A Differential 3rd-order Bessel Filter

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The circuit in **Fig.1** has been presented in Linear Audio as part of a differential line drive system, with only the final result of the analysis of the system transfer function. Here we show the complete analysis and the procedure of finding the suitable component values from the system poles, and verify the design by calculating the frequency domain and time domain (unit step response) performance.

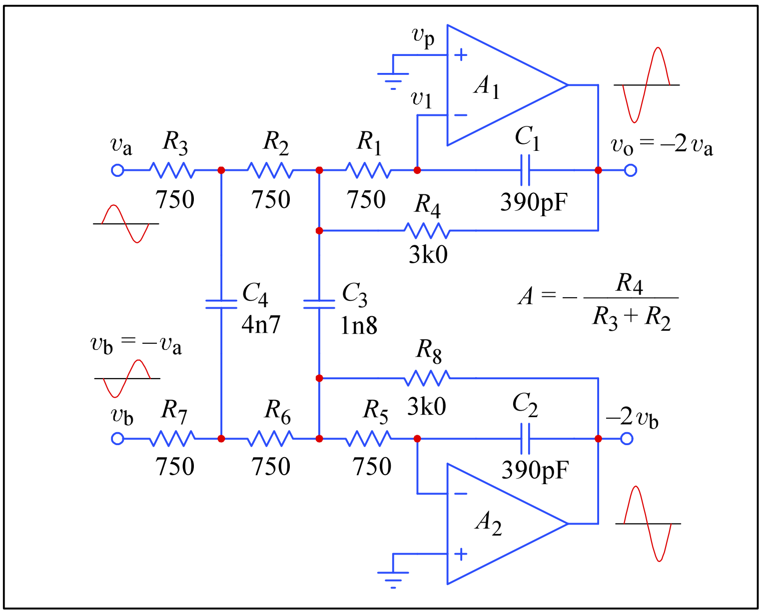

Fig.1: Differential third-order Bessel filter with a 50 kHz cut off, and a gain of two, inverting.

To make the circuit analysis easier, we cut the differential filter in half, taking only the upper arm, and grounding the lower side of capacitors C_3 and C_4. In the analysis we shall follow the node labels as are indicated in **Fig.3**.

IMPORTANT: In a differential filter the capacitors C_3 and C_4 are driven by the resistors in both arms, so in order to have the same system time constants those capacitors should be of double value in the single-ended circuit, as indicated in **Fig.3**.

Real or Ideal Opamps?

First we need to show that we are not making a big mistake if, instead of a real opamp, having a limited gain and bandwidth, we use an idealized model in the circuit analysis. A good audio grade opamp has a very high input resistance (>1MΩ), and a very low input capacitance (~1pF), so at audio frequencies (and even a decade beyond) its loading on the filter components can be neglected. To show this more clearly let us compare the amplifier's open loop performance to the highest circuit time constant set by R_1 and C_1 in the feedback loop.

But before we set off, let us make a short digression and consider the general polynomial form of the system transfer function as required by the circuit theory. This will hopefully clarify a couple of points, which sadly often remain obscure even in some of the best textbooks. The general Cauchy n^{th}-order polynomial form, by which we describe the performance of a circuit with n reactive components (capacitors or inductors) in the frequency domain is:

$$F_n(s) = \frac{1}{(s - s_1)(s - s_2)\cdots(s - s_n)} \tag{1}$$

Here $s_1, s_2, \ldots, s_n$ are the circuit poles set by the various system time constants. In general, a circuit may also have some zeros, $(s - z_1)$ etc., in the numerator, but here we are dealing with pole-only circuits.

When the expression (1) is multiplied, the last polynomial coefficient, which is a product of all the poles, $s_1 s_2 \cdots s_n$, will determine the system gain, which will obviously be different for systems with different number of poles. This is undesirable, because we want to treat the real (DC) gain of the circuit separately from its frequency dependence. We therefore normalize the characteristic circuit polynomial by dividing it with the same polynomial evaluated at $s = 0$ (DC). So the frequency dependence is:

$$\frac{F_n(s)}{F_n(0)} = \frac{\dfrac{1}{(s - s_1)(s - s_2)\cdots(s - s_n)}}{\dfrac{1}{(-s_1)(-s_2)\cdots(-s_n)}} = \frac{(-s_1)(-s_2)\cdots(-s_n)}{(s - s_1)(s - s_2)\cdots(s - s_n)} \tag{2}$$

In this way it is obvious that for $s = 0$, as well as for very low frequencies, the frequency response will be equal to 1 (it will deviate from unity as s approaches the lowest of the poles and progressively onward). Consequently the frequency independent gain can be attributed to a separate factor multiplying the expression (2).

The amplifier's transfer function $A(s)$ can be modeled by assuming its DC gain to be A_0, and its single dominant pole $s_0 = -\omega_0$. Every pole causes an additional 90° phase shift at high frequencies, so in order to remain stable at closed feedback loop, all other amplifier poles must be at very high frequencies, where the amplifier's gain is less than unity. This prevents the feedback from becoming positive with enough gain to cause oscillations. On the other hand, any pole defined by external circuitry must be suitably damped, so that the resistive (energy dissipating) part of the poles dominate, and the poles are positioned on the left hand side of the complex plane (they have their real part negative, whilst their imaginary part can be either zero or forming complex conjugate pairs). Poles too close to the imaginary axis (very low real part value) or poles in the right hand side of the complex plane (positive real part) are sure signs of trouble.

With the frequency variable $s = j\omega$, and the non-inverting input grounded, $v_\mathrm{p} = 0$, we have the open-loop gain determined by:

$$v_\mathrm{o} = (v_\mathrm{p} - v_\mathrm{n}) \cdot A(s) = (v_\mathrm{p} - v_\mathrm{n}) \cdot A_0 \frac{-s_0}{s - s_0} = -v_\mathrm{n} A_0 \frac{-s_0}{s - s_0} \tag{3}$$

From (1) we can express the open loop gain as a function of the complex frequency $s = \sigma + j\omega$ in the Laplace space:

$$A(s) = \frac{v_\mathrm{o}}{v_\mathrm{n}} = -A_0 \frac{-s_0}{s - s_0} \tag{4}$$

and along the imaginary frequency axis:

$$A(j\omega) = -A_0 \frac{\omega_0}{j\omega + \omega_0} \tag{5}$$

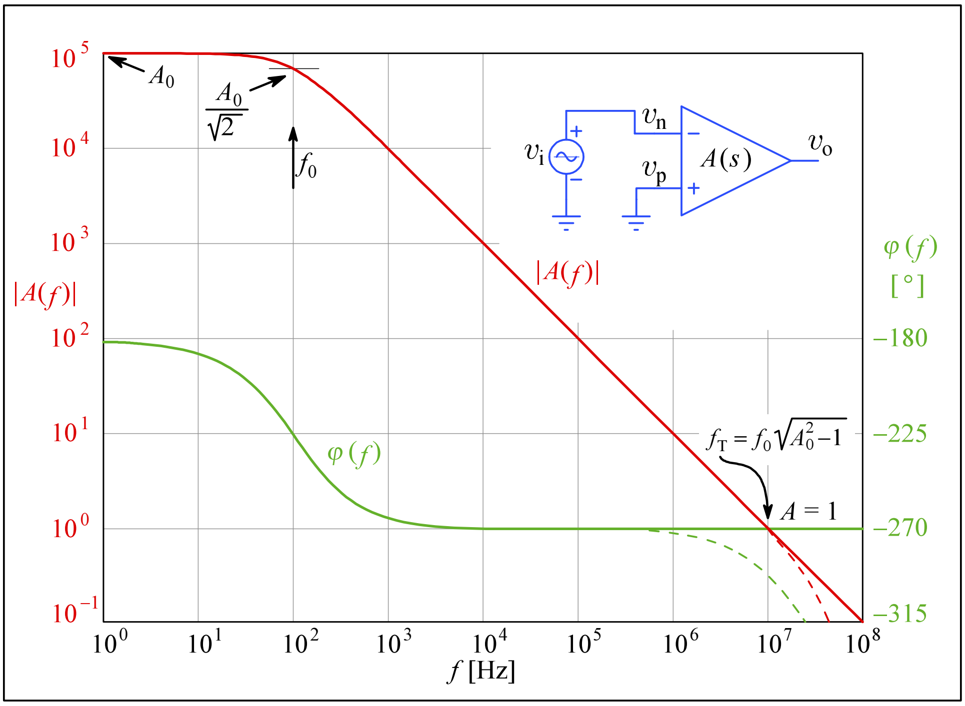

Fig.2: A typical open loop gain magnitude $|A(f)|$ and phase $\varphi(f)$ of an inverting amplifier. At the dominant pole frequency f_0 the magnitude is lower by $1/\sqrt{2}$, and falls off by a factor of 10 for each decade increase of the frequency. The dashed lines show the influence of a possible second pole, which for a stable unity gain closed loop must be higher than the transition frequency f_T. Note that for practical reasons we usually show the graphs as functions of the full cycle frequency (in Hz), instead of the angular frequency ω (in rad/s).

