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Applied Fitting Theory IV
Formulas for Track Fitting

This note introduces some mathematical algorithms that I have developed over the years for fitting and projecting tracks in solenoidal magnetic fields. I have spent several years developing these formulas but so far the documentation has existed only in hand-written notes and Fortran programs. Hopefully, this note will provide a useful reference for anyone interested in these details. Many of the equations have been implemented in the FTMONTE package, and I have made every effort to make sure that the formulas are correct, although there might be an error or two lurking somewhere. I have built in various cross checks in my programs to verify that the Fortran code is correct, especially the derivatives.

1. Helix equations of motion

1.1. Track representations

I use two track representations:

1. The C or canonical representation is used in track fitting; the 5 parameters describe the entire shape of the helix. $\alpha_C = (c, \phi_0, D, \lambda, z_0)$, where $c = 1/2R$, R being the radius of curvature, ϕ_0 is the ϕ of the track momentum at the distance of closest approach to the origin, D is the signed impact parameter in the $x-y$ plane, $\lambda = \cot\theta$, where θ is the polar angle measured from the $+z$ axis and z_0 is the z of the track at the point of closest approach to the origin in the $x-y$ plane.

2. The W representation is used in kinematic fitting and for finding the 4–momentum at a particular point on the helix. $\alpha_W = (p_x, p_y, p_z, E, x, y, z)$, i.e., a 4–momentum plus a point at which the 4–momentum is evaluated.

1.2. Equations of motion

I begin by defining the following useful symbols:

$$
\begin{aligned}
a &= -0.2998BQ \quad (B = \text{mag. field},\ Q = \text{charge}) \\
p_\perp &= \sqrt{p_x^2 + p_y^2} \\
u &= p_x/p_\perp \\
v &= p_y/p_\perp \\
u_0 &= p_{0x}/p_\perp = \cos\phi_0 \\
v_0 &= p_{0y}/p_\perp = \sin\phi_0 \\
\rho &= \frac{1}{R} \quad R = \text{radius of curvature}
\end{aligned}
$$

Assume that the particle has charge Q and is moving in a magnetic field of strength B. The particle moves along a helix with a signed radius of curvature R. R (and ρ) is positive for counterclockwise motion. The trajectory of a helix is governed by the following equations, valid when B is along $\vec{z}$.

$$
\begin{aligned}
p_x &= p_{0x}\cos\rho s_\perp - p_{0y}\sin\rho s_\perp \\
p_y &= p_{0y}\cos\rho s_\perp + p_{0x}\sin\rho s_\perp \\
p_z &= p_{0z} \\
E &= E_0 \\
x &= x_0 + \frac{p_{0x}}{a}\sin(\rho s_\perp) - \frac{p_{0y}}{a}(1-\cos(\rho s_\perp)) \\
y &= y_0 + \frac{p_{0y}}{a}\sin(\rho s_\perp) + \frac{p_{0x}}{a}(1-\cos(\rho s_\perp)) \\
z &= z_0 + \lambda s_\perp
\end{aligned}
$$

where (x_0, y_0, z_0) is a known point on the helix, $(p_{0x}, p_{0y}, p_{0z}, E_0)$ is its 4–momentum vector there and $\rho = 2c = -0.2998BQ/p_\perp \equiv a/p_\perp$, They are functions of $s_\perp$, the arc length in the $x - y$ plane from (x_0, y_0, z_0) to the point x, y, z we are trying to find. If (x_0, y_0, z_0) is the point of closest approach to the origin, then p_{0x}, p_{0y}, p_{0z}, E_0, x_0, y_0 and z_0 can be written in terms of the 5 canonical parameters:

$$
\begin{aligned}
p_{0x} &= p_\perp u_0 = \frac{a}{2c}\cos\phi_0 \\
p_{0y} &= p_\perp v_0 = \frac{a}{2c}\sin\phi_0 \\
p_{0z} &= p_\perp \lambda = \frac{a}{2c}\lambda \\
E_0 &= \sqrt{p_\perp^2(1+\lambda^2)+m^2} \\
x_0 &= -Dv_0 = -D\sin\phi_0 \\
y_0 &= +Du_0 = +D\cos\phi_0 \\
z_0 &= z_0
\end{aligned}
$$

It is easy to see that D is positive when the ϕ of the position is greater than that of the direction of motion at the point of closest approach. Note that $r_{\min} = |D|$ and that the point of closest approach, reached when $s_\perp = 0$ is just (x_0, y_0). The track reaches its maximum radius $r_{\max} = |D + 1/c|$ when $\rho s_\perp = \pi$. This occurs at the point $(x_{\max}, y_{\max})$, where

$$
\begin{aligned}
x_{\max} &= -\left(D + \frac{1}{c}\right) v_0, \\
y_{\max} &= +\left(D + \frac{1}{c}\right) u_0.
\end{aligned}
$$

The center of the circle is found to be at

$$
\begin{aligned}
x_c &= -\left(\frac{1}{2c} + D\right) v_0, \\
y_c &= +\left(\frac{1}{2c} + D\right) u_0.
\end{aligned}
$$

The quantity $1 + \rho D = 1 + 2cD$ is always positive; this fact turns out to be very useful when using the track parameters as we will see later.

A pair of conserved quantities can be derived from the (x, y, z) equations by substituting the equations for p_x and p_y:

$$p_x + ay = p_{0x} + ay_0 = \text{const}$$
$$p_y - ax = p_{0y} - ax_0 = \text{const}$$

1.3. Track coordinates as a function of radius

Sometimes it is useful to express ϕ, z and $s_\perp$ as a function of $r = \sqrt{x^2 + y^2}$:

$$\begin{aligned}
\sin(\phi - \phi_0) &= \frac{rc + (D/r)(1 + cD)}{1 + 2cD} = \frac{D}{r} + \frac{(c/r)(r^2 - D^2)}{1 + 2cD}, \\
\cos(\phi - \phi_0) &= \pm \frac{\sqrt{(1 - D^2/r^2)[(1 + cD)^2 - r^2c^2]}}{1 + 2cD}, \\
z - z_0 &= \lambda s_\perp, \\
\sin cs_\perp &= c\sqrt{\frac{r^2 - D^2}{1 + 2cD}},
\end{aligned}$$

where the sign of $\cos(\phi - \phi_))$ is chosen depending on whether the particle is on the outgoing (+1) or incoming (−1) branch of the circle. This ambiguity is fundamental; the particle always passes through the same radius going out and coming in.

The momentum and position coordinates at the radius r are given by

$$\begin{aligned}
p_x &= p_{0x}(1 - 2B^2) - p_{0y}2\epsilon B\sqrt{1 - B^2} \\
p_y &= p_{0y}(1 - 2B^2) + p_{0x}2\epsilon B\sqrt{1 - B^2} \\
p_z &= p_{0z} \\
x &= x_0 + \frac{u_0}{\rho}2\epsilon B\sqrt{1 - B^2} - \frac{v_0}{\rho}2B^2 \\
y &= x_0 + \frac{v_0}{\rho}2\epsilon B\sqrt{1 - B^2} + \frac{u_0}{\rho}2B^2 \\
z &= z_0 + \lambda s_\perp \\
s_\perp &= \begin{cases} (1/c)\sin^{-1} B & \text{for } \epsilon = +1 \\ (1/c)(\pi - \sin^{-1} B) & \text{for } \epsilon = -1 \end{cases}
\end{aligned}$$

where $B = c\sqrt{(r^2 - D^2)/(1 + 2cD)} = \sin \frac{1}{2}\rho s_\perp = \sin cs_\perp$ and $\epsilon = +1(-1)$ corresponds to the outgoing (incoming) branch.

