The features of both analogue and digital filters have been used together to improve the bandwidth of samplers. Erik Margan illustrates by example the improvements to be obtained by treating the combination as a single filter.

Antialiasing with mixed-mode filters

Analogue and digital filtering in combination can be used in sampling systems to improve system bandwidth, while retaining high out-of-band signal and noise rejection for effective antialiasing, without the need to increase the sampling frequency. Alternatively, less complicated, lower order filters can be used for attaining the same performance. A method of optimising the filter requirements is discussed.

As an example, suppose the input signal is to be sampled to 12-bit accuracy with a sampling frequency of 2MHz. In this case, frequencies above the Nyquist frequency (1MHz) should be attenuated by at least 2^{12}, or about 72dB. Assume also that constraints such as amplifier bandwidth and phase margin, component tolerances, layout parasitics, thermal effects, etc, limit the filter design to a 6th-order type.

Normally, Chebyshev or elliptic (Cauer) filter types are used for effective antialiasing, since these provide sharp cut-off and the procedure described here is not required. However, for a perfect transient performance or to preserve a high degree of phase coherence in complex signals, the filter must be of the linear-phase type, leading to a Bessel-type filter[2], an all-pole equi-ripple phase filter (±0.05°) or other filter types that can be compensated via phase equalisers.

The use of phase equalisers is limited to band-pass filters, since it is difficult to match the filter phase in wide bandwidth. Bessel filters have a smooth knee in the frequency domain, which makes them a poor choice for anti-aliasing applications. On the other hand, in contrast to the equi-ripple phase types, they can be built from a cascade of relatively low-Q sections, which makes them relatively insensitive to component tolerances. Most importantly, their time-domain performance is ideal.

Although a Bessel filter will be used in the example, calculating the stop-band asymptote of a 6th order Butterworth filter that satisfies the no-alias requirement gives a simple relation from which the required system asymptotes can easily be calculated. The frequency f_A at which the nth order Butterworth system reaches the required attenuation A can be calculated from,

$$f_A = 10^{\frac{\log_{10}\left(A^2-1\right)}{2n}} \qquad (1)$$

Equation 1 assumes a normalised system, with its –3dB cut-off frequency f_C=1 and the response at zero frequency A_0=1. Taking $A=2^{12}$ and n=6 results in f_A=4.

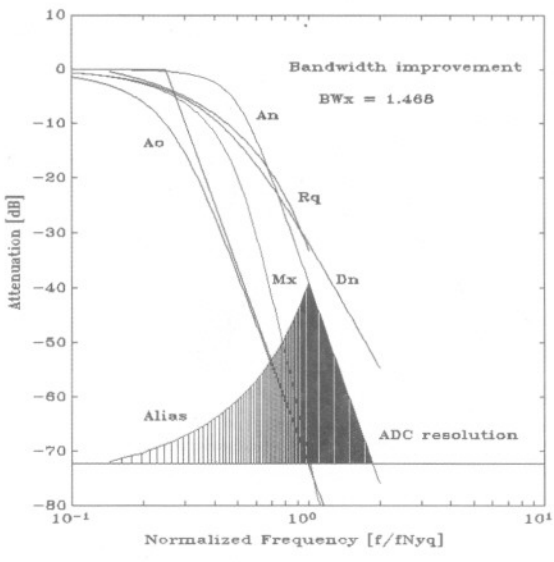

Fig. 1. Mixed-mode filter bandwidth improvement. Frequency scale normalised to the Nyquist frequency (0.5 of the sampling frequency). Attenuation scale normalised to the system gain at dc. Dotted curve A_o is the response of the original 6th-order analogue-only filter, reaching the 12-bit a-to-d converter resolution limit of –72dB at the Nyquist frequency. If the analogue filter bandwidth is moved upward (A_x), so that the converter resolution limit will be reached at $1.87f_{Nyq}$, the dark-shaded part area from f_{Nyq} to $1.87f_{Nyq}$ will generate an alias spectrum from f_{Nyq} to $0.13f_{Nyq}$ (light-shaded). The alias spectrum envelope, flipped about the frequency axis, determines the minimum required attenuation dashed line R_q of the digital filter D_x, which would make the alias spectrum envelope equal to the a-to-d converter resolution limit. The resulting mixed-mode filter response M_x will have its –3dB cut-off frequency 1.468 times higher than A_o.