The magnitude (absolute value) of the transfer function is the square root of the product of the gain by its own complex conjugate:

$$\left|\frac{v_\mathrm{o}}{v_\mathrm{n}}\right| = \sqrt{A(j\omega)\cdot A(-j\omega)} = A_0\sqrt{\left(\frac{\omega_0}{j\omega+\omega_0}\right)\left(\frac{\omega_0}{-j\omega+\omega_0}\right)}$$

$$= A_0\sqrt{\frac{\omega_0^2}{\omega^2+\omega_0^2}} = A_0\frac{1}{\sqrt{1+\left(\dfrac{\omega}{\omega_0}\right)^2}} \tag{6}$$

Fig.2 shows graphically the expression (7). At DC ($\omega = 0$) the gain magnitude equals A_0. When the input frequency becomes $\omega = \omega_0$ the open loop gain falls to $A_0/\sqrt{2}$. Above ω_0 the gain magnitude decreases by a factor of 10 for every 10-fold increase in frequency (or $-20\,\mathrm{dB}/10f$, or $-6\,\mathrm{dB}/2f$). And when $\omega = \omega_0\sqrt{A_0^2-1}$, the open loop gain decreases to unity (and decreasing further beyond that frequency). This frequency is the 'transition frequency', often labeled ω_T.

Of course, the gain magnitude is not the whole story, the change of the phase angle of the output signal referred to the input is also important. From equation (1) the phase angle is expressed as the arctangent of the ratio of the imaginary to the real part of the gain function:

$$\varphi(s) = \arctan\left(\frac{\Im\{A(s)\}}{\Re\{A(s)\}}\right) \tag{7}$$

To separate the real and the imaginary part we need to rationalize the denominator, so we multiply both the numerator and the denominator by the complex conjugate of the denominator:

$$A(j\omega) = -A_0\frac{\omega_0(-j\omega+\omega_0)}{(j\omega+\omega_0)(-j\omega+\omega_0)} = -A_0\frac{\omega_0}{\omega^2+\omega_0^2}(\omega_0 - j\omega) \tag{8}$$

Now the phase along the imaginary axis is:

$$\varphi(j\omega) = \arctan\left(\frac{-A_0\dfrac{\omega_0}{\omega^2+\omega_0^2}\cdot(-\omega)}{-A_0\dfrac{\omega_0}{\omega^2+\omega_0^2}\cdot\omega_0}\right) = \arctan\left(\frac{-\omega}{\omega_0}\right) \tag{9}$$

The phase plot indicates the initial phase inversion ($-180°$) at low frequencies, since the signal is being fed to the amplifier's inverting input. However, note that the positive direction of the phase rotation is defined as counterclockwise, so the negative frequency sign, $-\omega$ in (9), indicates a clockwise rotation. With the frequency increasing towards the dominat pole the phase angle rotates by a further $-45°$ to $-225°$, and another $-45°$ beyond, reaching $-270°$ two decades higher.

Good audio grade opamps usually have $f_0 = \omega_0/2\pi \approx 100\,\mathrm{Hz}$, and $A_0 \approx 10^5$, thus $f_\mathrm{T} = \omega_\mathrm{T}/2\pi \approx 10\,\mathrm{MHz}$. Observe that R_1 and C_1 in our circuit set the shortest time constant, equivalent to $f = 1/2\pi R_1 C_1 \approx 647\,\mathrm{kHz}$, which is almost 20× lower than ω_T, so there will be still some 24 dB of feedback at that frequency. Thus we can be sure that neither the amplifier's input impedance, nor its gain and bandwidth limitations will influence much the $R_1 C_1$ time constant, and at audio frequencies the situation will be better still.

Circuit Analysis

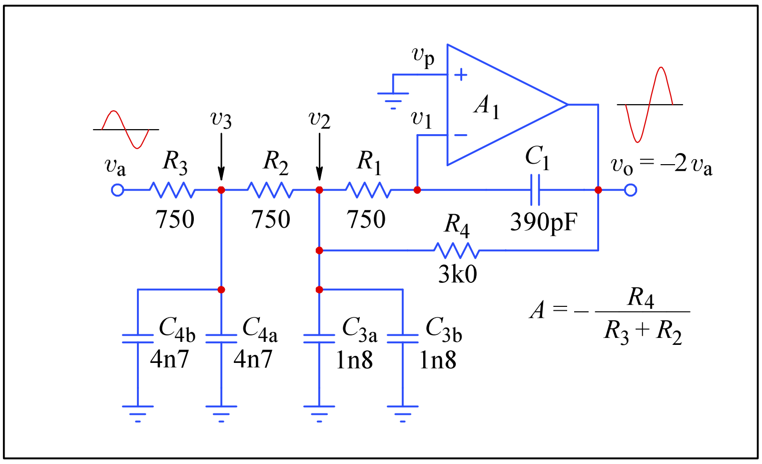

Fig.3: The single-ended filter used for analysis. Note the doubled C_3 and C_4.

We want to find the closed loop response of the circuit in **Fig.3**. With a real opamp the current sum equation at node v_1 would be $(v_2 - v_1)/R_1 = (v_1 - v_\mathrm{o})sC_1$, and we would have to use equation (3) to express v_1 by v_o. But because at the top of the audio band the open loop gain is still about 500, we can use an ideal amplifier model, and set $v_1 \approx v_\mathrm{p} = 0$, which simplifies the v_1 node equation to:

$$\frac{v_2}{R_1} = -v_\mathrm{o} s C_1 \tag{10}$$

From (10) we express v_2:

$$v_2 = -v_\mathrm{o} s C_1 R_1 \tag{11}$$

Next, the current sum at the node v_2 is:

$$\frac{v_3 - v_2}{R_2} = \frac{v_2}{R_1} + \frac{v_2 - v_\mathrm{o}}{R_4} + v_2 s C_3 \tag{12}$$

We can separate the voltages:

$$v_3 = v_2 \left(1 + \frac{R_2}{R_1} + \frac{R_2}{R_4} + s C_3 R_2 \right) - v_\mathrm{o} \frac{R_2}{R_4} \tag{13}$$

By substituting v_2 from (11) we obtain:

$$v_3 = -v_o sC_1R_1\left(1+\frac{R_2}{R_1}+\frac{R_2}{R_4}+sC_3R_2\right)-v_o\frac{R_2}{R_4} \tag{14}$$

By separating the voltage variables, and grouping the various powers of s we get:

$$v_3 = -v_o\left[sC_1R_1\left(1+\frac{R_2}{R_1}+\frac{R_2}{R_4}\right)+s^2C_1C_3R_1R_2+\frac{R_2}{R_4}\right] \tag{15}$$

Finally we set the current sum equation at node v_3:

$$\frac{v_a-v_3}{R_3}=\frac{v_3-v_2}{R_2}+v_3sC_4 \tag{16}$$

This we rearrange as:

$$v_a = v_3\left(1+\frac{R_3}{R_2}+sC_4R_3\right)+v_o sC_1R_1\frac{R_3}{R_2} \tag{17}$$

We now substitute v_3 from (15) and group the various powers of s:

$$v_a = -v_o\left(Ps^3+Qs^2+Rs+S\right) \tag{18}$$

where the polynomial coefficients are:

$$P = C_1C_3C_4R_1R_2R_3 \tag{19}$$

$$Q = C_1C_3R_1R_2\left(1+\frac{R_3}{R_2}\right)+C_1C_4R_1R_3\left(1+\frac{R_2}{R_1}+\frac{R_2}{R_4}\right) \tag{20}$$

$$R = C_1R_1\left(1+\frac{R_2}{R_1}+\frac{R_3}{R_1}+\frac{R_2}{R_4}+\frac{R_3}{R_4}\right)+C_4R_3\frac{R_2}{R_4} \tag{21}$$

$$S = \frac{R_2}{R_4}\left(1+\frac{R_3}{R_2}\right) \tag{22}$$

The system transfer function is:

$$\frac{v_o}{v_a} = -\frac{1}{Ps^3+Qs^2+Rs+S} \tag{23}$$

In order to obtain a canonical polynomial expression, we need to make the coefficient at the highest power of s equal to 1, so we must divide all the coefficients by P. By substituting $K_2 = Q/P$, $K_1 = R/P$, and $K_0 = S/P$ we obtain:

$$\frac{v_o}{v_a} = -\frac{\dfrac{1}{C_1C_3C_4R_1R_2R_3}}{s^3+K_2s^2+K_1s+K_0} \tag{23}$$

where we have:

$$K_2 = \frac{1}{C_4R_3}\left(1+\frac{R_3}{R_2}\right)+\frac{1}{C_3R_2}\left(1+\frac{R_2}{R_1}+\frac{R_2}{R_4}\right) \tag{24}$$

$$K_1 = \frac{1}{C_3 C_4 R_2 R_3}\left(1 + \frac{R_2}{R_1} + \frac{R_3}{R_1} + \frac{R_2}{R_4} + \frac{R_3}{R_4}\right) + \frac{1}{C_1 C_3 R_1 R_4} \tag{25}$$

$$K_0 = \frac{1}{C_1 C_3 C_4 R_1 R_2 R_3} \cdot \frac{R_2}{R_4}\left(1 + \frac{R_3}{R_2}\right) \tag{26}$$

