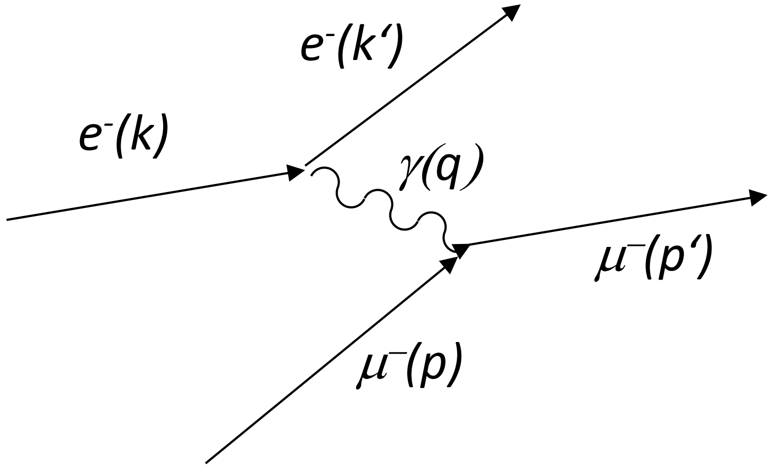
2.7 Electromagnetic interaction of Dirac particles

2.7.1 $e^- \mu^- \to e^- \mu^-$ scattering

Knowing that the Dirac equation describes relativistic particles with spin ½ let us calculate the cross section for the electromagnetic interaction between two such particles, an electron and a muon. The Feynman diagram of such a process is



and the matrix element is $$T_{fi} = -i\int j_\nu^e \left(-\frac{1}{q^2}\right) j_\mu^\nu d^4x$$

where j_ν^e and j_μ^ν denote the electromagentic current of the electron and muon, respectively:

$$j_\nu^e = -e\overline{u}(k')\gamma_\nu u(k) e^{i(k'-k)x}$$
$$j_\mu^\nu = -e\overline{u}(p')\gamma^\nu u(p) e^{i(p'-p)x}$$

Inserting the currents into the matrix element expression we obtain

$$T_{fi} = -ie^2 \overline{u}(k')\gamma_\nu u(k)\left(-\frac{1}{q^2}\right)\overline{u}(p')\gamma^\nu u(p)\int e^{i(k'+p'-k-p)x} d^4x =$$

$$= -i\left[-e\overline{u}(k')\gamma_\nu u(k)\right]\left(-\frac{1}{q^2}\right)\left[-e\overline{u}(p')\gamma^\nu u(p)\right](2\pi)^4\delta^4(k'+p'-k-p)$$

where in the last line we used the definition of a 4-dimensional delta function (the latter is just a consequence of energy and momentum conservation in the process). It is again custom to separate the delta function out of the matrix element by defining the amplitude for the process $\mathfrak{M}$ as

$$T_{fi} = -(2\pi)^4\delta^4(k'+p'-k-p)\mathfrak{M}$$

with

$$-i\mathrm{M} = \left[-e\overline{u}(k')\gamma^\kappa u(k)\right]\left(-\frac{g^{\kappa\nu}}{q^2}\right)\left[-e\overline{u}(p')\gamma^\nu u(p)\right]$$

Before proceeding with the calculation we need to determine what kind of the cross section we wold like to determine. The involved particles carry spin. Quite often the spin orientation (which in principle can be measured, i.e. one can distinguish between positive and negative helicity states) of particles is not measured. In this case one talks about the unpolarized cross section. It is defined as

$$\left|\overline{\mathcal{M}}\right|^2 = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{\substack{\text{allspin} \\ \text{orientations}}} \left|\mathcal{M}\right|^2$$

Factors *$(2s_i+1)$* represent possible spin states of the incoming particles (*a* and *b*). In the unpolarized cross section one averages over those possible spin orientations. For a particle of spin ½ this factor equals 2 (two possible spin orientations). The sum in the expression runs over all possible spin orientations of the spinors involved in the amplitude $\mathcal{M}$. It should be noted that the sum runs over amplitudes squared whch is a consequence of the fact that in principle the spin orientations can be measured. The sum involves currents, for example the electron current $\overline{u}(k')\gamma^\kappa u(k)$ which in the amplitude squared enters twice:

$$\left|\mathcal{M}\right|^2 \propto \left[\overline{u}(k')\gamma^\kappa u(k)\right]\left[\overline{u}(k')\gamma^\sigma u(k)\right]^+$$