Similarly, we can calculate the angle α between the track direction and the radius vector

$$\begin{aligned}
\sin\alpha &= \frac{xp_y - yp_x}{rp_\perp} = rc - \frac{D}{r}(1+cD), \\
\cos\alpha &= \frac{xp_x + yp_y}{rp_\perp} = (1+2cD)\cos(\phi-\phi_0) = \epsilon\sqrt{(1-D^2/r^2)[(1+cD)^2 - r^2c^2]}.
\end{aligned}$$

Note that $\sin\alpha \simeq rc$ and $s_\perp \simeq (1/c)\sin^{-1} rc$ for $D \ll r$ and $cD \ll 1$. Also, the sign of ϵ can be determined if one knows the position and momentum components at any given point on the helix from the formula for $\cos\alpha$.

1.4. Computing track parameters from instantaneous position and momentum

This is similar to the classic problem of finding planetary orbital elements given the instantaneous position and velocity vectors. Assume we are given $\alpha_W = (p_x, p_y, p_z, E, x, y, z)$. First we find λ and c:

$$\begin{aligned}
\lambda &= \cot\theta = p_z/\sqrt{p_x^2 + p_y^2} \equiv p_z/p_\perp \\
c &= \rho/2 = -0.1499BQ/p_\perp \equiv a/2p_\perp.
\end{aligned}$$

Take the equations of motion above and solve for $\sin\rho s_\perp$, $\cos\rho s_\perp$:

$$\begin{aligned}
(p_\perp + aD)\sin\rho s_\perp &= \rho(xp_x + yp_y), \\
(p_\perp + aD)\cos\rho s_\perp &= p_\perp - \rho(xp_y - yp_x),
\end{aligned}$$

which yields $s_\perp$ and

$$\begin{aligned}
\cos\phi_0 &= \frac{p_x + ay}{T}, \\
\sin\phi_0 &= \frac{p_y - ax}{T}, \\
D &= \frac{1}{a}[T - p_\perp] = \frac{-2(xp_y - yp_x) + a(x^2+y^2)}{T + p_\perp}, \\
z_0 &= z - \lambda s_\perp,
\end{aligned}$$

where $T = \sqrt{p_\perp^2 - 2a(xp_y - yp_x) + a^2(x^2+y^2)}$. The second form for D is useful for particles with small curvature. For the case when $a = 0$ (zero curvature), the equations simplify to

$$
\begin{aligned}
c &= \rho = 0 \\
\cos\phi_0 &= p_x/p_\perp \\
\sin\phi_0 &= p_y/p_\perp \\
D &= (yp_x - xp_y)/p_\perp \\
\lambda &= p_z/p_\perp \\
z_0 &= z - \lambda s_\perp \quad (s_\perp = \frac{xp_x + yp_y}{p_\perp})
\end{aligned}
$$

2. Useful formulas for track fitting in a Solenoidal Field

I present here some useful formulas for calculating the quantities directly measured in the most common tracking devices and their partial derivatives with respect to the 5 helix parameters. These values are needed by the track fitting routines described in this document. I assume that the magnetic field is along the z axis. See the first section for the definition of some of the symbols used here.

The measurments needed are (1-2) the distance of closest approach to axial and stereo drift chamber wires, (3) z as measured by cathode strips in drift chambers, (4-5) the distance along $r-\phi$ and $r-z$ silicon planes and (6-7) the distance along x and y in silicon disks located at fixed z positions. The Fortran code for these cases is stored in `libFTLIB`. The above quantities are computed in `FTDRF1-7` and `FTDF1-7`, the analytic derivatives are calculated in `FTDRV1-7` and the numerical derivatives (absolutely essential for checking the analytic expressions) are computed in `FTDRN1-7`.

2.1. Axial drift chamber wires

Consider an axial wire at position (x_w, y_w) and define (x, y, z) as the point on the helix which is closest to (x_w, y_w) (we need only consider the $x-y$ motion) and call d_w the signed distance of closest approach of the track to the wire. Now use the fact that the normal to the circle at (x, y) lies along the radius vector and passes through (x_w, y_w). The $x-y$ direction cosines of the track at that point are $(u, v) = (p_x/p_\perp, p_y/p_\perp)$ so the normal to the circle has direction cosines $(-v, u)$. Writing the equation of the normal as

$$x + d_w v = x_w,$$
$$y - d_w u = y_w,$$

we get, after a little manipulation,

$$(1 + \rho d_w)\cos\rho s_\perp = 1 - \rho v_0 \Delta_x + \rho u_0 \Delta_y,$$
$$(1 + \rho d_w)\sin\rho s_\perp = -\rho u_0 \Delta_x - \rho v_0 \Delta_y,$$

where $\Delta_x = x_0 - x_w$ and $\Delta_y = y_0 - y_w$. This allows us to solve for $s_\perp$ and

$$d_w = \frac{1}{\rho}\left[\sqrt{1 + 2\rho\Delta_w} - 1\right] = \frac{2\Delta_w}{1 + \sqrt{1 + 2\rho\Delta_w}} \simeq \Delta_w(1 - \frac{1}{2}\rho\Delta_w)$$

where $\Delta_w = u_0\Delta_y - v_0\Delta_x + \frac{1}{2}\rho(\Delta_x^2 + \Delta_y^2)$ is a very good approximation to d_w. The sign of d_w is positive when the track has a larger ϕ than that of the wire.

The derivatives of d_w with respect to the 5 helix parameters are

$$\frac{\partial d_w}{\partial c} = (\Delta_x^2 + \Delta_y^2)(1 - \rho\Delta_w) - \Delta_w^2,$$
$$\frac{\partial d_w}{\partial \phi_0} = -(\Delta_x u_0 + \Delta_y v_0)(1 + \rho D)(1 - \rho\Delta_w),$$
$$\frac{\partial d_w}{\partial D} = (1 - \rho\Delta_x v_0 + \rho\Delta_y u_0)(1 - \rho\Delta_w),$$
$$\frac{\partial d_w}{\partial \lambda} = 0,$$
$$\frac{\partial d_w}{\partial z_0} = 0.$$

These reduce, when $\rho = 0$, to

$$\frac{\partial d_w}{\partial c} = (\Delta_x u_0 + \Delta_y v_0)^2,$$
$$\frac{\partial d_w}{\partial \phi_0} = -(\Delta_x u_0 + \Delta_y v_0),$$
$$\frac{\partial d_w}{\partial D} = 1,$$
$$\frac{\partial d_w}{\partial \lambda} = 0,$$
$$\frac{\partial d_w}{\partial z_0} = 0.$$

2.2. **Stereo drift chamber wires**

Stereo wires couple three equations containing terms $\cos\rho s_\perp$, $\sin\rho s_\perp$ and $s_\perp$, which makes them impossible to solve analytically. Let z_p be the z coordinate of a point on the wire, $\tan\tau$ be the tangent of the stereo angle τ and (x_w, y_w) be the coordinates of the wire at $z_p = 0$ ((r_w, ϕ_w) in polar coordinates). The coordinates of any point (x_p, y_p) on the wire are

$$x_p = x_w + z_p \tan\tau \sin\phi_s,$$
$$y_p = y_w - z_p \tan\tau \cos\phi_s,$$

where the CLEO drift chamber is wire in such a way that $\phi_s = \phi_w$. The (unnormalized) direction cosines of the wire and of the track at the point of closest approach to the wire are

$$\vec{\eta}_w = (\tan\tau\sin\phi_s, -\tan\tau\cos\phi_s, 1),$$
$$\vec{\eta}_t = (u, v, \lambda) = (u_0\cos\rho s_\perp - v_0 \sin\rho s_\perp, v_0\cos\rho s_\perp - u_0\sin\rho s_\perp, \lambda).$$