In the normalized canonical polynomial form, the numerator must be equal to the coefficient K_0. But by doing so we increase the numerator by $(R_2 + R_3)/R_4$, so we must multiply the whole transfer function by an inverse of this factor:

$$\frac{v_\mathrm{o}}{v_\mathrm{a}} = -\frac{R_4}{R_2 + R_3} \cdot \frac{K_0}{s^3 + K_2 s^2 + K_1 s + K_0} \tag{27}$$

It is immediately obvious that the system DC gain must be:

$$A_0 = -\frac{R_4}{R_2 + R_3} \tag{28}$$

System Poles

Now that we have the transfer function expressed by the characteristic polynomial, and its coefficients determined, we must relate those coefficients to the system poles. By observing the relations in a general 3rd-order form:

$$F_3(s) = A_0 \frac{(-s_1)(-s_2)(-s_3)}{(s - s_1)(s - s_2)(s - s_3)} \tag{29}$$

$$= A_0 \frac{(-s_1)(-s_2)(-s_3)}{s^3 + s^2(-s_1 - s_2 - s_3) + s(s_1 s_2 + s_1 s_3 + s_2 s_3) + (-s_1)(-s_2)(-s_3)} \tag{30}$$

we can equate the coefficients K_2, K_1, K_0 with the appropriate combinations of system poles (note that $K_3 = 1$):

$$K_2 = -s_1 - s_2 - s_3 \tag{31}$$

$$K_1 = s_1 s_2 + s_1 s_3 + s_2 s_3 \tag{32}$$

$$K_0 = (-s_1)(-s_2)(-s_3) \tag{33}$$

In order to express the poles $s_{1,2,3}$ by the polynomial coefficients $K_{2,1,0}$ we have to solve a system of three equations with three unknowns. An easy way to accomplish the task would be by realizing that in (32) we have a sum and a product of two poles, say s_2 and s_3, and those are readily available from the other two equations. Specifically we reorder (32) as:

$$K_1 = s_1(s_2 + s_3) + s_2 s_3 \tag{34}$$

Then from (33) we have:

$$s_2 s_3 = -\frac{K_0}{s_1} \tag{35}$$

and from (31) we have:

$$s_2 + s_3 = -K_2 - s_1 \tag{36}$$

With (35) and (36) we return to (34) to obtain:

$$K_1 = s_1(-K_2 - s_1) - \frac{K_0}{s_1} \tag{37}$$

which, after a little reordering can be written as a general 3rd-order equation:

$$s_1^3 + K_2 s_1^2 + s_1 K_1 + K_0 = 0 \tag{38}$$

Various forms of cubic equations have been solved already in the XVI century by the Venetian mathematician *Noccolò Fontana Tartaglia*, and published in 1545 by *Gerolamo Cardano*, using purely algebraic expressions. Later *François Viète* independently discovered the trigonometric solutions. Here we shall use the general algebraic form.

A general solution can have either three real roots, or three coincident real roots, or one real and two complex conjugate roots. In order to shorten the long expressions let us replace the polynomial coefficients as follows:

$$\begin{aligned} a &= K_2 \\ b &= K_1 \\ c &= K_0 \end{aligned} \tag{39}$$

Since there is a rather long common term, we shall replace it by the symbol D:

$$D = \sqrt[3]{36ab - 108c - 8a^3 + 12\sqrt{12b^3 - 3a^2b^2 - 54abc + 81c^2 + 12a^3c}} \tag{40}$$

The real polynomial root is then:

$$r_1 = \frac{D}{6} - \frac{6}{D}\left(\frac{b}{3} - \frac{a^2}{9}\right) - \frac{a}{3} \tag{41}$$

and the two complex-conjugate roots are:

$$r_{2,3} = -\frac{D}{12} + \frac{3}{D}\left(\frac{b}{3} - \frac{a^2}{9}\right) - \frac{a}{3} \pm j\frac{\sqrt{3}}{2}\left[\frac{D}{6} + \frac{6}{D}\left(\frac{b}{3} - \frac{a^2}{9}\right)\right] \tag{42}$$

It should be pointed out that the expression under the square root in D must be non-negative in order to be applicable to solutions with at least one real root, as required by any realizable electronic circuit. So once the real solution is known, with $s_1 = r_1$, we can go back to equations (35) and (36) and solve a quadratic equation, using its well known general solution (known already to Babylonian mathematicians). From (36) we have:

$$s_2 = -K_2 - s_1 - s_3 \tag{43}$$

And from (35) we have:

$$s_2 = -\frac{K_0}{s_1 s_3} \tag{44}$$

Thus by equating (43) to (44):

$$-\frac{K_0}{s_1 s_3} = -K_2 - s_1 - s_3 \tag{45}$$

we have the quadratic equation for s_3:

$$s_3^2 + s_3(K_2 + s_1) - \frac{K_0}{s_1} = 0 \tag{46}$$

with the general solution:

$$s_{3_{1,2}} = \frac{-(K_2 + s_1) \pm \sqrt{(K_2 + s_1)^2 - 4\left(-\frac{K_0}{s_1}\right)}}{2} \tag{47}$$

We obtain here either two real solutions (if the expression under the square root is positive), or two coincident real solutions (if the expression under the square root is zero), or a complex conjugate pair (if the expression under the square root is negative). It is not necessary to solve for s_2 from either (43) or (44), because if we assign s_{3_1} to s_3 then $s_2 = s_{3_2}$.

Actually, we may spare ourselves all that trouble, since $s_2 = r_2$ and $s_3 = r_3$.

It is now possible to express the poles $s_{1,2,3}$ by the actual RC time constants from the coefficients $K_{2,1,0}$ in equations (24–26). However, this is a tedious work, and does not offer much insight in the working of our circuit, and anyway, those expressions will be different for different types of 3rd-order filters, so we may as well be satisfied by numerical values for the poles $s_{1,2,3}$.

However, we still have to find a way of calculating the circuit components, as well as find such values which will conform to the desired Bessel 3rd-order system. Apparently the expressions given by the polynomial coefficients $K_{2,1,0}$ are more simple to work with.

But first let us see the relations for the Bessel poles and the polynomial coefficients of a 3rd-order system.

The general form for Bessel coefficients of order n can be calculated as:

$$c_i = \left. \frac{(2n - i)!}{2^{n-i}\, i!\, (n - i)!} \right|_{i=0 \cdots n} \tag{48}$$

For a 3rd-order system ($n=3$, and $i=0,1,2,3$) the coefficients are:

$$c_0 = 15 \tag{49}$$
$$c_1 = 15 \tag{50}$$
$$c_2 = 6 \tag{51}$$
$$c_3 = 1 \tag{52}$$

Our characteristic polynomial is now $s^3 + c_2 s^2 + c_1 s + c_0$, and we want to find its roots. We can again use (40–42), by making the following substitutions:

$$\begin{aligned} a &= c_2 \\ b &= c_1 \\ c &= c_0 \end{aligned} \tag{53}$$

then by solving equations (40–42) we arrive at the Bessel poles of a 3rd-order system:

$$r_1 = -2.3222 \qquad [\text{rad/s}] \tag{54}$$
$$r_2 = -1.8389 - j1.7544 \qquad [\text{rad/s}] \tag{55}$$
$$r_3 = -1.8389 + j1.7544 \qquad [\text{rad/s}] \tag{56}$$

Note that the Bessel poles were derived on the assumption that the envelope delay is normalized to unity, and not the bandwidth as in the Butterworth case! The bandwidth (the angular frequency at which the magnitude response falls to $1/\sqrt{2}$) will thus be somewhat higher than 1 rad/s, so we shall have to account for this, in addition to the actual bandwidth, when specifying the values of resistors and the capacitors.

Now for Butterworth systems the bandwidth is simply equal to the n^{th} root of the absolute value of the product of all the n poles. Because all Butterworth poles lie on a circle, the bandwidth is also equal to the absolute value of any single pole.

Unfortunately for Bessel systems there is no simple formula to relate the pole values to the system bandwidth. Bessel poles lie on a family of ellipses becoming larger with increasing system order, but all with the near focus at the complex plane origin and the other focus on the positive part of the real axis.

To obtain the bandwidth we must input the values of $r_{1,2,3}$ into equation (29), calculate the frequency response magnitude for a range of frequencies, then find the frequency at which the magnitude is down by $1/\sqrt{2}$, and then divide all the polynomial roots by that frequency. For a greater precision it might be necessary to reiterate this process once or twice.

Using that procedure the bandwidth of the 3rd-order Bessel system normalized to a unit envelope delay is $\omega_{3\text{N}} \approx 1.7556$ rad/s. By dividing the roots $r_{1,2,3}$ by $\omega_{3\text{N}}$ a bandwidth of 1 rad/s will result.