Note that the indices of the gamma matrices are different; the first gamma four-vector is multiplied by the corresponding four-vector in the muon current and analogously the second one. Writing out the current product above

$$\left[\overline{u}(k')\gamma^{\kappa}u(k)\right]\left[\overline{u}(k')\gamma^{\sigma}u(k)\right]^{+}=\left[\overline{u}(k')\gamma^{\kappa}u(k)\right]\left[u^{+}(k)\underbrace{\gamma^{\sigma+}\gamma^{0+}}_{=\gamma^{\sigma+}\gamma^{0}=\gamma^{0}\gamma^{\sigma}}u(k')\right]=$$

$$=\left[\overline{u}(k')\gamma^{\kappa}u(k)\right]\left[u^{+}(k)\gamma^{0}\gamma^{\sigma}u(k')\right]=\left[\overline{u}(k')\gamma^{\kappa}u(k)\right]\left[\overline{u}(k)\gamma^{\sigma}u(k')\right]$$

The sum over spin orientations implies

$$\sum_{s,s'=1,2}\left[\overline{u}^{(s')}(k')\gamma^{\kappa}u^{(s)}(k)\right]\left[\overline{u}^{(s)}(k)\gamma^{\sigma}u^{(s')}(k')\right]=$$

$$=\sum_{s,s'=1,2}\left[\overline{u}_{\alpha}{}^{(s')}(k')\gamma_{\alpha\beta}{}^{\kappa}u_{\beta}{}^{(s)}(k)\right]\left[\overline{u}_{\delta}{}^{(s)}(k)\gamma_{\delta\varepsilon}{}^{\sigma}u_{\varepsilon}{}^{(s')}(k')\right]$$

where in the last line we explicitly wrote out the components of the spinors and gamma matrices to be multiplied (sum over the repeated indicies is implied). Each of the factors $u_i{}^{(s)}$ and $\gamma_{ij}{}^{\kappa}$ is now a simple scalar and their products are commutative. Hence we can move the last factor $u_{\varepsilon}{}^{(s')}(k')$ to the beginning of the product thus obtaining

$$\sum_{s,s'=1,2}\underbrace{u_{\varepsilon}{}^{(s')}(k')\overline{u}_{\alpha}{}^{(s')}(k')}_{\not{k}'+m_e}\gamma_{\alpha\beta}{}^{\kappa}\underbrace{u_{\beta}{}^{(s)}(k)\overline{u}_{\delta}{}^{(s)}(k)}_{\not{k}+m_e}\gamma_{\delta\varepsilon}{}^{\sigma}$$

The notation below the line denotes what we obtain by applying the completness relation, with m_e denoting the mass of electron. The sum is thus

$$\left(\not{k}'+m_e\right)_{\varepsilon\alpha}\gamma_{\alpha\beta}{}^{\kappa}\left(\not{k}+m_e\right)_{\beta\delta}\gamma_{\delta\varepsilon}{}^{\sigma}$$

Examining the matrix indices we realize that the above product is just the trace of the expression,

$$Tr\left[\left(\not{k}'+m_e\right)\gamma^{\kappa}\left(\not{k}+m_e\right)\gamma^{\sigma}\right]$$

The same can of course be obtained for the other (muon) current in the spin averaged amplitude. The latter reads

$$\left|\overline{\mathfrak{M}}\right|^2=\frac{1}{2\cdot 2}\frac{e^4}{q^4}Tr\left[\left(\not{k}'+m_e\right)\gamma^{\kappa}\left(\not{k}+m_e\right)\gamma^{\sigma}\right]Tr\left[\left(\not{p}'+m_{\mu}\right)\gamma_{\kappa}\left(\not{p}+m_{\mu}\right)\gamma_{\sigma}\right]$$

It may be of some comfort to know that once we are aware of this result it is not necessary to repeat the derivation each time when calculating amplitudes for various processes. Already from the form of the amplitude $\mathfrak{M}$ on p. ??? we can directly guess the expression for the spin averaged amplitude above.

In proceeding with the calculation of $\left|\overline{\mathfrak{M}}\right|^2$ we use some known identities in calculation of traces without the need for explicit matrix multiplication. This identities are called the trace theorems.