The line which is normal to both the wire and the helix at its point of closest approach has direction cosines

$$\vec{u}_d = \frac{\vec{\eta}_w \times \vec{\eta}_t}{|\vec{\eta}_w \times \vec{\eta}_t|} = (v + \lambda'\cos\phi_s, -u + \lambda'\sin\phi_s, G_1\tan\tau)/u_d$$
$$u_d = \sqrt{1 + 2\lambda' H_1 + \lambda'^2 + G_1^2\tan^2\tau}$$

where u_d is a normalization factor and

$$\lambda' = \lambda\tan\tau,$$
$$C_1 = u_0\cos\phi_s + v_0\sin\phi_s = \cos(\phi_0 - \phi_s),$$
$$S_1 = v_0\cos\phi_s - u_0\sin\phi_s = \sin(\phi_0 - \phi_s),$$
$$G_1 = C_1\cos\rho s_\perp - S_1\sin\rho s_\perp = \cos(\phi_0 - \phi_s + \rho s_\perp),$$
$$H_1 = C_1\sin\rho s_\perp + S_1\cos\rho s_\perp = \sin(\phi_0 - \phi_s + \rho s_\perp).$$

Just as in the axial wire case we write the equation of the normal:

$$x + d_w(v + \lambda'\cos\phi_s)/u_d = x_p,$$
$$y - d_w(u - \lambda'\sin\phi_s)/u_d = y_p,$$
$$z + d_w G_1 \tan\tau/u_d = z_p,$$

which can be seen to reduce to the axial case by setting the stereo angle τ to zero. Note that the line normal to the track and the wire lies outside the $x - y$ plane. Inserting the equations of motion for x, y and z we get, after some manipulation, the equations we want to solve

$$(1 + \rho d_w/u_d) \cos \rho s_\perp = +A_1 - \frac{\rho d_w}{u_d} \lambda' S_1$$

$$(1 + \rho d_w/u_d) \sin \rho s_\perp = -A_2 - \frac{\rho d_w}{u_d} \lambda' C_1$$

$$z_p = z_0 + \lambda s_\perp - \frac{d_w}{u_d} G_1 \tan \tau$$

where $A_1 = 1 - \rho\Delta_x v_0 + \rho\Delta_y u_0$, $A_2 = \rho\Delta_x u_0 + \rho\Delta_y v_0$, $\Delta_x = x_0 - x_w - z_p \tan\tau \sin\phi_s$ and $\Delta_y = y_0 - y_w + z_p \tan\tau \cos\phi_s$.

Squaring, adding and collecting terms leads to the quadratic equation

$$\frac{d_w^2}{u_d^2}(1 - \lambda'^2) + \frac{d_w}{u_d}\frac{J}{\rho} - \frac{2\Delta_w}{\rho},$$

where as before $\Delta_w = u_0\Delta_y - v_0\Delta_x + \frac{1}{2}\rho(\Delta_x^2 + \Delta_y^2)$ and $J = 1 + \lambda'(A_1 S_1 - A_2 C_1)$. Note that the definition of Δ_w here differs slightly from the axial case because it has z dependent terms in Δ_x and Δ_y. The solution is

$$\begin{aligned} d_w &= \frac{u_d/\rho}{1 - \lambda'^2} J \left(-1 + \sqrt{1 + 2(1 - \lambda'^2)\rho\Delta_w/J^2} \right), \\ &= \frac{u_d}{J} \frac{2\Delta_w}{1 + \sqrt{1 + 2(1 - \lambda'^2)\rho\Delta_w/J^2}}, \\ &\simeq \frac{u_d}{J} \Delta_w (1 - \frac{1}{2}(1 - \lambda'^2)\rho\Delta_w/J^2). \end{aligned}$$

The first two expressions for d_w are exact but still depend on $\cos\rho s_\perp$, $\sin\rho s_\perp$ and $z_p \simeq z_0 + \lambda s_\perp$. However, the latter expressions are all multiplied by terms proportional to $\tan\tau$ so they only modify the final result slightly. For highest precision, I recommend determining $s_\perp$ using the following procedure. First, treat the wire as axial and solve $\sin cs_\perp = c\sqrt{(r_w^2 - D^2)/(1 + 2cD)}$ for $s_\perp$. This yields $z \simeq z_0 + \lambda s_\perp$ which can be used to get more precise values for Δ_x and Δ_y. These can in turn be used to obtain better values for $\cos\rho s_\perp$ and $\sin\rho s_\perp$ (and hence $s_\perp$) using the $(1 + \rho d_w/u_d)\cos\rho s_\perp$ and $(1 + \rho d_w/u_d)\sin\rho s_\perp$ equations above with Δ_w substituting for d_w in this first iteration. The values obtained for

$\cos\rho s_\perp$, $\sin\rho s_\perp$ and $s_\perp$ are accurate enough to be plugged into the above equation for d_w. One can iterate this procedure once more if necessary (using the value of d_w just obtained to get precise values of $\cos\rho s_\perp$, $\sin\rho s_\perp$ and z_p) but this is probably not necessary.

An alternative approach is to take the three coupled equations for $\cos\rho s_\perp$, $\sin\rho s_\perp$ and z_p and expand the three unknowns $s_\perp$, z_p and d_w to first order. For this work it is useful to multiply the first two equations by $\cos\rho s_\perp$ and $\sin\rho s_\perp$ and add and subtract:

$$\begin{aligned} d_w &= \frac{u_d}{\rho}(A_c - 1) - d_w\lambda' H_1, \\ 0 &= -\frac{u_d}{\rho}A_s - d_w\lambda' G_1, \\ z_p &= z_0 + \lambda s_\perp - \frac{d_w}{u_d}G_1 \tan\tau, \end{aligned}$$

where the terms A_c and A_s are defined by $A_c = A_1\cos\rho s_\perp - A_2\sin\rho s_\perp$ and $A_s = A_1\sin\rho s_\perp + A_2\cos\rho s_\perp$. The solution is started by assuming axial wires, i.e., $\tan\tau = 0$. Collecting d_w terms and expanding to first order we get

$$\begin{aligned} \delta d_w &= \frac{(X_u A_c - X_u + u_d G_1\lambda')\delta s_\perp}{1 + \lambda' H_1}, \\ \delta s_\perp &= \frac{\Delta_s - G_1\lambda'\delta d_w}{u_d + \rho d_w + X_u A_s + u_d H_1\lambda'}, \\ \delta z_s &\simeq \lambda\delta s_\perp, \end{aligned}$$

where $X_u = (G_1/u_d)\,[\lambda' - H_1\tan^2\tau]$, $\Delta_c = (u_d/\rho)(A_c - 1) - d_w(1 + H_1\lambda')$ and $\Delta_s = -(u_d/\rho)A_s - d_w G_1\lambda'$.

To calculate the derivatives of d_w with respect to the five track parameters we use the approximate form of the d_w solution:

$$d_w \simeq \frac{u_d}{J}\Delta_w(1 - \frac{1}{2}(1 - \lambda'^2)\rho\Delta_w/J^2),$$

and ignore derivatives of u_d and J because they are multiplied by terms of order $\Delta_w\tan\tau$ which are very small for typical drift chambers. The derivatives with respect to c, ϕ_0 and D are similar to the axial case except that for highest accuracy one must include the z_p

term in the definitions of Δ_x and Δ_y. The derivatives of d_w can all be expressed in terms of those of Δ_w:

$$\frac{\partial d_w}{\partial \alpha_i} = \frac{u_d}{J}\left(\frac{\partial \Delta_w}{\partial \alpha_i}(1-\rho\Delta_w/J^2) - \delta_{i1}\Delta_w^2/J^2\right).$$

The values of $\partial\Delta_w/\partial\alpha_i$ are calculated below, ignoring terms of order $\tan^2\tau$ and $\rho d_w \tan\tau$.