For audio work we need a 50 kHz bandwidth, or $\omega_\text{H} = 2\pi \cdot 50000$ rad/s, so we must multiply the roots by a factor $\omega_\text{H}/\omega_{3\text{N}}$. Thus our poles will be:

$$s_{1,2,3} = r_{1,2,3} \cdot \frac{\omega_\text{H}}{\omega_{3\text{N}}} \tag{43-57}$$

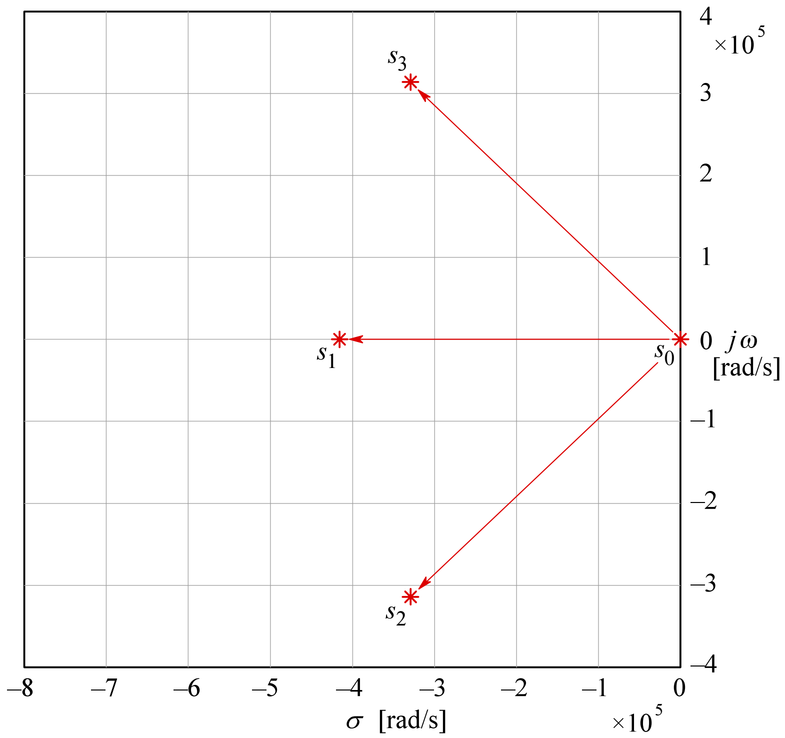

Fig.4: The layout of the system poles $s_{1,2,3}$. The pole at the complex plane origin, $s_0 = 0$, is owed to the Laplace transform of the input unit step signal. The system poles are on the left hand side (negative real part) of the complex plane. The poles s_2 and s_3 form a complex conjugate pair, the imaginary parts are controlled dominantly by the ratio C_3/C_1. The real pole s_1 is controlled dominantly by C_4.

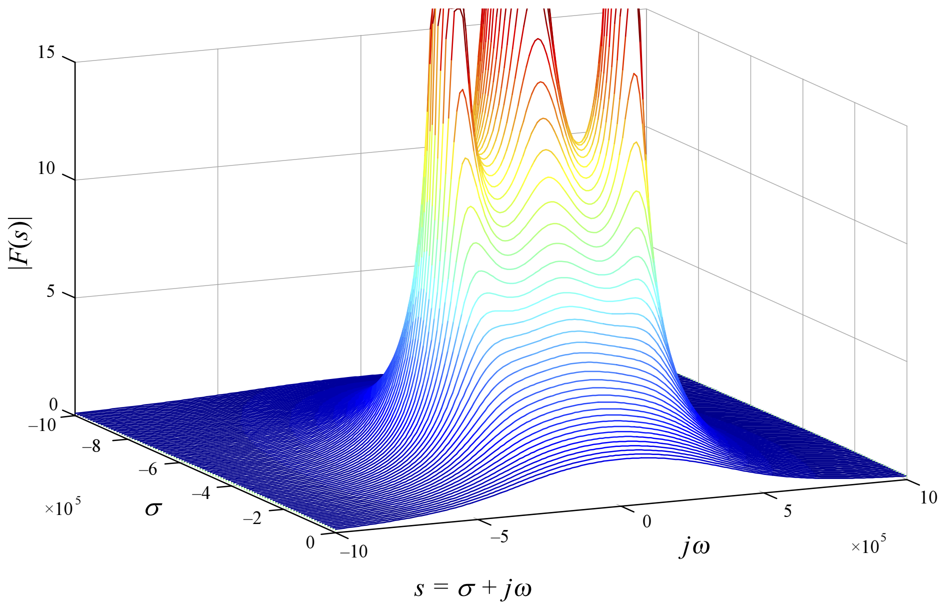

Fig.5: A ‘waterfall’ plot of the magnitude (absolute value) of the transfer function, $|F(s)|$, above the complex plane. The approximately Gaussian shape of the surface along the imaginary axis is the frequency response $|F(j\omega)|$ in linear (negative and positive) frequency scale. Above each pole the magnitude goes to infinity. The shape of each curve shows what would the frequency response look like if the poles were moved closer to the imaginary axis.

Calculating the Resistors and Capacitors

Now we have everything necessary to calculate the values of resistors and capacitors. However, there are 4 resistors and 3 capacitors in the filter, but we have a system of only 3 equations with 3 unknowns in (31–33). Fortunately, the circuit time constants are defined by appropriate RC products, which allows us to reduce the number of unknowns.

We have already established that the system gain $A_0 = R_4/(R_2 + R_3)$. In order to have a gain of 2 (to compensate the loss owed to the $600\,\Omega$ line impedance matching), we can make $R_2 = R_3 = R$, and $R_4 = 4R$. The remaining resistor R_1 can be of any suitable value, so to minimize the system variations we set it also to $R_1 = R$. This determines the resistor ratios, and by making $R = 1\,\Omega$ we can disregard it in the equations and determine the capacitance ratios. Once we do that, we can increase R to any suitable value and decrease the capacitors in proportion.

Note that by using the poles from (57) in relations for the polynomial coefficients $K_{2,1,0}$ (31–33) we would have to solve the 3-equation system once to express the poles by the coefficients, and then again to find the RC component values from the poles. It is thus more economic here to express the components directly by the coefficients, so we have to solve the system of equations only once.

From (24), (25) and (26) we have the normalized component values:

$$6 = \frac{1}{C_4 R_3}\left(1 + \frac{R_3}{R_2}\right) + \frac{1}{C_3 R_2}\left(1 + \frac{R_2}{R_1} + \frac{R_2}{R_4}\right) \tag{58}$$

$$15 = \frac{1}{C_3 C_4 R_2 R_3}\left(1 + \frac{R_2}{R_1} + \frac{R_3}{R_1} + \frac{R_2}{R_4} + \frac{R_3}{R_4}\right) + \frac{1}{C_1 C_3 R_1 R_4} \tag{59}$$

$$15 = \frac{1}{C_1 C_3 C_4 R_1 R_2 R_3} \cdot \frac{R_2}{R_4}\left(1 + \frac{R_3}{R_2}\right) \tag{60}$$

By replacing $R_1 = R_2 = R_3 = R$ and $R_4 = 4R$ (as required for a gain of 2):

$$6 = \frac{1}{C_4 R}\left(1 + \frac{R}{R}\right) + \frac{1}{C_3 R}\left(1 + \frac{R}{R} + \frac{R}{4R}\right) \tag{61}$$

$$15 = \frac{1}{C_3 C_4 R R}\left(1 + \frac{R}{R} + \frac{R}{R} + \frac{R}{4R} + \frac{R}{4R}\right) + \frac{1}{C_1 C_3 4 R R} \tag{62}$$

$$15 = \frac{1}{C_1 C_3 C_4 R R R} \cdot \frac{R}{4R}\left(1 + \frac{R}{R}\right) \tag{63}$$

By making $R = 1$ we have:

$$6 = \frac{1}{C_4}(1+1) + \frac{1}{C_3}\left(1+1+\frac{1}{4}\right) \tag{64}$$

$$15 = \frac{1}{C_3 C_4}\left(1+1+1+\frac{1}{4}+\frac{1}{4}\right) + \frac{1}{4C_1 C_3} \tag{65}$$

$$15 = \frac{1}{C_1 C_3 C_4} \cdot \frac{1}{4}(1+1) \tag{66}$$

Consequently:

$$6 = \frac{2}{C_4} + \frac{9}{4C_3} \tag{67}$$

$$15 = \frac{7}{2C_3 C_4} + \frac{1}{4C_1 C_3} \tag{68}$$

$$15 = \frac{1}{C_1 C_3 C_4} \cdot \frac{1}{2} \tag{69}$$

From (69) we can express, say, C_1:

$$C_1 = \frac{1}{30 C_3 C_4} \tag{70}$$

and insert it into (68);

$$15 = \frac{7}{2C_3 C_4} + \frac{1}{4\dfrac{1}{30 C_3 C_4} C_3} \tag{71}$$

From (71) we express C_3

$$C_3 = \frac{7}{15 C_4 (2 - C_4)} \tag{72}$$

We insert C_3 from (72) into (67):

$$6 = \frac{2}{C_4} + \frac{9}{4\dfrac{7}{15 C_4 (2 - C_4)}} \tag{73}$$

Now our only variable is C_4. By reordering we arrive at the 3rd-order polynomial:

$$C_4^3 - 2C_4^2 + \frac{168}{135} C_4 - \frac{56}{135} = 0 \tag{74}$$

We can solve (74) by the same relations as for the roots $r_{1,2,3}$ (40–42).

