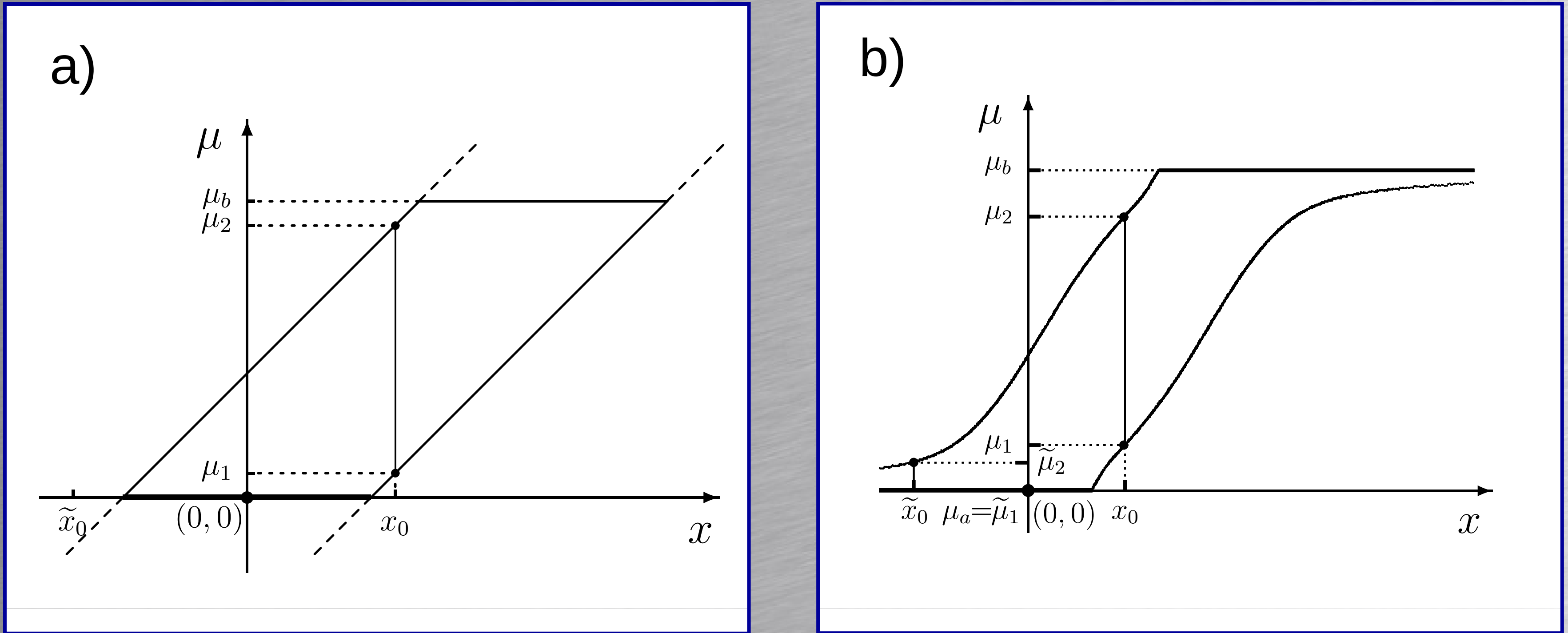
b) intervals from the likelihood - ratio ordering principle.

Important because of CLT :

$$n \to \infty : \bar{x}_n \sim N\left(\langle x \rangle, \sqrt{\frac{Var(x)}{n}}\right) \approx N\left(\langle x \rangle, \sqrt{\frac{s_n^2}{n}}\right)$$

## 4. Intervals for constrained parameters.

**Example 3. Confidence intervals for** $x \sim N(0 \le \mu \le 3.92, \sigma = 1)$:



## 5. Confidence intervals for discrete distributions.

For discrete $F_I(n \mid \mu)$, equations

$$F_I(n_a \mid \mu) = \alpha\,,\; F_I(n_b \mid \mu) = \alpha + \gamma$$

do not have solutions for all $\mu$.

Construct the shortest CI's whose coverage $\geq \gamma$ (conservative CI's).

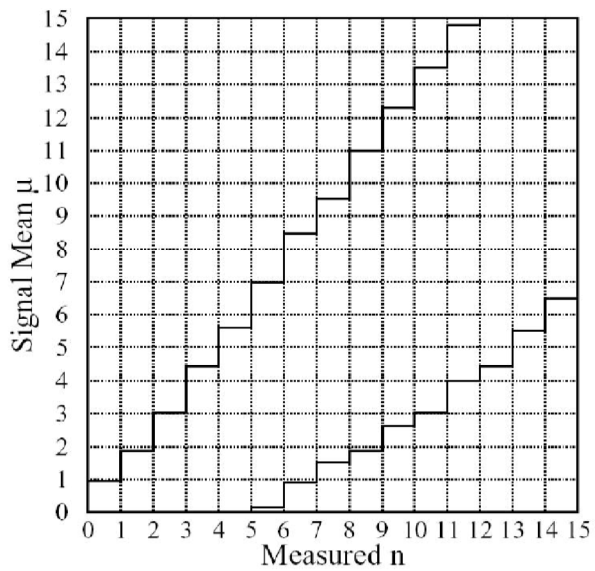



FIG. 7. Confidence belt based on our ordering principle, for 90% C.L. confidence intervals for unknown Poisson signal mean $\mu$ in the presence of Poisson background with known mean $b = 3.0$.