$$\begin{aligned}
\frac{\partial \Delta_w}{\partial c} &= \Delta_x^2 + \Delta_y^2 + \lambda'\frac{\partial s_\perp}{\partial c}(C_1+\rho\Delta_n)\\
\frac{\partial \Delta_w}{\partial \phi_0} &= -u_0\Delta_x - v_0\Delta_y + \lambda'\frac{\partial s_\perp}{\partial \phi_0}(C_1+\rho\Delta_n)\\
\frac{\partial \Delta_w}{\partial D} &= 1 + \lambda'\frac{\partial s_\perp}{\partial D}(C_1+\rho\Delta_n)\\
\frac{\partial \Delta_w}{\partial \lambda} &= \tan\tau\left(s_\perp + \lambda\frac{\partial s_\perp}{\partial \lambda}\right)(C_1+\rho\Delta_n)\\
\frac{\partial \Delta_w}{\partial z_0} &= \tan\tau\left(1 + \lambda\frac{\partial s_\perp}{\partial z_0}\right)(C_1+\rho\Delta_n)
\end{aligned}$$

where $\Delta_n = -\Delta_x \sin\phi_s + \Delta_y\cos\phi_s$ and the derivatives with respect to $s_\perp$ are given by

$$\begin{aligned}
\frac{\partial s_\perp}{\partial c} &= -\frac{2}{\rho}(D\sin\rho s_\perp - (x_w u + y_w v) + s_\perp),\\
\frac{\partial s_\perp}{\partial \phi_0} &= -(x_w v - y_w u),\\
\frac{\partial s_\perp}{\partial D} &= -\sin\rho s_\perp,\\
\frac{\partial s_\perp}{\partial \lambda} &= -s_\perp H_1\tan\tau,\\
\frac{\partial s_\perp}{\partial z_0} &= -H_1\tan\tau,
\end{aligned}$$

where $u = u_0\cos\rho s_\perp - v_0\sin\rho s_\perp$ and $v = v_0\cos\rho s_\perp + u_0\sin\rho s_\perp$. $\partial s_\perp/\partial c = 0$ at $\rho = c = 0$.

2.3. **Cathode strips**

Cathodes actually measure the z position of a hit on the nearest anode wire layer. Recall from the previous discussion of axial drift chamber layers

$$(1+\rho d_w)\cos\rho s_\perp = 1 - \rho v_0\Delta_x + \rho u_0\Delta_y,$$

$$(1+\rho d_w)\sin\rho s_\perp = -\rho u_0\Delta_x - \rho v_0\Delta_y,$$

where $\Delta_x = x_0 - x_w$ and $\Delta_y = y_0 - y_w$ and (x_w, y_w) is the coordinate of the corresponding anode wire. These equations can be solved trivially for $s_\perp$ and $z = z_0 + \lambda s_\perp$ (recall that

$1+\rho d_w$ is always positive). For highest accuracy in calculating derivatives, it is necessary to solve for d_w using the results in the section on axial wires:

$$d_w = \frac{1}{\rho}\left[\sqrt{1+2\rho\Delta_w} - 1\right] = \frac{2\Delta_w}{1+\sqrt{1+2\rho\Delta_w}} \simeq \Delta_w(1-\frac{1}{2}\rho\Delta_w)$$

where $\Delta_w = u_0\Delta_y - v_0\Delta_x + \frac{1}{2}\rho(\Delta_x^2+\Delta_y^2)$ To find derivatives, we eliminate d_w by multiplying the first two equations by $\sin\rho s_\perp$ and $\cos\rho s_\perp$ and subtract to get

$$\sin\rho s_\perp(1-\rho v_0\Delta_x + \rho u_0\Delta_y) + \cos\rho s_\perp(\rho u_0\Delta_x + \rho v_0\Delta_y) = 0$$

Setting as before $A_1 = 1-\rho\Delta_x v_0 + \rho\Delta_y u_0$ and $A_2 = \rho\Delta_x u_0 + \rho\Delta_y v_0$ we get

$$\begin{aligned}
\frac{\partial z}{\partial c} &= -2\frac{\lambda}{\rho^2}\frac{\rho s_\perp(1+\rho d_w) - \sin\rho s_\perp}{1+\rho d_w} \\
\frac{\partial z}{\partial \phi_0} &= -\frac{\lambda}{\rho}\frac{1+\rho d_w - (1+\rho D)\cos\rho s_\perp)}{1+\rho d_w} \\
\frac{\partial z}{\partial D} &= -\frac{\lambda\sin\rho s_\perp}{1+\rho d_w} \\
\frac{\partial z}{\partial \lambda} &= s_\perp \\
\frac{\partial z}{\partial z_0} &= 1
\end{aligned}$$

For the special case $\rho = 0$, $s_\perp = -(u_0\Delta_x + v_0\Delta_y)$ and the derivatives reduce to

$$\begin{aligned}
\frac{\partial z}{\partial c} &= 0 \\
\frac{\partial z}{\partial \phi_0} &= \lambda(D-d_w) \\
\frac{\partial z}{\partial D} &= 0 \\
\frac{\partial z}{\partial \lambda} &= s_\perp \\
\frac{\partial z}{\partial z_0} &= 1
\end{aligned}$$

2.4. **Silicon barrel** ($r-\phi$)

The silicon barrel detector is composed of silicon planes arranged in a polygonal fashion in ϕ around the origin. The planes are assumed to be parallel to z and have small overlaps at the edges where they meet. Assume that each plane is defined by its outward normal $\vec{\eta} = (\cos\beta, \sin\beta, 0)$ and any point on the plane $\vec{x}_c$. For this discussion it is convenient to take $\vec{x}_c$ as the point at which the normal, proceeding from the origin, strikes the plane. The quantity measured by the electronics (up to a constant offset) is the distance along the plane in the $r-\phi$ direction from the point $\vec{x}$ where the track hits the plane and $\vec{x}_c$.

The equation of the plane is $\vec{\eta}\cdot(\vec{x}-\vec{x}_c) = 0$ or $x\cos\beta + y\sin\beta - \Delta = 0$, where $\Delta = \vec{\eta}\cdot\vec{x}_c$ is the perpendicular distance from the plane to the origin. It is easier to solve the equation in terms of "local" coordinates $(x_\beta, y_\beta, z_\beta)$ where the equation of the plane is $x_\beta = 0$. The transformation equations are

$$
\begin{aligned}
x_\beta &= x\cos\beta + y\sin\beta - \Delta,\\
y_\beta &= y\cos\beta - x\sin\beta,\\
z_\beta &= z,
\end{aligned}
$$

The equations of the helix can then be written $\rho x_{0\beta} + u_{0\beta}\sin\rho s_\perp - v_{0\beta}(1-\cos\rho s_\perp) = 0$, where the rotated quantities are

$$
\begin{aligned}
u_{0\beta} &= u_0\cos\beta + v_0\sin\beta = \cos(\phi_0-\beta),\\
v_{0\beta} &= v_0\cos\beta - u_0\sin\beta = \sin(\phi_0-\beta),\\
x_{0\beta} &= x_0\cos\beta + y_0\sin\beta - \Delta = -Dv_{0\beta} - \Delta,\\
y_{0\beta} &= y_0\cos\beta - x_0\sin\beta = -Du_{0\beta}.
\end{aligned}
$$

The quantity d_t lies in the plane and increases along some direction $\hat{d}$ defined to be perpendicular to the orientation of the strips. Here we assume that the strips measure the distance along $r-\phi$, i.e. $\hat{d} = (-\sin\beta, \cos\beta, 0)$. For z measurements $\hat{d} = (0,0,1)$ which will be exploited later.

$$\begin{aligned}
d_t &= \hat{d}\cdot(\vec{x}-\vec{x}_c),\\
&= -x\sin\beta + y\cos\beta\\
&= y_{0\beta} + \frac{v_{0\beta}}{\rho}\sin\rho s_\perp + \frac{u_{0\beta}}{\rho}(1-\cos\rho s_\perp),\\
&= y_{0\beta} + \frac{u_{0\beta}}{\rho} - \frac{1}{\rho}\cos(\phi_0-\beta+\rho s_\perp).
\end{aligned}$$