However, here we need only the real solution, because once we determine C_4 we already have real relations for finding C_1 and C_3. Our coefficients in (74) are now:

$$a = -2 \qquad b = \frac{168}{135} \qquad c = -\frac{56}{135} \tag{75}$$

We insert these into (41) and the resulting r_1 will be our normalized C_4. By putting the numbers into a mathematical computer program, such as Matlab, MathCAD, Mathematica, or other suitable software, we obtain:

$$C_4 = 1.2815\,\mathrm{F} \tag{76}$$

It is useful to make a program for equations (40–42) since the general solutions can be used for any other 3rd-order system calculation.

With the value of C_4 from (76) we return to (72) and calculate C_3:

$$C_3 = \frac{7}{30C_4\left(1-\frac{C_4}{2}\right)} = 0.5068\,\mathrm{F} \tag{77}$$

and with the value of C_3 we return to (70) and obtain C_1:

$$C_1 = \frac{1}{30C_3C_4} = 0.0513\,\mathrm{F} \tag{78}$$

Remember, these values have been calculated by assuming a Bessel system with the envelope delay of 1 s with the upper cut off frequency $\omega_{3\mathrm{N}} \approx 1.75\,\mathrm{rad/s}$ and with a normalized value $R = 1\Omega$. Because $\omega = 1/RC$, and because we require an actual cut off frequency $\omega_{\mathrm{H}} = 2\pi \cdot 50\,000\,\mathrm{rad/s}$ (or $f_{\mathrm{H}} = 50\,\mathrm{kHz}$), we now have to multiply these capacitor values by $\omega_{3\mathrm{N}}/\omega_{\mathrm{H}}$, and divide them by the actual value of R to obtain the required values for a system upper cut off frequency equal to ω_{H}.

We can choose any standard value of R (making sure that our amplifiers are capable of driving such an impedance), and then check if the resulting capacitor values are within their tolerance limits to their standard values. Without a special order the capacitors available on the market are usually of a 10% tolerance, with the E12 set of values, whilst the resistors are easily obtainable with 1% values and the E48 set. Therefore there is a good chance of finding easily several suitable R values.

One such suitable value is $R = 750\,\Omega$. This makes $R_4 = 3000\,\Omega$, and the capacitors:

$$\begin{aligned} C_4 &= 9.548\,\mathrm{nF} \\ C_3 &= 3.776\,\mathrm{nF} \\ C_1 = C_2 &= 382\,\mathrm{pF} \end{aligned} \tag{79}$$

As already emphasized, in the **differential filter** C_4 and C_3 are being driven by the resistors in both filter arms, so they effectively see a double resistance. Thus the differential filter uses C_3 and C_4 of half the value of the single-ended example.

We can approximate the values to the following nearest standard values:

$$C_4 = 4.7\,\mathrm{nF}$$
$$C_3 = 1.8\,\mathrm{nF}$$
$$C_1 = C_2 = 390\,\mathrm{pF} \tag{80}$$

Note that for our purpose it is not particularly important to have the upper cut off frequency exactly $50\,\mathrm{kHz}$, it is much more important to have the envelope delay substantially flat up to the filter cut off frequency. With a flat envelope delay all the relevant frequencies will pass through the system with equal delay, and that preserves the shape of the time-domain transient (step and impulse) response as close as possible to the ideal for the chosen bandwidth.

Design Verification

Because the circuit is very simple we can verify the design by building it on a proto-board, and measure its performance by inserting a low frequency square wave to the input and observe the waveform on an oscilloscope, checking the rise time (the 10% to 90% transition should be of the order of 10–12 μs) and the amount of overshoot (this should be ideally 0.4%, not exceeding 1%).

We can also build the filter model in one of the many circuit simulation programs and run the AC and transient simulations.

We can also verify the design by calculating the magnitude, the phase angle and the envelope delay as functions of frequency and plot the results. We can also check the design by calculating and plotting the transient (step) response. The procedure example is shown here briefly.

Frequency Domain Performance

As already shown in equation (6), the magnitude (absolute value) of the transfer function is calculated by taking the square root of the product of the transfer function with its own complex conjugate. We take the transfer function from (27) and express it as a function of purely imaginary frequency $j\omega$:

$$F(j\omega) = A_0 \frac{K_0}{(j\omega)^3 + (j\omega)^2 K_2 + j\omega K_1 + K_0} \tag{81}$$

The gain A_0 is the same as in (28), and the coefficients $K_{2,1,0}$ are the same as in equations (24–26). Taking the appropriate power of the imaginary unit yields:

$$F(j\omega) = A_0 \frac{K_0}{-j\omega^3 - \omega^2 K_2 + j\omega K_1 + K_0} \tag{82}$$

We need to separate the real and imaginary parts in the denominator:

$$F(j\omega) = A_0 \frac{K_0}{(-\omega^2 K_2 + K_0) + j(-\omega^3 + \omega K_1)} \tag{83}$$

We rationalize the denominator by multiplying both the numerator and the denominator by the complex conjugate of the denominator:

$$F(j\omega) = \frac{A_0 K_0 [(-\omega^2 K_2 + K_0) - j(-\omega^3 + \omega K_1)]}{[(-\omega^2 K_2 + K_0) + j(-\omega^3 + \omega K_1)][(-\omega^2 K_2 + K_0) - j(-\omega^3 + \omega K_1)]} \tag{84}$$

By multiplying the terms in the denominator we obtain:

$$F(j\omega) = A_0 K_0 \frac{(-\omega^2 K_2 + K_0) - j(-\omega^3 + \omega K_1)}{(-\omega^2 K_2 + K_0)^2 + (-\omega^3 + \omega K_1)^2} \tag{85}$$

So the magnitude is:

$$M(\omega) = A_0 \sqrt{F(j\omega)F(-j\omega)} \tag{86}$$

$$= A_0 K_0 \frac{\sqrt{(-\omega^2 K_2 + K_0)^2 + (-\omega^3 + \omega K_1)^2}}{(-\omega^2 K_2 + K_0)^2 + (-\omega^3 + \omega K_1)^2}$$

$$= A_0 K_0 \frac{\sqrt{\omega^6 + \omega^4 (K_2^2 - 2K_1) + \omega^2 (K_1^2 - 2K_2 K_0) + K_0^2}}{\omega^6 + \omega^4 (K_2^2 - 2K_1) + \omega^2 (K_1^2 - 2K_2 K_0) + K_0^2} \tag{87}$$

To obtain the phase as the function of frequency use (85), and take the arctangent of the imaginary to real ratio:

$$\varphi(\omega) = \arctan \frac{\Im\{F(j\omega)\}}{\Re\{F(j\omega)\}} \tag{88}$$

$$\varphi(\omega) = \arctan \frac{A_0 K_0 \dfrac{-(-\omega^3 + \omega K_1)}{(-\omega^2 K_2 + K_0)^2 + (-\omega^3 + \omega K_1)^2}}{A_0 K_0 \dfrac{(-\omega^2 K_2 + K_0)}{(-\omega^2 K_2 + K_0)^2 + (-\omega^3 + \omega K_1)^2}} \tag{89}$$

$$= \arctan \frac{\omega^3 - \omega K_1}{-\omega^2 K_2 + K_0} \tag{90}$$

The envelope delay is calculated as the phase derivative in frequency. Because the phase angle is measured in radians and the angular frequency in radians per second, the envelope delay is measured in seconds.

$$\tau_e(\omega) = \frac{d\varphi(\omega)}{d\omega} \tag{91}$$

By using the phase expression of (90) the envelope delay is:

$$\tau_{\mathrm{e}}(\omega) = \frac{d}{d\omega}\left(\arctan\frac{\omega^3-\omega K_1}{-\omega^2 K_2+K_0}\right) \tag{92}$$

$$= \frac{1}{1+\left(\frac{\omega^3-\omega K_1}{-\omega^2K_2+K_0}\right)^2}\,\frac{(2\omega^2-K_1)(-\omega^2K_2+K_0)+(\omega^3-\omega K_1)(-2\omega K_2)}{(-\omega^2K_2+K_0)^2}$$

$$= \frac{1}{1+\left(\frac{\omega^3-\omega K_1}{-\omega^2K_2+K_0}\right)^2}\,\frac{-4\omega^4K_2+\omega^2(3K_2K_1+2K_0)-K_1K_0}{(-\omega^2K_2+K_0)^2}$$

$$= \frac{(-\omega^2K_2+K_0)^2}{(-\omega^2K_2+K_0)^2+(\omega^3-\omega K_1)^2}\,\frac{-4\omega^4K_2+\omega^2(3K_2K_1+2K_0)-K_1K_0}{(-\omega^2K_2+K_0)^2}$$

$$= \frac{-4\omega^4K_2+\omega^2(3K_2K_1+2K_0)-K_1K_0}{\omega^6+\omega^4(K_2^2-2K_1)+\omega^2(K_1^2-2K_2K_0)+K_0^2} \tag{93}$$