Now calculate the 6th-order Bessel system polynomial coefficients (see the Bessel panel), divide them by $\sqrt[n]{d_0}$ to normalise the system to have the same stop-band asymptote as the Butterworth filter and extract the polynomial roots[3] to get the poles.

Since f_A must be equal to the Nyquist frequency, denormalise the system by taking the inverse value of f_A, which gives the Butterworth bandwidth relative to the Nyquist frequency f_{Nyq}, equal to 250kHz. The poles of the Bessel filter must also be divided by f_A, resulting in a –3dB bandwidth of 144kHz. This is the reference figure for the analogue-only antialiasing filter. If this figure is not high enough and if the choice of the analogue-to-digital converter limits the maximum sampling frequency, use mixed-mode filtering to expand the system bandwidth.

Analogue/digital filters

The idea of using mixed-mode filtering comes from the fact that the total system frequency response is a simple multiplication of the analogue and digital filter frequency responses. Transforming the digital z-domain response is trans-

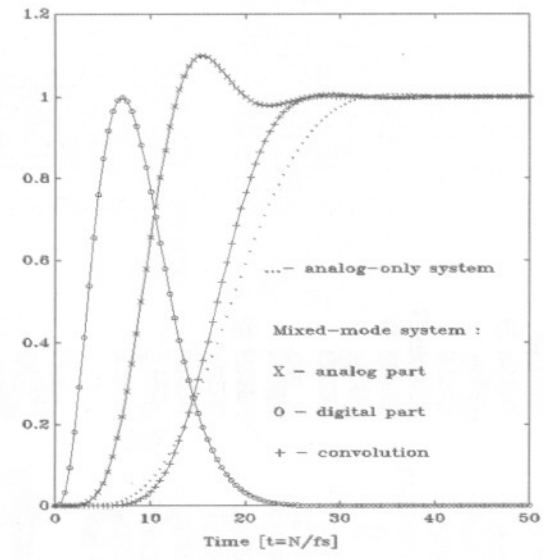

Fig. 2. Time-domain representation of the mixed-mode filter performance. Convolving the analogue filter step response with the digital filter impulse response gives the perfect step response with a rise time shorter than the analogue-only filter.

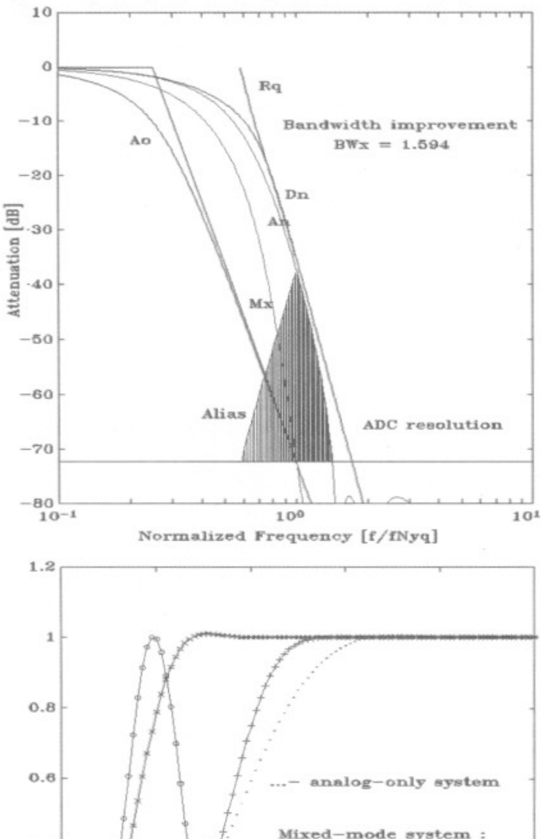

Fig. 3. Example of mixed-mode filtering, using zeros with analogue filter. Zeros are at 1.5,2.0 and 4.0 times f_{Nyk}. Analogue/digital filter can be now moved up by 2.37, while still resulting in a relatively narrow alias spectrum and giving total bandwidth improvement of nearly 1.6.