Specifically for the above example the following trace theorem can be used:

$$Tr\left[(\not{k}'+m_e)\gamma^\kappa(\not{k}+m_e)\gamma^\sigma\right]=4\left[k'^\kappa k^\sigma+k'^\sigma k^\kappa-(k'\cdot k-m_e^2)g^{\kappa\sigma}\right]$$

Upon using the same theorem for the traces of the muon current we obtain

$$\left|\overline{\mathfrak{M}}\right|^2=\frac{8e^4}{q^4}\left[(k'p')(kp)+(k'p)(kp')-m_e^2p'p-m_\mu^2k'k+2m_e^2m_\mu^2\right]$$

The most difficult part of the cross section calculation is by this accomplished. We obtained the spin averaged amplitude expressed in terms of four-momenta products (it should be noted that the products of four-vectors are Lorentz invariant).

To obtain the cross section from the spin averaged amplitude we need to add a few further factors. We defined the differential cross section (p. ???) as

$$\frac{d\sigma}{d\Omega}=\frac{dW_{fi}/d\Omega}{\rho_i v_i},\ \frac{dW_{fi}}{d\Omega}=\frac{2\pi}{\hbar}\left|T_{fi}\right|^2\frac{d\rho_f}{d\Omega}$$

Density of final states ρ_f was obtained from

$$d^3N=V\frac{d^3p}{(2\pi\hbar)^3}=\frac{1}{\rho}\frac{d^3p}{(2\pi\hbar)^3}$$

where we used $\rho = 1/V$ to denote the probability density in Schrödinger equation.

The density of final states for relativistic particles must be written using

$$d^3N = \frac{V}{2E}\frac{d^3p}{(2\pi\hbar)^3}$$, taking into account the normalization to *2E* particles in volume *V*.

ρ_f is proportional to d^3p/E. A differential Lorentz transformation (in *x* direction) is

$$dp_x' = \gamma(dp_x - \beta dE)$$
$$dp_y' = dp_y, \quad dp_z' = dp_z$$
$$dE' = \gamma(dE - \beta dp_x)$$

and the Lorent transformed d^3p/E factor is

$$\frac{d^3p'}{E'} = \frac{\gamma(dp_x - \beta dE)dp_y dp_z}{\gamma(E - \beta p_x)} = \frac{dp_x(1 - \beta dE/dp_x)}{E - \beta p_x} dp_y dp_z =$$
$$= \frac{dp_x(1 - \beta E/p_x)}{E(1 - \beta p_x/E)} dp_y dp_z = \frac{d^3p}{E}$$

where we used $E = \sqrt{(p_x^2+p_y^2+p_z^2+m^2)}$ and hence $dE/d^3p_x = p_x/E$. The density of final states proportional to d^3p/E is Lorentz invariant.

The last factor needed for the cross section is the density of incoming particles, $\rho_i v_i$. If the initial particle *a* is moving and particle *b* is resting (in a target) then

$$\rho_i v_i = \underbrace{\frac{2E_a}{V} v_a}_{\substack{current\,density\\ of\ particles\ a}} \quad \underbrace{\frac{2E_b}{V}}_{\substack{density\ of\\ target\ particles}}$$

If both initial state particles are moving, then

$$\rho_i v_i \equiv F = \frac{2E_a}{V} \frac{2E_b}{V} \left| \vec{v}_a - \vec{v}_b \right|$$

Taking into account $\frac{v}{c} = \beta = \frac{\gamma m v}{\gamma m c} = \frac{\gamma m v c}{\gamma m c^2} = \frac{cp}{E}$ the velocity difference can be written as

$$\vec{v}_a - \vec{v}_b = \frac{\vec{p}_a E_b - \vec{p}_b E_a}{E_a E_b} \qquad \text{and} \qquad F \propto \left| \vec{p}_a E_b - \vec{p}_b E_a \right| = \sqrt{\left(p_a p_b\right)^2 - m_a^2 m_b^2}$$

The latter expression can be written in explicitly Lorentz invariant form as shown.

The differential cross section can thus be written as a product of Lorentz invariant factors,

$$d\sigma = \frac{|\mathfrak{M}|^2}{F} dQ \quad ,$$

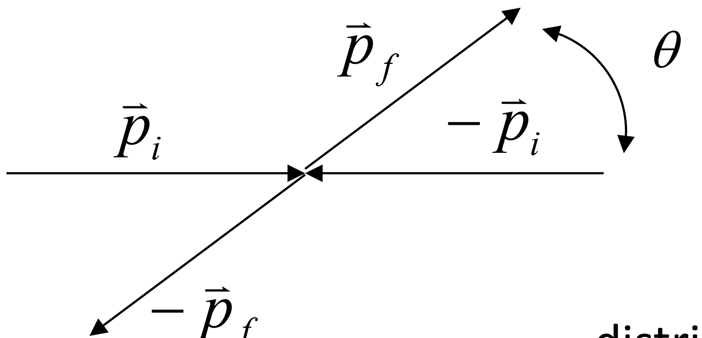
with individual factors for the process $a\, b \to c\, d$ written as