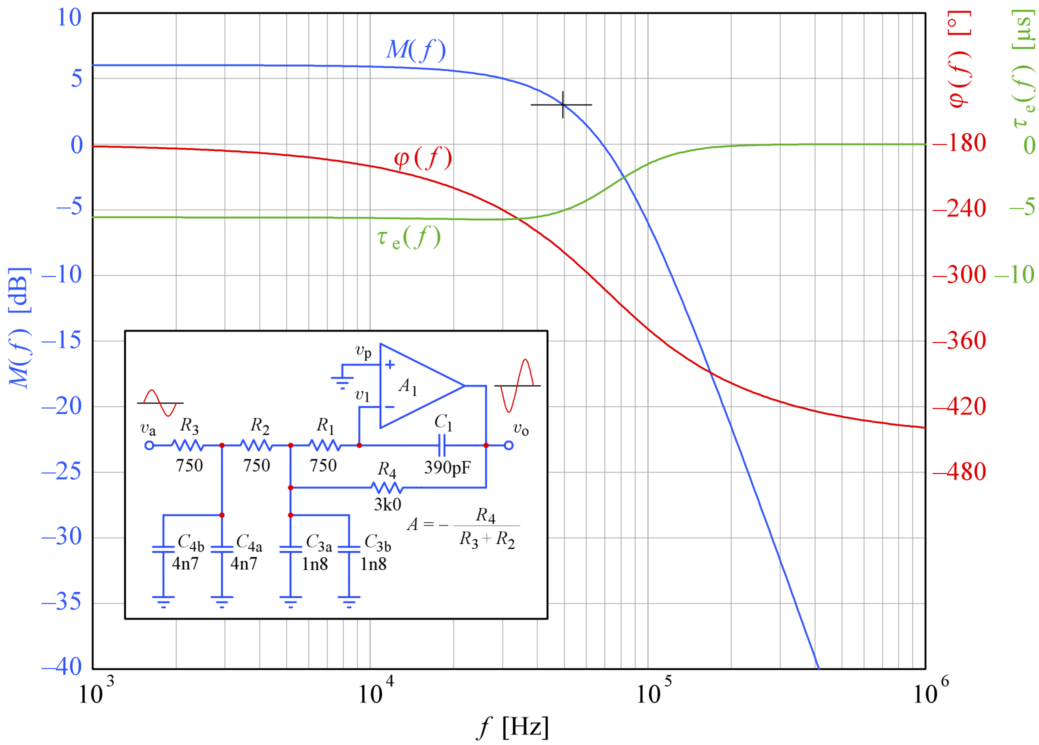

Fig.6: Frequency domain performance of the (single ended) 3rd-order Bessel filter. Note the gain of 2 (+6 dB), the $-18\,\mathrm{dB}/2f$ roll off, the phase inversion ($-180°$) at low frequencies, and the envelope delay of ~6 µs being essentially flat up to the cut off frequency (~50 kHz). The graphs were plotted using actual component values rounded to the standard values. The difference from the ideal Bessel filter is mostly notable from the envelope delay, which increases very slightly between 5×10^3 and 3×10^4 Hz.

Time Domain Performance (Transient Response)

It is beyond the scope of this text to provide the full mathematical background on the forward and inverse Laplace transform, the complex number theory, the contour integration of Cauchy or Laurent types of complex functions, and the theory of residues, which are necessary for understanding the relationship between the frequency domain and time domain performance of electronic circuits. Here we shall only give a ‘cookery book recipe’ for obtaining the (linear) time domain performance via residue calculation. More information can be found in standard mathematical textbooks, and in a compressed form suitable for electronics engineers in **Ref.6**.

A *residue* (*residuum* in Latin, meaning a remainder) of the transfer function is found by eliminating one of its poles and calculating the value of the transfer function at that pole from the remaining poles. By doing so for all the poles we obtain all the residues and **the sum of all the residues is the time domain response of the system**.

It is worth mentioning that the residue pairs of complex conjugated pole pairs are also complex conjugated, thus summing them up results in the double value of the real part only (imaginary components, being of opposite sign, cancel by the addition). Thus only $n/2$ residues need to be calculated for even order systems, and $1+(n-1)/2$ for odd order systems. Here we offer a full calculation as an example.

The calculation of residues is easy if we express the transfer function by (29) with the values of the poles from (57). Then the general form for a k^{th} residue is:

$$R_k = \lim_{s \to s_k} (s - s_k) \frac{\prod_{i=1}^{n}(-s_i)}{\prod_{i=1}^{n}(s - s_i)} \mathrm{e}^{st} \tag{94}$$

Note: The expression (94) is valid only for functions containing simple non-repeating poles. For functions containing multiple (coincident) poles, the procedure is different. If a function contains m coincident poles, say $s_{i+1} = \ldots = s_{i+m} = a$, there is a term $(s-a)^m$ in the characteristic polynomial of the transfer function, and to obtain the residue at the pole $s = a$ the limiting process should be performed on the $(m-1)^{\text{th}}$ derivative:

$$R_k = \lim_{s \to a} \frac{1}{(m-1)!} \left[\frac{d^{(m-1)}}{ds^{(m-1)}} (s-a)^m G(s) \right]$$

Note that circuits containing coincident poles (equal RC components) are never optimal in any sense, except maybe in the case of a 2nd-order system, if critical damping is absolutely necessary. In practice it is easy to disregard optimization and use equal RC components throughout the system, but in such a way we sacrifice too much, since in order to achieve the same bandwidth in a system with n stages those RC components must be selected for the cut off frequency higher by $\sqrt{n}$, with a consequent increase in system noise caused by the larger bandwidth in the first stages. It is therefore always preferable to use staggered (complex conjugated) poles wherever possible. This is especially true for multi-pole Bessel systems, since (owed to the flat envelope delay) their theoretical overshoot of the step response never exceeds 0.5%, a design goal also easily achievable in practice.

So for systems with simple staggered poles we can always use equation (94) for the evaluation of the residues.

One more important fact to note is that the sum of residues of the transfer function $F(s)$ will give us the system's impulse response. The system's time domain response to an input signal is equal to the convolution integral of that signal with the system's impulse response. In the frequency domain the convolution integral is transformed into a simple multiplication of the transformed signal by the transformed impulse response. Because the Laplace transform of a unit impulse $\delta(t)$ (the Dirac's delta) is equal to $D(s) = 1$, and the multiplication by 1 does not change the transfer function, it is obvious that the inverse Laplace transform of the transfer function $F(s)$ must return the system's impulse response $f(t)$.

However for the step response the situation is different. The Laplace transform of the Heaviside's unit step $h(t) = 1|_{t>0}$ is equal to $H(s) = 1/s$, thus to obtain the step response we need the residues of the composite function $G(s) = \frac{1}{s}F(s)$. Because of this there will be an additional residue owed to the pole at $s = 0$. The meaning of this pole and its residue is easily comprehended by realizing that frequency is inverse of time, thus $s = 0$ means $t = \infty$. In other words, the pole at the complex plane origin determines the final value of the system's output to which it settles to when all transient phenomena have vanished, after some long time (several times longer than the largest system's time constant). For low pass systems this will be the system's DC value, equal to 1 for unit gain systems, and equal to A_0 for systems with a DC gain of A_0. For band pass and high pass systems this value will usually be zero (or a small DC offset).

In spite of this being a general rule, it is a good practice to check the result!

Here we now calculate the step response. We shall use the following numerical values of the poles, taken from (54–56) and multiplied by the ratio of the desired to the normalized system cut off frequency, as in (57):

$$s_1 = r_1 \frac{\omega_{\mathrm{H}}}{\omega_{3\mathrm{N}}} = -2.3222 \frac{2\pi \cdot 5{\times}10^4}{1.7556} = -4.1555{\times}10^5 \,\mathrm{rad/s} \tag{95}$$

$$\begin{aligned} s_2 = r_2 \frac{\omega_{\mathrm{H}}}{\omega_{3\mathrm{N}}} &= (-1.8389 - j1.7544) \frac{2\pi \cdot 5{\times}10^4}{1.7556} \\ &= (-3.2907 - j3.1394){\times}10^5 \,\mathrm{rad/s} \end{aligned} \tag{96}$$

$$\begin{aligned} s_3 = r_3 \frac{\omega_{\mathrm{H}}}{\omega_{3\mathrm{N}}} &= (-1.8389 + j1.7544) \frac{2\pi \cdot 5{\times}10^4}{1.7556} \\ &= (-3.2907 + j3.1394){\times}10^5 \,\mathrm{rad/s} \end{aligned} \tag{97}$$

Since we want the step response we must multiply our transfer function $F(s)$, equation (29) by the Laplace transform of the input unit step signal, $1/s$, so we shall have an additional pole at the complex plane origin:

$$s_0 = 0 \,\mathrm{rad/s} \tag{98}$$

and our new function will be:

$$G(s) = \frac{1}{s} F(s) = \frac{1}{s} A_0 \frac{(-s_1)(-s_2)(-s_3)}{(s-s_1)(s-s_2)(s-s_3)} \tag{99}$$