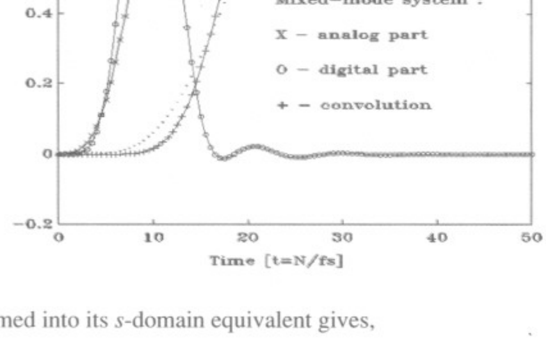

Fig. 4. Time-domain performance of the Fig. 3 mixed-mode filter, using zeros and poles in the analogue section. Note better rise-time of the mixed-mode step response.

formed into its s-domain equivalent gives,

$$H(s) = A(s) \times D(s) \qquad (2)$$

That is also true for the reverse case (i.e. a system formed from a digital filter, a d-to-a converter and analogue filter). In the time-domain, Eq. 2 becomes the convolution integral of

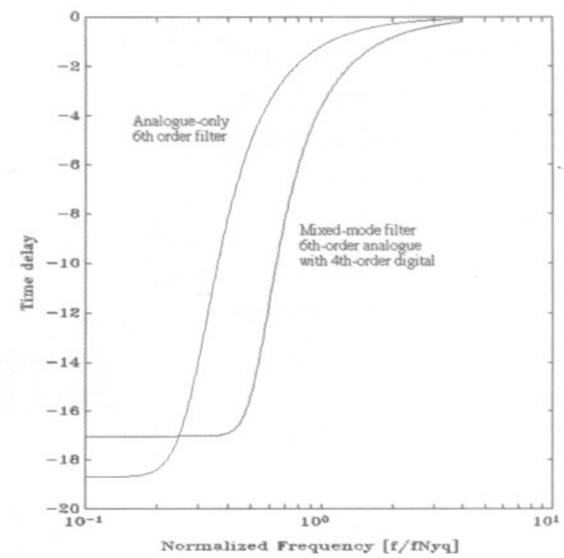

Fig. 5. Time-delay (phase vs frequency derivative) of all-pole mixed-mode filter is constant up to a frequency more than double that of analogue-only filter.

the analogue signal with the digital filter impulse response and convolution is exactly the process performed by digital filtering, the digital filter coefficients representing the sampled equivalent of the impulse response[3].

However, as is well known from analogue filters, cascading two separately optimised filters reduces the total system bandwidth more than one would like. It is thus better to use a single filter system but of higher order. Since the limit is a 6th-order analogue filter, calculate a 10th-order filter, assign six of its poles to the analogue part and the remaining four to the digital part. A higher order filter has a steeper stop-band and so its bandwidth can be higher while still satisfying the antialiasing condition, but how much higher is not yet known. **Figure 1** shows the optimisation criterion.

Dotted curve A_o is the 6th-order analogue-only reference system, shown along with its pass-band and stop-band asymptotes. A_x and D_x are the analogue and digital part of the mixed-mode filter M_x, which is a 10th-order Bessel filter. Of its ten poles (arranged as five complex-conjugate pairs), six of them, in three pairs, have been assigned to the analogue filter A_x and the remaining four in two pairs to D_x.

Since A_x is of the same order as A_o, its stop-band slope is the same as the reference, allowing easy calculation of the effect of increasing its bandwidth. In Fig. 1, it has been increased by 1.87 and the line-shaded frequency band between the Nyquist frequency f_{Nyq} and $1.87f_{Nyq}$ will, when sampled, be reflected into the dot-shaded alias spectrum between f_{Nyq} and $(2\text{-}1.87)f_{Nyq}$. The difference, in dB, between the a-to-d converter resolution level and the alias spectral envelope gives the minimum required attenuation (shown as the dashed line R_q) that the digital filter must have to suppress the alias spectrum below the ADC resolution level.

From Fig. 1, one could conclude that optimal performance is reached whenever the mixed-mode response reaches the a-to-d converter resolution level at the Nyquist frequency, but be warned that this will not be so in the majority of cases. Instead, the optimum is achieved by iteration – first, shift upward the analogue and digital frequency responses (the poles multiplied by a factor between 1 and 2), then calculate the alias spectral envelope, take the difference between the a-to-d converter resolution level and the alias envelope and finally compare it to the frequency response of the digital filter. If the filter is much below the required level, repeat the process; if it is above the required level, multiply the poles by