# 6. On the shortest confidence intervals.

**Example 4 (Exponential family):** Given $t_1$, chose $\alpha = \text{const.}$ such that the lenght of the confidence interval $(\tau_a, \tau_b)$ will be minimal.

For $\gamma = 0.2 : \alpha = 0.740976 \Rightarrow (\tau_a, \tau_b) = (0.353 t_1, 0.740 t_1)$.

**Note :** $(\tau_a, \tau_b)$ does not contain $\hat{\tau} = t_1$, but contains $t_1 / 2$.

**Example 4 (cont'd):** $x \equiv \ln x, \mu \equiv \ln \tau \Rightarrow f_{I'}(x \mid \mu) = e^{x-\mu} \exp\{-e^{x-\mu}\}$.

For $\gamma = 0.2 : \alpha = 0.527573 \Rightarrow (\mu_a, \mu_b) = (\ln t_1 - 0.263, \ln t_1 + 0.288)$.

**Note :** $(\mu_a, \mu_b) \neq (\ln \tau_a, \ln \tau_b)$.

**Example 5 (Hypothesis testing):** $H : \tau = \tau_1$. $H$ rejected at confidence (significance) level $\gamma$ if $\tau_1$ is outside the shortest confidence interval $(\tau_a, \tau_b)$ whose coverage is $\gamma$.

The choice of parametrization depends on what you want, accept or reject $H$ (ideology!).

## V. Hypothesis testing:

1. Basic definitions,
2. Errors of the first and the second kind,
3. Neyman-Pearson Lemma,
4. Uniformly most powerful tests.

# 1. Basic definitions.

H. Frank, S.C. Althoen (1994), *Statistics*, Chaps. 9-11, pp. 326-480.

**Inference :** what is the value of parameter $\theta$?

**Test of a hypothesis :** is $\theta_0$ the value of parameter $\theta$?

**Test (null) hypothesis** $H_0 : \theta = \theta_0$.

**Alternative hypothesis** $H_1 : \theta = \theta_1$ $(\theta_1 \neq \theta_0)$ or $\theta > \theta_0$.

**Test** $W$ **:** a numerical index that is expected to take the value $w_0$ if $H_0$ is correct and is expected some other value if $H_1$ is correct.

**Test statistic** $W$ **:** $W = W(x_1, x_{2,} \ldots)$ .

**Rejection (critical) region** $R_C$ **:** the region of the values of a test $W$ that are unlikely (??) if $H_0$ is correct but relatively likely (??) if $H_1$ is correct.

**Critical value (significance level, size of** $R_C$**) :** $\alpha \equiv \Pr_I(R_C \mid \theta_0)$ .

**Confidence level** $Cl \equiv 1-\alpha$ .

**Decision :** if $w \in R_C$ abandon $H_0$ in favor of $H_1$ at confidence level $Cl$;
if $w \notin R_C$, $H_1$ is rejected.

## 2. Errors of the first and second kind.

**Error I (false positive):** rejecting $H_0$ when it is correct.

**Error II (false negative):** rejecting $H_1$ when it is correct (i.e., fail to reject $H_0$ when it is indeed false).

$\alpha = P(\text{Error I})$, $\beta \equiv P(\text{Error II})$.

**Power of the test:** $1-\beta$ (probability that a test will reject a false $H_0$).

## 3. Neyman-Pearson Lemma.

**A best** $R_C$ **of size** $\alpha$ **:** $P(R_C \mid H_0) = \alpha$ , $P(R_C \mid H_1) \geq P(Q_C \mid H_1)$
for all $Q_C$ for which $P(Q_C \mid H_0) = \alpha$ .

**Lemma 1 (Neyman - Pearson).** $H_0 : \theta = \theta_0,\ H_1 : \theta = \theta_1$ ,

$$W(x_1,\ldots,x_n;\theta_0,\theta_1) = \frac{L(x_1,\ldots,x_n;\theta_0)}{L(x_1,\ldots,x_n;\theta_1)},$$

$$R_C = \{(x_1,\ldots,x_n) : W(x_1,\ldots,x_n;\theta_0,\theta_1) \leq \eta\}$$

$\Rightarrow R_C$ is a best critical region of size $\alpha$, i.e.,
$W(x_1,\ldots,x_n;\theta_0,\theta_1)$ is a **most powerful test**
of size $\alpha = P(R_C \mid H_0)$.

## 4. Uniformly most powerful test.

**Uniformly most powerful test of size $\alpha$.**

$H_0 : \theta = \theta_0, H_1 : \theta \in A; (\theta_0 \notin A)$,

$W\left(x_1, \ldots, x_n; \theta_0, \theta_1\right)$ most powerfull for all $\theta_1 \in A$

$\Rightarrow W\left(x_1, \ldots, x_n; \theta_0, \theta_1\right)$ is a uniformly most powerful test of size $\alpha$ for alternatives in $A$.

**Example 6 (UMPT for Normal distr.) :** $\{X_1, \ldots, X_n\}$ i.i.d., $X_i \sim N(\mu, 1)$,

$$H_0 : \mu = \mu_0, H_1 : \mu > \mu_0 \Rightarrow \frac{L(x_1, \ldots, x_n; \mu_0)}{L(x_1, \ldots, x_n; \mu_1)}; \mu_1 > \mu_0, \text{ is UMPT.}$$

**Example 7 :** There is no UMPT if $H_0 : \mu = \mu_0, H_1 : \mu \neq \mu_0$.

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