$$F = 4\sqrt{(p_a p_b)^2 - m_a^2 m_a^2}$$

$$dQ = (2\pi)^4 \delta^4 (k' + p' - k - p) \frac{d^3 p_c}{(2\pi)^3 2E_c} \frac{d^3 p_d}{(2\pi)^3 2E_d}$$

$$-i\mathfrak{M} = \left[j_{ca}^{\mu}\right] \left[-i \frac{g^{\mu\nu}}{q^2}\right] \left[j_{db}^{\nu}\right]$$

The above ingredients of the differential cross section take specifically compact form if written in the center-of-mass frame (CMS) of the initial and final state particles:



$$\vec{p}_a = \vec{p}_i = -\vec{p}_b$$
$$\vec{p}_c = \vec{p}_f = -\vec{p}_d$$

If we are interested in the angular distribution of final state particles we can write (note that p_i and p_f are not 4-vectors but the magnitudes of the corresponding 3-momenta):

$$d^3 p_c = p_f^2 dp_f d\Omega, \quad d\Omega = 2\pi \sin\theta d\theta$$

and integrate over d^3p_d:

$$\int \frac{d^3 p_d}{2E_d} \underbrace{\delta^4(p_c + p_d - p_a - p_b)}_{=\delta(E_c+E_d-E_a-E_b)\delta^3(\vec{p}_c+\vec{p}_d-\vec{p}_a-\vec{p}_b)} = \frac{1}{2E_d}\delta(E_c + E_d - E_a - E_b)$$

Denoting the CMS collission energy by E $(=E_a+E_b)$ we have

$$dQ = \frac{1}{4\pi^2}\frac{p_f^2 dp_f d\Omega}{4E_c E_d}\delta(E_c + E_d - E)$$

$$E = E_c + E_d = \sqrt{p_f^2 + m_c^2} + \sqrt{p_f^2 + m_d^2}$$

$$\frac{dE}{dp_f} = \frac{p_f}{E_c} + \frac{p_f}{E_d}$$

$$dQ = \frac{1}{4\pi^2}\frac{p_f}{4(E_c + E_d)} d\Omega dE \delta(E_c + E_d - E)$$

Upon integration over the energy E we get

$$dQ = \frac{1}{4\pi^2} \frac{p_f}{4E} d\Omega$$

Substitution of p_a and p_b expressed in terms of p_i into the expression for F yields

$$F = 4 p_i E$$

The differential cross section in CMS is

$$\frac{d\sigma}{d\Omega dE} = \frac{\left|\mathfrak{M}\right|^2 p_f}{64\pi^2 p_i (E_c + E_d)^2} \delta(E_c + E_d - E)$$

and the trivial integration over E (because of the delta function) yields

$$\frac{d\sigma}{d\Omega} = \frac{\left|\mathfrak{M}\right|^2 p_f}{64\pi^2 p_i E^2}$$

In the ultrarelativistic limit $m_x << p_x$ and $p_i = p_f$,

$$\frac{d\sigma}{d\Omega} = \frac{\left|\mathfrak{M}\right|^2}{64\pi^2 E^2}$$

Returning to our example of the $e^-\mu^- \to e^-\mu^-$ scattering, we can now write the differential cross sectiomn in the CMS in the ultrarelativistic limit :

$$k=(E/2,\vec{p}_i),\quad k'=(E/2,\vec{p}_f),\quad p=(E/2,-\vec{p}_i),\quad p'=(E/2,-\vec{p}_f)$$

$$k'p=(E^2/4)+\vec{p}_i\vec{p}_f=(E^2/4)(1+\cos\theta)$$

$$kp=(E^2/4)+\vec{p}_i\vec{p}_i=E^2/2$$

$$q^2=(k'-k)^2=(0,p_f-p_i)^2=-(E^2/2)(1-\cos\theta)$$

$$\frac{d\sigma}{d\Omega}=\frac{e^4}{32\pi^2E^2}\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}$$

In rewriting the expression to obtain the units m^2 as expected for the cross section one should take into account $\alpha = e^2/4\pi\hbar c$ to obtain

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2(\hbar c)^2}{2E^2}\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}$$