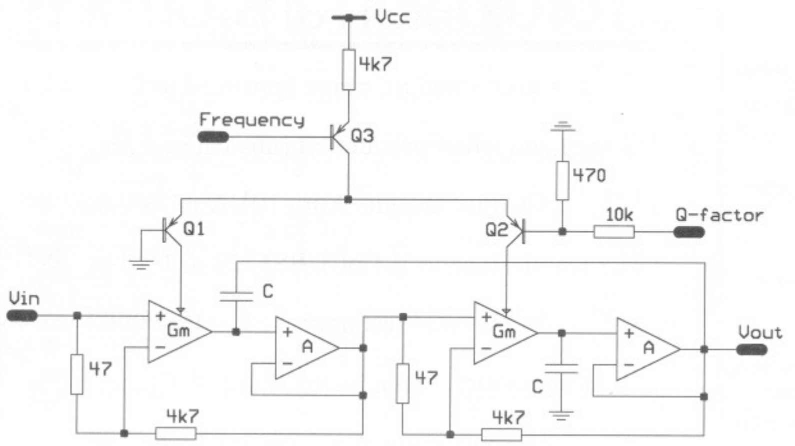

Fig. 6. Two-pole, voltage-controlled filter example. Cascade of three such sections needed for the six-pole example of Figs 1 and 2. This is a classic Sallen-Key configuration in which the resistors have been replaced by transconductance amplifier's g_m and each g_m-C pair buffered. Buffer op-amps ACFB must be of the wide-band type (i.e., with current feedback) to prevent parasitic transfer function zeros. Q and frequency of each two-pole section must be adjusted separately, in accordance to the poles selected. Resistive dividers of 4.7kΩ and 47Ω keep the OTAs in the linear range and prevent slew-rate limiting for large signals.

a factor lower than 1 and test the result again.

From the shape of the alias spectral envelope it is clear that there is no point in making the digital filter of high order. Likewise, it is advantageous to choose the poles having smaller imaginary part for the digital filter, since this results in a smoother response and consequently greater bandwidth improvement factor. In this example, the mixed-mode system has its –3dB cut-off frequency at 211.5kHz, which is 1.468 times the all-analogue filter bandwidth.

Splitting the filter poles between the analogue and digital part may also be taken into consideration; designers of systems that must operate in real time will look for the pole selection that gives the digital filter a more symmetrical impulse response – every other complex-conjugate pole pair is assigned to the digital filter. This property of symmetry can then be exploited to reduce the required filter coefficients (and consequently the number of multiplications) by half, speeding-up the digital filtering process.

On the other hand, when the available analogue gain-bandwidth product is critical, the designer may prefer to assign the poles with the lower imaginary part to the analogue filter, but at some expense to the bandwidth improvement.

Figure 2 shows the time-domain behavior of the same filters used to produce Fig. 1, with the time scale normalised to the sampling period and the markers on the curves corresponding to actual samples. Analogue step response, with its notable overshoot, convolved with the digital impulse response gives a perfect step response with a rise time shorter than that of the analogue-only filter (the dotted curve).

From Fig. 1 it is also obvious that all-pole filters can not achieve a bandwidth improvement greater than about 1.5, since this would require the analogue filter asymptote to approach the sampling frequency at the a-to-d converter resolution level, extending the alias spectrum towards dc, where it would be hard to eliminate. If the analogue filter is designed to have some stop-band zeros at the sampling frequency and its first few multiples, a greater bandwidth improvement will be possible. One such case is shown in **Fig. 3** and **Fig. 4**, where a six-pole, six-zero analogue filter is combined with an eight-pole equivalent digital filter. Zeros are at 1.5 , 2.0 and 4.0 times f_{Nyq}, which were not chosen for optimum pass-to-stop band transition, but for narrowing the alias band.

While the bandwidth improvement in both cases may seem small, it will be appreciated by those who use spectrum analysis daily. It must be noted that the resulting improvement in phase linearity is even greater than in bandwidth, since the additional extension comes from the use of a higher order filter. **Figure 5** shows how the all-pole, mixed-mode system time-delay, i.e. the phase *vs* frequency derivative,

$$t_D = \frac{d\varphi}{d\varpi}$$

remains constant up to a frequency more than double that in the analogue-only filter.

If the a-to-d converter system is to be used with different sampling frequencies, the digital filter part can be left unchanged, but the analogue filter must be frequency-shifted accordingly; transconductance operational amplifiers used for frequency control offer the best way of doing this[4]. **Figure 6** shows an example of a two-pole filter section, with separately adjustable frequency and Q.