The equation of the plane is $x_\beta = 0$ or, writing it in terms of $\sin(\phi_0-\beta+\rho s_\perp)$,

$$\begin{aligned}
0 &= \rho x_{0\beta} - v_{0\beta} + u_{0\beta}\sin\rho s_\perp + v_{0\beta}\cos\rho s_\perp,\\
&= \rho x_{0\beta} - v_{0\beta} + \sin(\phi_0-\beta+\rho s_\perp),
\end{aligned}$$

allowing d_t to be expressed as

$$\begin{aligned}
d_t &= y_{0\beta} + \frac{1}{\rho}u_{0\beta} - \frac{1}{\rho}\sqrt{1-(v_{0\beta}-\rho x_{0\beta})^2},\\
&= y_{0\beta} - \frac{2v_{0\beta}x_{0\beta}-\rho x_{0\beta}^2}{u_{0\beta}+\sqrt{1-(v_{0\beta}-\rho x_{0\beta})^2}},
\end{aligned}$$

where the second form is useful when ρ is small. We can also solve for $s_\perp$ using

$$\begin{aligned}
\sin\rho s_\perp &= \rho x_{0\beta}u_{0\beta} + \rho(y_{0\beta}-d_t)v_{0\beta},\\
\cos\rho s_\perp &= 1-\rho x_{0\beta}v_{0\beta} + \rho(y_{0\beta}-d_t)u_{0\beta}.
\end{aligned}$$

In the special case $\rho = 0$, $d_t = y_{0\beta} - x_{0\beta}(v_{0\beta}/u_{0\beta}) = y_{0\beta} + v_{0\beta}s_\perp$, where $s_\perp = -x_{0\beta}/u_{0\beta}$.

The derivatives of d_t with respect to the five helix parameters, $\partial d_t/\partial\alpha_i$, are best expressed in terms of $\partial s_\perp/\partial\alpha_i$. From the equation of the plane $\rho x_{0\beta} + u_{0\beta}\sin\rho s_\perp - v_{0\beta}(1-\cos\rho s_\perp) = 0$ we derive

$$\begin{aligned}
\frac{\partial s_\perp}{\partial\rho} &= -\frac{1}{\rho}\frac{x_{0\beta}+s_\perp u_\beta}{u_\beta},\\
\frac{\partial s_\perp}{\partial\phi_0} &= \frac{1}{\rho}\frac{y_{0\beta}+(1/\rho)(u_{0\beta}-u_\beta)}{u_\beta} = \frac{d_t}{u_\beta},\\
\frac{\partial s_\perp}{\partial D} &= \frac{v_{0\beta}}{u_\beta},
\end{aligned}$$

where $u_\beta = u_{0\beta}\cos\rho s_\perp - v_{0\beta}\sin\rho s_\perp = \cos(\phi_0+\rho s_\perp+\beta)$ and $v_\beta = v_{0\beta}\cos\rho s_\perp + u_{0\beta}\sin\rho s_\perp =$

$\sin(\phi_0 + \rho s_\perp + \beta)$. For $\rho = 0$ these simplify to

$$\frac{\partial s_\perp}{\partial \rho} = \frac{1}{2} s_\perp^2 \frac{v_{0\beta}}{u_{0\beta}},$$
$$\frac{\partial s_\perp}{\partial \phi_0} = \frac{y_{0\beta} + v_{0\beta} s_\perp}{u_{0\beta}} = \frac{d_t}{u_{0\beta}},$$
$$\frac{\partial s_\perp}{\partial D} = \frac{v_{0\beta}}{u_{0\beta}},$$

Now we can write the derivatives of d_t with respect to the five helix parameters:

$$\frac{\partial d_t}{\partial c} = 2\frac{u_\beta - u_{0\beta}}{\rho^2} + 2\left(\frac{s_\perp}{\rho} + \frac{\partial s_\perp}{\partial \rho}\right) v_\beta,$$
$$\frac{\partial d_t}{\partial \phi_0} = \Delta + v_\beta \frac{\partial s_\perp}{\partial \phi_0},$$
$$\frac{\partial d_t}{\partial D} = u_{0\beta} + v_\beta \frac{\partial s_\perp}{\partial D},$$
$$\frac{\partial d_t}{\partial \lambda} = 0,$$
$$\frac{\partial d_t}{\partial z_0} = 0.$$

When $\rho = 0$, these simplify to

$$\frac{\partial d_t}{\partial c} = u_{0\beta} s_\perp^2 + 2 v_{0\beta} \frac{\partial s_\perp}{\partial \rho},$$
$$\frac{\partial d_t}{\partial \phi_0} = \Delta + v_{0\beta} \frac{\partial s_\perp}{\partial \phi_0},$$
$$\frac{\partial d_t}{\partial D} = u_{0\beta} + v_{0\beta} \frac{\partial s_\perp}{\partial D},$$
$$\frac{\partial d_t}{\partial \lambda} = 0,$$
$$\frac{\partial d_t}{\partial z_0} = 0.$$

2.5. **Silicon barrel ($r - z$)**

We use the same notation as for the silicon barrel in $r - \phi$ except that the quantity measured by the electronics (up to an offset) is $z = \hat{d} \cdot (\vec{x} - \vec{x}_c)$, where $\hat{d} = (0, 0, 1)$. We must simultaneously solve the equation of the plane and the helix equations. This yields, as in the $r - \phi$ case (see the last section for notation)

$$
\begin{aligned}
0 &= \rho x_{0\beta} + u_{0\beta} \sin \rho s_\perp - v_{0\beta}(1 - \cos \rho s_\perp), \\
z &= z_0 + \lambda s_\perp.
\end{aligned}
$$

Although it is possible to solve the first equation for $s_\perp$, which can be used to find z, it is simpler to use the formulas derived in the previous section for $r - \phi$ silicon measurements:

$$
\begin{aligned}
d_t &= y_{0\beta} + \frac{1}{\rho} u_{0\beta} - \frac{1}{\rho}\sqrt{1 - (v_{0\beta} - \rho x_{0\beta})^2}, \\
&= y_{0\beta} - \frac{2 v_{0\beta} x_{0\beta} - \rho x_{0\beta}^2}{u_{0\beta} + \sqrt{1 - (v_{0\beta} - \rho x_{0\beta})^2}}, \\
\sin \rho s_\perp &= \rho x_{0\beta} u_{0\beta} + \rho(y_{0\beta} - d_t) v_{0\beta}, \\
\cos \rho s_\perp &= 1 - \rho x_{0\beta} v_{0\beta} + \rho(y_{0\beta} - d_t) u_{0\beta}.
\end{aligned}
$$

where d_t is the distance along $r - \phi$ from the point where the normal, proceding from the origin, passes through the plane.