Our first residue for $s \to 0$ will be:

$$R_0(t) = \lim_{s \to 0} (s - s_0) \frac{1}{s} A_0 \frac{-s_1 s_2 s_3}{(s-s_1)(s-s_2)(s-s_3)} \, \mathrm{e}^{st} \tag{100}$$

$$= A_0 \frac{-s_1 s_2 s_3}{(0-s_1)(0-s_2)(0-s_3)} \, \mathrm{e}^{0t} \tag{101}$$

$$= A_0 \tag{102}$$

because the product $(s - s_0)\frac{1}{s} = (s - 0)\frac{1}{s} = s\frac{1}{s} = 1$ before the limiting process, and after the limiting we have the product of the poles in both the numerator and the denominator, which equals 1, and also $\mathrm{e}^0 = 1$. The second residue for $s \to s_1$ will be:

$$R_1(t) = \lim_{s \to s_1} (s - s_1) \frac{1}{s} A_0 \frac{(-s_1)(-s_2)(-s_3)}{(s-s_1)(s-s_2)(s-s_3)} \, \mathrm{e}^{st} \tag{103}$$

$$= \lim_{s \to s_1} \frac{1}{s} A_0 \frac{-s_1 s_2 s_3}{(s-s_2)(s-s_3)} \, \mathrm{e}^{st} \tag{104}$$

$$= \frac{1}{s_1} A_0 \frac{-s_1 s_2 s_3}{(s_1-s_2)(s_1-s_3)} \, \mathrm{e}^{s_1 t} \tag{105}$$

$$= -A_0 \frac{s_2 s_3}{(s_1-s_2)(s_1-s_3)} \, \mathrm{e}^{s_1 t} \tag{106}$$

because $(s - s_1)$ cancels before the limiting, and $s_1$cancels after the limiting. Likewise for $s \to s_2$:

$$R_2(t) = \lim_{s \to s_2} (s - s_2) \frac{1}{s} A_0 \frac{(-s_1)(-s_2)(-s_3)}{(s-s_1)(s-s_2)(s-s_3)} \, \mathrm{e}^{st} \tag{107}$$

$$= \lim_{s \to s_2} \frac{1}{s} A_0 \frac{-s_1 s_2 s_3}{(s-s_1)(s-s_3)} \, \mathrm{e}^{st} \tag{108}$$

$$= \frac{1}{s_2} A_0 \frac{-s_1 s_2 s_3}{(s_2-s_1)(s_2-s_3)} \, \mathrm{e}^{s_2 t} \tag{109}$$

$$= -A_0 \frac{s_1 s_3}{(s_2-s_1)(s_2-s_3)} \, \mathrm{e}^{s_2 t} \tag{110}$$

and finally for $s \to s_3$:

$$R_3(t) = \lim_{s \to s_3} (s - s_3) \frac{1}{s} A_0 \frac{(-s_1)(-s_2)(-s_3)}{(s - s_1)(s - s_2)(s - s_3)} \, \mathrm{e}^{st} \tag{111}$$

$$= \lim_{s \to s_3} \frac{1}{s} A_0 \frac{-s_1 s_2 s_3}{(s - s_1)(s - s_2)} \, \mathrm{e}^{st} \tag{112}$$

$$= \frac{1}{s_3} A_0 \frac{-s_1 s_2 s_3}{(s_3 - s_1)(s_3 - s_2)} \, \mathrm{e}^{s_3 t} \tag{113}$$

$$= -A_0 \frac{s_1 s_2}{(s_3 - s_1)(s_3 - s_2)} \, \mathrm{e}^{s_3 t} \tag{114}$$

The time domain response is the sum of all residues:

$$g(t) = \sum_{i=0}^{n} R_i(t) = R_0(t) + R_1(t) + R_2(t) + R_3(t) \tag{115}$$

which in our case means:

$$g(t) = A_0 \left[1 - \frac{s_2 s_3 \mathrm{e}^{s_1 t}}{(s_1 - s_2)(s_1 - s_3)} - \frac{s_1 s_3 \mathrm{e}^{s_2 t}}{(s_2 - s_1)(s_2 - s_3)} - \frac{s_1 s_2 \mathrm{e}^{s_3 t}}{(s_3 - s_1)(s_3 - s_2)} \right] \tag{116}$$

We may now insert the appropriate numbers into the computer and calculate the step response for the required time range and resolution. According to a well known relation, which has been derived for a simple RC low pass system, the step rise time from 10% to 90% of the final amplitude is related to the cut off frequency as:

$$t_r \approx \frac{0.35}{f_{\mathrm{H}}} \tag{117}$$

Since Bessel systems follow closely the frequency response of the 1st-order system down to $-3\,$dB, the same approximation is valid for Bessel systems of any order. Thus, for a $50\,$kHz cut off we expect a rise time of about $7\,\mu$s, and since the system must settle close to the final value within some 5× larger time, it will be enough to select a time range from 0 to $35\,\mu$s. A resolution of $0.1\,\mu$s should be suitable, giving us a total of 350 time samples (longer time range and/or smaller time increments can be chosen at will, though this means more time samples and thus longer calculation).

However, before leaving the subject, there is another important insight to realize, which is difficult to see from the complex exponential expression in (116). In order to see this let us write the poles in terms of their real and imaginary part.

The pole s_1 is real, and $s_{2,3}$ form a complex conjugate pair:

$$\begin{aligned} s_1 &= \sigma_1 \\ s_2 &= \sigma_2 - j\omega_2 \\ s_3 &= \sigma_2 + j\omega_2 \end{aligned} \tag{118}$$

Here the real and imaginary components represent the numerical values as in (95–97). The time domain response can then be written as:

$$
\begin{aligned}
g(t) = A_0 \Bigg[1 &- \frac{(\sigma_2 - j\omega_2)(\sigma_2 + j\omega_2)}{(\sigma_1 - \sigma_2 + j\omega_2)(\sigma_1 - \sigma_2 - j\omega_2)} \mathrm{e}^{\sigma_1 t} \\
&- \frac{\sigma_1(\sigma_2 + j\omega_2)}{(\sigma_2 - j\omega_2 - \sigma_1)(\sigma_2 - j\omega_2 - \sigma_2 - j\omega_2)} \mathrm{e}^{(\sigma_2 - j\omega_2)t} \\
&- \frac{\sigma_1(\sigma_2 - j\omega_2)}{(\sigma_2 + j\omega_2 - \sigma_1)(\sigma_2 + j\omega_2 - \sigma_2 + j\omega_2)} \mathrm{e}^{(\sigma_2 + j\omega_2)t} \Bigg]
\end{aligned} \tag{119}
$$

As we sum and multiply the various terms we obtain:

$$
\begin{aligned}
g(t) = A_0 \Bigg[1 &- \frac{\sigma_2^2 + \omega_2^2}{(\sigma_1 - \sigma_2)^2 + \omega_2^2} \mathrm{e}^{\sigma_1 t} \\
&- \frac{\sigma_1(\sigma_2 + j\omega_2)}{\sigma_2 - \sigma_1 - j\omega_2} \cdot \frac{\mathrm{e}^{(\sigma_2 - j\omega_2)t}}{-2j\omega_2} \\
&- \frac{\sigma_1(\sigma_2 - j\omega_2)}{\sigma_2 - \sigma_1 + j\omega_2} \cdot \frac{\mathrm{e}^{(\sigma_2 + j\omega_2)t}}{2j\omega_2} \Bigg]
\end{aligned} \tag{120}
$$

To reorder this properly we must rationalize the denominators of the last two terms, so we multiply those numerators and denominators by the denominator's complex conjugate:

$$
\begin{aligned}
g(t) = A_0 \Bigg[1 &- \frac{\sigma_2^2 + \omega_2^2}{(\sigma_1 - \sigma_2)^2 + \omega_2^2} \mathrm{e}^{\sigma_1 t} \\
&- \frac{\sigma_1(\sigma_2 + j\omega_2)(\sigma_2 - \sigma_1 + j\omega_2)}{(\sigma_2 - \sigma_1 - j\omega_2)(\sigma_2 - \sigma_1 + j\omega_2)} \cdot \frac{\mathrm{e}^{(\sigma_2 - j\omega_2)t}}{-2j\omega_2} \\
&- \frac{\sigma_1(\sigma_2 - j\omega_2)(\sigma_2 - \sigma_1 - j\omega_2)}{(\sigma_2 - \sigma_1 + j\omega_2)(\sigma_2 - \sigma_1 - j\omega_2)} \cdot \frac{\mathrm{e}^{(\sigma_2 + j\omega_2)t}}{2j\omega_2} \Bigg]
\end{aligned} \tag{121}
$$

The denominators are now real (except for the exponential terms, which will be dealt with later):

$$
\begin{aligned}
g(t) = A_0 \Bigg[1 &- \frac{\sigma_2^2 + \omega_2^2}{(\sigma_1 - \sigma_2)^2 + \omega_2^2} \mathrm{e}^{\sigma_1 t} \\
&- \frac{\sigma_1(\sigma_2 + j\omega_2)(\sigma_2 - \sigma_1 + j\omega_2)}{(\sigma_2 - \sigma_1)^2 + \omega_2^2} \cdot \frac{\mathrm{e}^{(\sigma_2 - j\omega_2)t}}{-2j\omega_2} \\
&- \frac{\sigma_1(\sigma_2 - j\omega_2)(\sigma_2 - \sigma_1 - j\omega_2)}{(\sigma_2 - \sigma_1)^2 + \omega_2^2} \cdot \frac{\mathrm{e}^{(\sigma_2 + j\omega_2)t}}{2j\omega_2} \Bigg]
\end{aligned} \tag{122}
$$