Voltage at the base of Q_2 of about ±50mV dc sets the Q (the imaginary components of the pole pair) and the control voltage at the base of Q_3 (ranging from V_{CC}–0.7V to about +0.7V) sets the frequency; the magnitude of the pole pair – the ratio of the imaginary to the real component remains unchanged. A cascade of three such sections is needed for the six-pole analogue filter, each section being adjusted separately and the adjustments remaining in fixed proportions as the frequency control voltage is changed. A simpler, but less flexible, solution is to make all the transconductances equal and select the values of capacitors as required by the poles.

I built my experimental filter using RCA *CA 3080* operational transconductance apmlifiers and Comlinear *CLC 400* current-feedback devices. However, the Linear Technology *LT 1228*[5], which is a single-chip OTA with current feedback, is the natural choice. Transfer function of the filter in Fig. 6 is,

$$\frac{V_{out}}{V_{in}} = \frac{g_{m1}g_{m2}/(k^2C_1C_2)}{s^2 + sg_{m1}/(kC_1) + g_{m1}g_{m2}/(k^2C_1C_2)} \qquad (3)$$

where k is the attenuation of the OTA input resistive divider (1/101), and g_m is the OTA transconductance, set by the bias currents from the collectors of Q_1 and Q_2 . Comparing Eq. 3 with the general two-pole transfer function :

$$H(s) = \frac{p_1p_2}{(s-p_1)(s-p2)}$$

$$= \frac{p_1p_2}{s^2 + s(-p_1-p2) + p_1p_2} \qquad (4)$$

and normalising g_{m1}=g_{m2}=1 produces,

$$C_1 = \frac{1}{k(-p_1-p_2)}$$

and

$$C_2 = \frac{1}{k^2p_1p_2C_1} \qquad (5)$$

Aliasing

In theory, the bandwidth of the sampling system is equal to the Nyquist frequency, which is one-half of the a-to-d converter's sampling frequency. In practice, however, correct waveform spectrum can be found only if the input signal frequencies above the Nyquist frequency are attenuated to levels lower than the a-to-d converter resolution, to avoid 'aliasing' (if the signal contains discrete frequency components above the Nyquist frequency, or broadband noise). This is known in literature as the Shannon's sampling theorem (see Further Reading).

Aliasing can be best understood if the reader remembers the scene from Western movies, where the wheels of the stage coach seem to be rotating backwards, while the horses are running wild to escape from the desperados behind. What is perceived, is as if the wheels rotate with a frequency equal to the difference between the frequency at which the pictures were taken and the actual wheel rotation frequency.

A wheel, rotating at exactly the same frequency (or its integer multiple or submultiple) as the picture rate, would be perceived as stationary (remember the stroboscope effect). This is the same as if an a-to-d converter is sampling a signal of a frequency equal to its sampling frequency – such a signal can not be distinguished from a d.c. level. Likewise, a signal with a frequency slightly lower than the sampling frequency, could not be distinguished from a low frequency, equal to the difference of the two.

Table 1. Poles used in the example of Fig. 1 and 2.

Analogue-only system	Mixed-mode system Analogue	Digital
–0.1346 ±0.2494*i*	–0.3886 ±0.1534*i*	–0.4066 ±0.0510*i*
–0.1999 ±0.1405*i*	–0.2870 ±0.3657*i*	–0.3506 ± 0.2576*i*
–0.2273 ±0.0464*i*	–0.1826 ±0.4836*i*	

Alternatively, normalising $C_1=C_2=1$ produces,

$$g_{m1}=k(-p_1-p_2)$$

and

$$g_{m2}=\frac{k^2 p_1 p_2}{g_{m1}} \qquad (6)$$

Poles p_1 and p_2 are the suitable complex-conjugate pair of the mixed-mode filter poles.

Bessel filters

Bessel filters[2] are optimum in the sense that all the derivatives of the envelope (group) delay response are zero at origin, which results in a maximally flat envelope delay. This means that all the relevant frequencies pass through the system with equal time delay, resulting in a transient response with a minimal overshoot. In the complex frequency plane, a system with pure time delay may be represented by

$$H(s)=e^{-sT} \qquad (7)$$

First, normalise this by making $T=1$; then expand e^{-s} as a polynomial. However, if this is done using the Taylor series expression for e^x and if the polynomial degree exceeds 4, the resulting polynomial would not be of the Hurwitz type, since some of the poles would be in the right-half of the complex plane, making the system unstable. But there is another expression for e^{-s} that we can use:

$$e^{-s}=\frac{1}{\sinh s+\cosh s}=\frac{1/\sinh s}{1+\cosh s/\sinh s} \qquad (8)$$