It is easy to express the partial derivatives in terms of $\partial s_\perp / \partial \alpha_i$ defined in the last section:

$$
\begin{aligned}
\frac{\partial z}{\partial c} &= 2\lambda \frac{\partial s_\perp}{\partial \rho}, \\
\frac{\partial z}{\partial \phi_0} &= \lambda \frac{\partial s_\perp}{\partial \phi_0}, \\
\frac{\partial z}{\partial D} &= \lambda \frac{\partial s_\perp}{\partial D}, \\
\frac{\partial z}{\partial \lambda} &= s_\perp, \\
\frac{\partial z}{\partial z_0} &= 1.
\end{aligned}
$$

2.6. Silicon disk at fixed z position

This is a device placed at $z = z_D$ measuring (up to an offset) a coordinate in the xy plane of the disk. As in the case of the silicon barrel, the equation of the plane is $\vec{\eta} \cdot (\vec{x} - \vec{x}_c) = 0$, where $\vec{x}_c = (0, 0, z_D)$. The quantity measured is $x_\alpha = \hat{d} \cdot (\vec{x} - \vec{x}_c)$ where $\hat{d} = (\cos\alpha, \sin\alpha, 0)$ lies in the direction of increasing x_α. $\alpha = 0$ $(\pi/2)$ corresponds to x (y) measurements. We first define the rotated quantities

$$
\begin{aligned}
x_\alpha &= x\cos\alpha + y\sin\alpha, \\
y_\alpha &= y\cos\alpha - x\sin\alpha, \\
u_\alpha &= u\cos\alpha + v\sin\alpha, \\
v_\alpha &= v\cos\alpha - u\sin\alpha,
\end{aligned}
$$

and similarly for $x_{0\alpha}$, $y_{0\alpha}$, $u_{0\alpha}$ and $v_{0\alpha}$. From the helix equations we get $s_\perp = (z_D - z_0)/\lambda$ from which we compute

$$
x_\alpha = x_{0\alpha} + \frac{u_{0\alpha}}{\rho}\sin\rho s_\perp - \frac{v_{0\alpha}}{\rho}(1 - \cos\rho s_\perp),
$$

which reduces, when $\rho = 0$, to $x_\alpha = x_{0\alpha} + u_{0\alpha}s_\perp$.

The derivatives for x_α are easily calculated:

$$
\begin{aligned}
\frac{\partial x_\alpha}{\partial c} &= 2\frac{v_{0\alpha} - v_\alpha + \rho s_\perp u_\alpha}{\rho^2}, \\
\frac{\partial x_\alpha}{\partial \phi_0} &= -y_\alpha, \\
\frac{\partial x_\alpha}{\partial D} &= -v_{0\alpha}, \\
\frac{\partial x_\alpha}{\partial \lambda} &= -\frac{u_\alpha}{\lambda}s_\perp, \\
\frac{\partial x_\alpha}{\partial z_0} &= -\frac{u_\alpha}{\lambda}.
\end{aligned}
$$

When $\rho = 0$, these derivatives become

$$\frac{\partial x_\alpha}{\partial c} = -s_\perp^2 v_{0\alpha},$$
$$\frac{\partial x_\alpha}{\partial \phi_0} = -y_\alpha,$$
$$\frac{\partial x_\alpha}{\partial D} = -v_{0\alpha},$$
$$\frac{\partial x_\alpha}{\partial \lambda} = -\frac{u_{0\alpha}}{\lambda} s_\perp,$$
$$\frac{\partial x_\alpha}{\partial z_0} = -\frac{u_{0\alpha}}{\lambda}.$$

3. Projections of helix including errors

It is frequently necessary to project a track to a particular wire or detector, for example, to decide if a measurement should be added to the fit or just to provide a reference impact point for a calorimeter. An important consideration is the errors of the projection. In this section I will derive the errors for the following kinds of track projections: (1) axial and stereo drift chamber wire, (2) cathode strip, (3) silicon barrel, (4) silicon disk, and (5) surface of sphere of radius r.

As before, α represents the five track parameters, $\mathbf{V}_\alpha$ is the 5×5 covariance matrix and $\mathbf{A}$ represents the derivatives of the projected quantity with respect to the helix parameters. $\mathbf{V}_d$ is the covariance matrix of the projected quantities. $\mathbf{A}$ is either a 1×5 or 2×5 matrix, depending on the number of projected quantities. Calling generically the two projected quantities d_1 and d_2 (e.g., x and y), we get

$$\mathbf{A}^T = \begin{pmatrix} \partial d_1/\partial c & \partial d_2/\partial c \\ \partial d_1/\partial \phi_0 & \partial d_2/\partial \phi_0 \\ \partial d_1/\partial D & \partial d_2/\partial D \\ \partial d_1/\partial \lambda & \partial d_2/\partial \lambda \\ \partial d_1/\partial z_0 & \partial d_2/\partial z_0 \end{pmatrix}.$$

The covariance matrix of d_1 and d_2 is then given by $\mathbf{V}_d = \mathbf{A}\mathbf{V}_\alpha\mathbf{A}^T$.

The discussion is considerably simplified since most of the derivatives in **A** have already been calculated in Section 2.1–2.6. The only case we have not considered is projecting to a cylindrical surface of radius r.

3.1. Previous computations

1. Axial and stereo drift chamber wire (d_w, z). d_w and its derivatives were calculated for axial and stereo layers in Section 2.1 and 2.2 while the corresponding values for z were computed in Section 2.3.

2. Cathode strip (z). z and its derivatives were calculated in Section 2.3.

3. Silicon barrel (d_t, z). These quantities and their derivatives were calculated in Section 2.4 and 2.5.

4. Silicon disk (x, y). These quantities and their derivatives were calculated in Section 2.6.

3.2. Projecting to a cylindrical surface of radius r

The quantities we wish to calculate are ϕ and z, but the errors we want are $\delta d_t = r\sin\theta\delta\phi$ and δz, i.e., the errors in the $r - \phi$ and z distances. We exploit the formulas of Section 1.3 which express the helix coordinates as a function of r:

$$
\begin{aligned}
\sin(\phi - \phi_0) &= \frac{rc + (D/r)(1 + cD)}{1 + 2cD} = \frac{D}{r} + \frac{(c/r)(r^2 - D^2)}{1 + 2cD},\\
\cos(\phi - \phi_0) &= \epsilon\frac{\sqrt{(1 - D^2/r^2)[(1 + cD)^2 - r^2c^2]}}{1 + 2cD},\\
z - z_0 &= \lambda s_\perp,\\
\sin cs_\perp &= c\sqrt{\frac{r^2 - D^2}{1 + 2cD}},\\
s_\perp &= \begin{cases} \frac{1}{c}\sin^{-1} B & \text{for } \epsilon = +1 \\ \frac{1}{c}(\pi - \sin^{-1} B) & \text{for } \epsilon = -1 \end{cases}
\end{aligned}
$$

where $\epsilon = +1(-1)$ corresponds to the outgoing (incoming) branch of the helix.

The derivative matrix (transposed) is

$$\mathbf{A}^T = \begin{pmatrix} \frac{\sin\theta}{\cos(\phi-\phi_0)}\frac{r^2-D^2}{(1+2cD)^2} & \frac{1+cD}{1+2cD}\frac{\sin cs_\perp}{c^2} - \frac{s_\perp}{c} \\ r\sin\theta & 0 \\ \frac{\sin\theta}{\cos(\phi-\phi_0)}\frac{1+2cD-2c^2(r^2-D^2)}{(1+2cD)^2} & -\frac{\sin cs_\perp}{c}\left(\frac{D}{r^2-D^2} + \frac{c}{1+2cD}\right) \\ 0 & s_\perp \\ 0 & 1 \end{pmatrix}$$

which reduces, when $c = \rho = 0$, to

$$\mathbf{A}^T = \begin{pmatrix} \frac{\sin\theta}{\cos(\phi-\phi_0)}(r^2 - D^2) & -D\sqrt{r^2 - D^2} \\ r\sin\theta & 0 \\ \frac{\sin\theta}{\cos(\phi-\phi_0)} & -\frac{D}{\sqrt{r^2-D^2}} \\ 0 & s_\perp \\ 0 & 1 \end{pmatrix}$$

The covariance matrix for $(\delta d_t, \delta z)$ is $\mathbf{V}_d = \mathbf{A}\mathbf{V}_\alpha\mathbf{A}^T$.