The series for hyperbolic sine function has even powers of s and the hyperbolic cosine odd powers of s. When these polynomials are divided using long division, the poles of the resulting polynomial meet the stability requirement. Expressing this as a partial fraction expansion truncated at the nth fraction gives an nth-order Bessel system. This can be expressed as

$$H(s)=\frac{d_o}{B_n(s)} \qquad (9)$$

where

$$B_n(s)=\sum_{k=0}^{n} d_k s^k.$$

$Bn(s)$ is an nth order Bessel polynomial which, for different n, satisfies the relations,

$$B_0(s)=1$$
$$B_1(s)=s+1$$
$$B_n(s)=(2n-1)B_{n-1}(s)+s^2 B_{n-2}(s) \qquad (10)$$

The coefficients d_k of the resulting polynomial can be calculated as,

$$d_k=\frac{(2n-k)!}{2^{(n-k)}k!(n-k)!)}, \text{ for } k=0,1,2...n \qquad (11)$$

Roots of $B_n(s)$ are the poles of $H(s)$. Calculated in this way, the system is normalised to a time-delay of 1 for any n, which results in a bandwidth increasing with n. In these calculations, a different normalisation is used: the asymptote of the filter stop-band is made equal to that of the Butterworth filter of equal order, by dividing the polynomial coefficients d_k by $\sqrt[n]{d_0}$.

Bessel filter poles are found in the left-half of the complex plane, on a family of ellipses with one focus at the origin $0+0i$ and the other on the positive part of the real axis. **Table 1** shows the poles used in the example of Fig. 1 and Fig. 2. These values are given relative to the Nyquist frequency – to get the true values, multiply them by 1MHz.

Filter response calculation

In the frequency domain:

$$H(s)=\frac{\prod_{i=1}^{n}(-p_i)}{\prod_{i=1}^{n}(s-p_j)}\cdot\frac{\prod_{j=1}^{m}(s-zj)}{\prod_{j=1}^{m}(-zj)} \qquad (12)$$

where $s=j\omega$ and p_i are the poles and z_j are the zeros (if any).

Magnitude in decibels is

$$M(\omega)=20\log_{10}\sqrt{H(j\omega).H(-j\omega)} \qquad (13)$$

In the time domain, calculate the residue of each pole and sum the residues at each time point to get the impulse response. For the step response, each residue is multiplied by $1/s$ the Laplace transform of the input unit-step. The residue of the kth pole can be calculated as,

$$R_k(t)=\lim_{s\to p_k}(s-p_k)\cdot\frac{\prod_{i=1}^{n}(-p_i)}{\prod_{i=1}^{n}(s-p_i)}\cdot\frac{\prod_{j=1}^{m}(s-z_j)}{\prod_{j=1}^{m}(-z_j)}\cdot e^{p_k t} \qquad (14)$$

Terms $(s-p_k)$ cancel for $i=k$ before limiting. Next, make $s=p_k$, without using the limiting process. By doing so, the general applicability of Eq.14 is lost – it does not hold for systems containing coincident poles, but for all optimised system families the result is still valid. The time t can be chosen to start from 0 up to any desired time, in sampling period increments. Then:

$$f(t)=\sum_{k=1}^{n}R_k(t) \qquad (15)$$

In summary

From all this, one can see that mixed-mode (analogue plus digital) linear-phase filtering can be used effectively to extend the usable spectral bandwidth of sampled signals by about 50% and the phase coherence by more than 100%, while keeping the signal spectral resolution, the sampling frequency and the number of samples unchanged. ■

Further reading

1. Shannon, CE. Collected papers. IEEE Press, Cat.No.: PC 0331.
2. Thomson, WE. Networks With Maximally-Flat Delay. *Wireless Engineer*, vol. 29, October 1952, pp.256-263.
3. Oppenheim, AV and Schafer, RW. *Digital Signal Processing*. Prentice-Hall, 1975.
4. Azadet, K. Linear-phase, continuous-time video filters based on a mixed analogue/digital structure. *ECCTD '93-Circuit Theory and Design*, pp.73-78. Elsevier Sci. Publ.
5. Hickman, I. Versatile twin amplifier has many uses. *Electronics World +Wireless World*, December 1993, pp.1044-1048.