4. Detector Misalignments and Track Fitting

Fitting with real detectors is complicated by the fact that they often have built in misalignments, causing the track to be projected to an incorrect position which in turn pulls the fitted parameters from their true values. The problem is often arises when several detectors are used together to fit a single track. If the misalignments are small enough to be approximated by linear parameters they can be incorporated very easily into the fit. Moreover, by fitting many hundreds or thousands of tracks, the parameters describing the misalignments can be extracted analytically from the pattern of residuals using a least squares technique. This method for determining misalignment constants was described in CBX 91–73.[1]

We parametrize the misalignment parameters as three offsets along the coordinate axes, d_x, d_y, d_z, and three angles about these axes θ_x, θ_y, θ_z. The parameters are assumed to be small enough so that terms of second order, e.g., $d_x\theta_y$, can be neglected. In this approximation the order in which the displacements and rotations are applied does not matter. A point (x, y, z) is shifted by the amounts

$$\delta x = d_x + z\theta_y - y\theta_z,$$
$$\delta y = d_y + x\theta_z - z\theta_x,$$
$$\delta z = d_z + y\theta_x - x\theta_y.$$

Each of these shifts has a linear effect on the measurement that can be expressed in terms of a matrix $\mathbf{B}$ (following the notation introduced in CBX 91–73). We express δy_m, the shift in the measured quantity (detector dependent), as

$$\delta y_m = \mathbf{B} \begin{pmatrix} d_x \\ d_y \\ d_z \\ \theta_x \\ \theta_y \\ \theta_z \end{pmatrix}$$

where $\mathbf{B}$ must be determined separately for each detector as discussed below.

4.1. Axial drift chamber wire

The drift distance d_w to the wire is given by

$$d_w = \frac{1}{\rho}\left[\sqrt{1 + 2\rho\Delta_w} - 1\right] = \frac{2\Delta_w}{1 + \sqrt{1 + 2\rho\Delta_w}} \simeq \Delta_w(1 - \frac{1}{2}\rho\Delta_w)$$

where $\Delta_w = u_0\Delta_y - v_0\Delta_x + \frac{1}{2}\rho(\Delta_x^2 + \Delta_y^2)$, $\Delta_x = x_0 - x_w$ and $\Delta_y = y_0 - y_w$. One can add the corrections directly to the wire position or add them after the fact with the $\mathbf{B}$ matrix. $\mathbf{B}$ is computed as

$$\mathbf{B}^T = \begin{pmatrix} v_w \\ -u_w \\ 0 \\ z u_w \\ z v_w \\ -(x_w u_w + y_w v_w) \end{pmatrix} (1 - \rho\Delta_w).$$

where $u_w = u_0 + \rho\Delta_y$ and $v_w = v_0 - \rho\Delta_x$.

4.2. Stereo drift chamber wire

As in the axial drift chamber case, one can apply the misalignment parameters directly to the wire position rather than add them after the fact as a linear correction. The first approach is complicated slightly by the fact that the direction cosines of the wire must be rotated.

To calculate the effect of the misalignments, we write down the equation of the drift distance d_w from Section 2.2

$$d_w \simeq \frac{u_d}{J}\Delta_w(1 - \frac{1}{2}(1 - \lambda'^2)\rho\Delta_w/J^2),$$

where $\Delta_w = u_0\Delta_y - v_0\Delta_x + \frac{1}{2}\rho(\Delta_x^2 + \Delta_y^2)$, $\Delta_x = x_0 - x_w - z_p \tan\tau \sin\phi_s$ and $\Delta_y = y_0 - y_w + z_p \tan\tau \cos\phi_s$. Applying the displacements and rotations of the misalignments allows $\mathbf{B}$ to be computed as follows

$$\mathbf{B}^T = \begin{pmatrix} v_w \\ -u_w \\ -\tan\tau(u_w \cos\phi_s + v_w \sin\phi_s) \\ z_p u_w \\ z_p v_w \\ -(x_w + z_p \tan\tau\cos\phi_s)u_w - (y_w - z_p \tan\tau \sin\phi_s)v_w \end{pmatrix} \frac{u_d}{J}(1 - u_d\rho\Delta_w/J^2),$$

where $u_w = u_0 + \rho\Delta_y$ and $v_w = v_0 - \rho\Delta_x$ (see Section 2.2 for notation).

4.3. Cathode strips

Here the misalignment parameters are applied to the anode wire nearest to the cathode strips. The relevant equation of motion is $z = z_0 + \lambda s_\perp$, where $s_\perp$ satisfies (see Section 2.3)

$$(1 + \rho d_w)\cos\rho s_\perp = 1 - \rho v_0\Delta_x + \rho u_0\Delta_y,$$
$$(1 + \rho d_w)\sin\rho s_\perp = -\rho u_0\Delta_x - \rho v_0\Delta_y,$$

where $\Delta_x = x_0 - x_w$ and $\Delta_y = y_0 - y_w$, with (x_w, y_w) being the coordinate of the wire. As in the axial wire case it is probably simpler to add the corrections directly to the anode wire coordinates and then compute $s_\perp$ and z. Nevertheless, we can add the corrections after the fact using $\mathbf{B}$. First we eliminate d_w from the equations above and get an equation in $s_\perp$ only

$$\sin\rho s_\perp(1-\rho v_0\Delta_x+\rho u_0\Delta_y)+\cos\rho s_\perp(\rho u_0\Delta_x+\rho v_0\Delta_y)=0.$$

Applying small changes to the axial wire (see the discussion above) allows the derivatives of $s_\perp$ to be computed relative to each of the misalignment parameters, yielding

$$\mathbf{B}^T=\begin{pmatrix}-\lambda u/(1+\rho d_w)\\ -\lambda v/(1+\rho d_w)\\ -1\\ \lambda zv/(1+\rho d_w)-y_w\\ \lambda zu/(1+\rho d_w)+x_w\\ -\lambda(vx_w-uy_w)/(1+\rho d_w)\end{pmatrix}(1-\rho\Delta_w),$$

where $u=u_0\cos\rho s_\perp-v_0\sin\rho s_\perp$, $v=v_0\cos\rho s_\perp+u_0\sin\rho s_\perp$ and d_w was computed in Section 2.1.

4.4. **Silicon barrel**

This case is a little more complicated than the previous ones because it is not as straightforward to apply the misalignments directly to the measurements. Misalignments are also more important for silicon because the induced changes in the measurements are of the same order of magnitude as the errors. From Section 2.4 the equation of the plane is $\vec{\eta}\cdot(\vec{x}-\vec{x}_c)=0$ or $x\cos\beta+y\sin\beta-\Delta=0$, where $\vec{\eta}=(\cos\beta,\sin\beta,0)$ represents the direction cosines of the outward normal to the plane and $\vec{x}_c$ is an arbitrary point on the plane which we choose to be the point where the normal, passing through the origin, strikes the plane. $\Delta=\vec{\eta}\cdot\vec{x}_c$ is the perpendicular distance from the plane to the origin.

We first consider the case where the measured quantity is the "transverse distance" in the $r-\phi$ plane defined by $d_t=\hat{d}\cdot(\vec{x}-\vec{x}_c)$, where $\vec{x}$ is the point where the tracks hits the plane and $\hat{d}$ is a unit vector lying along the direction of increasing d_t. For $r-\phi$ measurements $\hat{d}=(\cos\beta,\sin\beta,0)$.

The effect of misalignments can be computed straightforwardly from the equations for d_t and the plane by applying the displacements and rotations to $\vec{\eta}$, $\hat{d}$ and $\vec{x}_c$ and including the change in $s_\perp$. Applying the misalignments to the quantities defining the plane yields

$$\vec{\eta}' = \begin{pmatrix} \cos\beta \\ \sin\beta \\ 0 \end{pmatrix} + \begin{pmatrix} -\theta_z \sin\beta \\ \theta_z \cos\beta \\ \theta_x \sin\beta - \theta_y \cos\beta \end{pmatrix},$$

$$\hat{d}' = \begin{pmatrix} -\sin\beta \\ \cos\beta \\ 0 \end{pmatrix} + \begin{pmatrix} -\theta_z \cos\beta \\ -\theta_z \sin\beta \\ \theta_x \cos\beta + \theta_y \sin\beta \end{pmatrix},$$

$$\vec{x}_c' = \begin{pmatrix} \Delta\cos\beta \\ \Delta\sin\beta \\ 0 \end{pmatrix} + \begin{pmatrix} d_x - \Delta\theta_z \sin\beta \\ d_y + \Delta\theta_z \cos\beta \\ d_z + \Delta(\theta_x \sin\beta - \theta_y \cos\beta) \end{pmatrix}.$$

The equation of the plane is modified to

$$x\cos\beta + y\sin\beta - \Delta = d_x\cos\beta + d_y\sin\beta - (y\cos\beta - x\sin\beta)\theta_z + z(\theta_y\cos\beta - \theta_x\sin\beta)$$

where (x, y, z) is the point where the track hits the plane. Since x, y and z are calculable in terms of $s_\perp$ and the quantities on the right are small, we can expand the left side about the solution for the perfectly aligned case and obtain $\delta s_\perp$, the change in $s_\perp$. The solution is (see Section 2.4 for notation)

$$\delta s_\perp = \frac{d_x\cos\beta + d_y\sin\beta - (y\cos\beta - x\sin\beta)\theta_z + z(\theta_y\cos\beta - \theta_x\sin\beta)}{u_\beta}.$$

To find the new measurement $d_t' = \hat{d}' \cdot (\vec{x}' - \vec{x}'_c)$ we apply the equations above and get

$$d_t' = d_t + v_\beta \delta s_\perp - d_y\cos\beta + d_x\sin\beta + (x\cos\beta + y\sin\beta)\theta_z + z(\theta_x\cos\beta + \theta_y\sin\beta),$$

from which we obtain $\mathbf{B}_{r-\phi}$

$$\mathbf{B}_{r-\phi}^T = \begin{pmatrix} (v_\beta \cos\beta + u_\beta \sin\beta)/u_\beta \\ -(u_\beta \cos\beta - v_\beta \sin\beta)/u_\beta \\ 0 \\ z(u_\beta \cos\beta - v_\beta \sin\beta)/u_\beta \\ z(v_\beta \cos\beta + u_\beta \sin\beta)/u_\beta \\ x(u_\beta \cos\beta + v_\beta \sin\beta)/u_\beta - y(v_\beta \cos\beta - u_\beta \sin\beta)/u_\beta \end{pmatrix} = \begin{pmatrix} v/u_\beta \\ -u/u_\beta \\ 0 \\ zu/u_\beta \\ zv/u_\beta \\ (xu - yv)/u_\beta \end{pmatrix}$$

The misalignments can also be expressed in terms of shifts and rotations local to the silicon plane. The advantage is that the displacements decouple from the rotations (since the axis of rotation is now the center of the silicon) and the formulas for the measurement shifts are more compact and understandable. To begin, imagine the silicon plane rotated about the z axis by $-\beta$ and then shifted so that its center is at $(0,0,0)$ and its normal is $(1,0,0)$. The local x_β, y_β and z_β axes are, respectively, along the silicon thickness, width and length. They are related to the global coordinates by

$$\begin{aligned} x_\beta &= x\cos\beta + y\sin\beta - \Delta \\ y_\beta &= y\cos\beta - x\sin\beta \\ z_\beta &= z \\ u_\beta &= u\cos\beta + v\sin\beta \\ v_\beta &= v\cos\beta - u\sin\beta \end{aligned}$$

The three displacements along the local axes are

$$\begin{aligned} d_{x\beta} &= d_x\cos\beta + d_y\sin\beta \\ d_{y\beta} &= d_y\cos\beta - d_x\sin\beta + \Delta\theta_z \\ d_{z\beta} &= d_z \end{aligned}$$

while the three rotations can be written

$$\begin{aligned} \theta_{x\beta} &= \theta_x\cos\beta + \theta_y\sin\beta \\ \theta_{y\beta} &= \theta_y\cos\beta - \theta_x\sin\beta \\ \theta_{z\beta} &= \theta_z \end{aligned}$$

With these definitions the equation for the plane becomes $x_\beta = 0$ and we get the simpified equations

$$\delta s_\perp = (d_{x\beta} - y_\beta \theta_{z\beta} + z_\beta \theta_{y\beta})/u_\beta,$$
$$d'_t = d_t + v_\beta \delta s_\perp - d_{y\beta} + z_\beta \theta_{x\beta}.$$

For measurements along z the $r - \phi$ discussion applies except that the measured quantity is $z = \hat{d} \cdot (\vec{x} - \vec{x}_c)$ with $\hat{d} = (0, 0, 1)$ When misalignments are applied, $s_\perp$ changes as described above while $\hat{d}$ moves to

$$\hat{d}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\theta_y \\ \theta_x \\ 0 \end{pmatrix},$$

so that the z measurement changes to

$$\begin{aligned} z' &= z + \lambda \delta s_\perp - d_z - x\theta_y + y\theta_x \\ &= z_\beta + \lambda \delta s_\perp - d_{z\beta} + \Delta\theta_{y\beta} - y_\beta \theta_{x\beta} \end{aligned}$$

which defines $\mathbf{B}_z$

$$\mathbf{B}_z^T = \begin{pmatrix} (\lambda/u_\beta)\cos\beta \\ (\lambda/u_\beta)\sin\beta \\ -1 \\ -z(\lambda/u_\beta)\sin\beta + y \\ z(\lambda/u_\beta)\cos\beta - x \\ (\lambda/u_\beta)(x\sin\beta - y\cos\beta) \end{pmatrix}$$

4.5. Silicon disk at fixed z position

This case is fairly easy since the disk sits at $z = z_D$ in the xy plane (see Section 2.6). The equation of the plane is $\vec{\eta} \cdot (\vec{x} - \vec{x}_c) = 0$, where $\vec{\eta} = (0, 0, 1)$ is the outward normal and $\vec{x}_c = (0, 0, z_D)$. The quantity measured is $x_\alpha = \hat{d} \cdot (\vec{x} - \vec{x}_c)$ where $\hat{d} = (\cos\alpha, \sin\alpha, 0)$ lies in the direction of increasing x_α. Under the effect of misalignments, $\vec{\eta}$, $\hat{d}$ and $\vec{x}_c$ become

$$\vec{\eta}' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\theta_y \\ \theta_x \\ 0 \end{pmatrix},$$

$$\hat{d}' = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} -\theta_z \sin\alpha \\ -\theta_z \cos\alpha \\ \theta_x \sin\alpha - \theta_y \cos\alpha \end{pmatrix},$$

$$\vec{x}_c' = \begin{pmatrix} 0 \\ 0 \\ z_D \end{pmatrix} + \begin{pmatrix} d_x + z_D\theta_y \\ d_y - z_D\theta_z \\ d_z \end{pmatrix}.$$

Since $s_\perp = (z_D - z)/\lambda$, the change in $s_\perp$ from misalignments is $\delta s_\perp = (d_z + y\theta_x - x\theta_y)/\lambda$, which yields the new value of x_α:

$$x_\alpha' = x_\alpha + u_\alpha \delta s_\perp - d_x \cos\alpha - d_y \sin\alpha - z_D(\theta_y \cos\alpha - \theta_x \sin\alpha) + \theta_z(y\cos\alpha - x\sin\alpha),$$

where $u_\alpha = u\cos\alpha + v\sin\alpha$ as defined in Section 2.6. These equations define $\mathbf{B}$:

$$\mathbf{B}^T = \begin{pmatrix} -d_x \cos\alpha \\ -d_x \cos\alpha \\ u_\alpha/\lambda \\ z\sin\alpha + y(u_\alpha/\lambda) \\ -z\cos\alpha - x(u_\alpha/\lambda) \\ y\cos\alpha - x\sin\alpha \end{pmatrix}$$

References

1. P. Avery, Applied Fitting Theory II: Determining Systematic Effects by Fitting, CBX 91–73.