Now we perform the multiplications in the numerators:

$$
g(t) = A_0 \Bigg\{ 1 - \frac{\sigma_2^2 + \omega_2^2}{(\sigma_1 - \sigma_2)^2 + \omega_2^2} \mathrm{e}^{\sigma_1 t}
$$

$$
\left.
\begin{aligned}
&-\frac{\sigma_1[\sigma_2^2-\sigma_1\sigma_2-\omega_2^2+j\omega_2(2\sigma_2-\sigma_1)]}{(\sigma_2-\sigma_1)^2+\omega_2^2}\cdot\frac{\mathrm{e}^{(\sigma_2-j\omega_2)t}}{-\,2j\omega_2}\\
&-\frac{\sigma_1[\sigma_2^2-\sigma_1\sigma_2-\omega_2^2-j\omega_2(2\sigma_2-\sigma_1)]}{(\sigma_2-\sigma_1)^2+\omega_2^2}\cdot\frac{\mathrm{e}^{(\sigma_2+j\omega_2)t}}{2j\omega_2}
\end{aligned}
\right\}
\tag{123}
$$

and we separate the real and imaginary parts, but we leave the imaginary unit in the denominators of the exponential parts, as we shall need them later:

$$
\begin{aligned}
g(t)=A_0\Bigg[1&-\frac{\sigma_2^2+\omega_2^2}{(\sigma_1-\sigma_2)^2+\omega_2^2}\mathrm{e}^{\sigma_1 t}\\
&-\frac{\sigma_1(\sigma_2^2-\sigma_1\sigma_2-\omega_2^2)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\cdot\frac{\mathrm{e}^{(\sigma_2-j\omega_2)t}}{-\,2j\omega_2}-j\frac{\sigma_1\omega_2(2\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\cdot\frac{\mathrm{e}^{(\sigma_2-j\omega_2)t}}{-\,2j\omega_2}\\
&-\frac{\sigma_1(\sigma_2^2-\sigma_1\sigma_2-\omega_2^2)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\cdot\frac{\mathrm{e}^{(\sigma_2+j\omega_2)t}}{2j\omega_2}+j\frac{\sigma_1\omega_2(2\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\cdot\frac{\mathrm{e}^{(\sigma_2+j\omega_2)t}}{2j\omega_2}\Bigg]
\end{aligned}
\tag{124}
$$

We group together the two real parts and the two imaginary parts:

$$
\begin{aligned}
g(t)=A_0\Bigg[1&-\frac{\sigma_2^2+\omega_2^2}{(\sigma_1-\sigma_2)^2+\omega_2^2}\mathrm{e}^{\sigma_1 t}\\
&-\frac{\sigma_1(\sigma_2^2-\sigma_1\sigma_2-\omega_2^2)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\left(\frac{\mathrm{e}^{(\sigma_2-j\omega_2)t}}{-\,2j\omega_2}+\frac{\mathrm{e}^{(\sigma_2+j\omega_2)t}}{2j\omega_2}\right)\\
&-j\frac{\sigma_1\omega_2(2\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\left(\frac{\mathrm{e}^{(\sigma_2-j\omega_2)t}}{-\,2j\omega_2}-\frac{\mathrm{e}^{(\sigma_2+j\omega_2)t}}{2j\omega_2}\right)\Bigg]
\end{aligned}
\tag{125}
$$

We can now cancel the imaginary unit in the last term:

$$
\begin{aligned}
g(t)=A_0\Bigg\{1&-\frac{\sigma_2^2+\omega_2^2}{(\sigma_1-\sigma_2)^2+\omega_2^2}\mathrm{e}^{\sigma_1 t}\\
&-\frac{\sigma_1(\sigma_2^2-\sigma_1\sigma_2-\omega_2^2)}{\omega_2\left[(\sigma_2-\sigma_1)^2+\omega_2^2\right]}\left(\frac{\mathrm{e}^{(\sigma_2-j\omega_2)t}}{-\,2j}+\frac{\mathrm{e}^{(\sigma_2+j\omega_2)t}}{2j}\right)\\
&-\frac{\sigma_1\omega_2(2\sigma_2-\sigma_1)}{\omega_2\left[(\sigma_2-\sigma_1)^2+\omega_2^2\right]}\left(\frac{\mathrm{e}^{(\sigma_2-j\omega_2)t}}{-\,2}-\frac{\mathrm{e}^{(\sigma_2+j\omega_2)t}}{2}\right)\Bigg\}
\end{aligned}
\tag{126}
$$

Now we separate the real and imaginary exponents:

$$
\begin{aligned}
g(t)=A_0\Bigg\{1&-\frac{\sigma_2^2+\omega_2^2}{(\sigma_1-\sigma_2)^2+\omega_2^2}\mathrm{e}^{\sigma_1 t}\\
&-\frac{\sigma_1(\sigma_2^2-\sigma_1\sigma_2-\omega_2^2)}{\omega_2\left[(\sigma_2-\sigma_1)^2+\omega_2^2\right]}\left(\frac{\mathrm{e}^{\sigma_2 t}\,\mathrm{e}^{-j\omega_2 t}}{-\,2j}+\frac{\mathrm{e}^{\sigma_2 t}\,\mathrm{e}^{j\omega_2 t}}{2j}\right)
\end{aligned}
$$

$$- \frac{\sigma_1(2\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\left(\frac{\mathrm{e}^{\sigma_2 t}\,\mathrm{e}^{-j\omega_2 t}}{-2} - \frac{\mathrm{e}^{\sigma_2 t}\,\mathrm{e}^{j\omega_2 t}}{2}\right)\Bigg\} \tag{127}$$

and we extract the common real exponential part:

$$\begin{aligned} g(t) = A_0\Bigg\{1 &- \frac{\sigma_2^2+\omega_2^2}{(\sigma_1-\sigma_2)^2+\omega_2^2}\mathrm{e}^{\sigma_1 t} \\ &- \frac{\sigma_1(\sigma_2^2-\sigma_1\sigma_2-\omega_2^2)}{\omega_2\left[(\sigma_2-\sigma_1)^2+\omega_2^2\right]}\mathrm{e}^{\sigma_2 t}\left(\frac{\mathrm{e}^{-j\omega_2 t}}{-2j}+\frac{\mathrm{e}^{j\omega_2 t}}{2j}\right) \\ &- \frac{\sigma_1(2\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\mathrm{e}^{\sigma_2 t}\left(\frac{\mathrm{e}^{-j\omega_2 t}}{-2}-\frac{\mathrm{e}^{j\omega_2 t}}{2}\right)\Bigg\} \end{aligned} \tag{128}$$

The imaginary exponential parts can be rearranged to form the Euler's representation of trigonometric functions, a sine and a cosine:

$$\begin{aligned} g(t) = A_0\Bigg\{1 &- \frac{\sigma_2^2+\omega_2^2}{(\sigma_1-\sigma_2)^2+\omega_2^2}\mathrm{e}^{\sigma_1 t} \\ &- \frac{\sigma_1(\sigma_2^2-\sigma_1\sigma_2-\omega_2^2)}{\omega_2\left[(\sigma_2-\sigma_1)^2+\omega_2^2\right]}\mathrm{e}^{\sigma_2 t}\,\frac{\mathrm{e}^{j\omega_2 t}-\mathrm{e}^{-j\omega_2 t}}{2j} \\ &+ \frac{\sigma_1(2\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\mathrm{e}^{\sigma_2 t}\,\frac{\mathrm{e}^{j\omega_2 t}+\mathrm{e}^{-j\omega_2 t}}{2}\Bigg\} \end{aligned} \tag{129}$$

and so we have a purely real time function:

$$\begin{aligned} g(t) = A_0\Bigg\{1 &- \frac{\sigma_2^2+\omega_2^2}{(\sigma_1-\sigma_2)^2+\omega_2^2}\mathrm{e}^{\sigma_1 t} \\ &- \frac{\sigma_1(\sigma_2^2-\sigma_1\sigma_2-\omega_2^2)}{\omega_2\left[(\sigma_2-\sigma_1)^2+\omega_2^2\right]}\mathrm{e}^{\sigma_2 t}\sin(\omega_2 t) \\ &+ \frac{\sigma_1(2\sigma_2-\sigma_1)}{(\sigma_2-\sigma_1)^2+\omega_2^2}\mathrm{e}^{\sigma_2 t}\cos(\omega_2 t)\Bigg\} \end{aligned} \tag{130}$$

What we have here are some real scaling coefficients in front of real exponential functions with negative exponents (decaying with time!), the last two multiplying (damping!) a sine and a cosine function.

The resulting step response is plotted in **Fig.7**, and it clearly shows that the design goal of obtaining a maximally steep rise time with minimal overshoot, characteristic of Bessel systems, has been met.

Of educational importance is **Fig.8**, where each row of equation (130) has been drawn separately in the red-green-blue order, with the gray curve the resulting sum (but without the multiplication by A_0). Note the purely exponential first row (red), and the heavily damped sine and cosine of the second (green) and third (blue) row. It is precisely the correct damping which ensures the desired response.

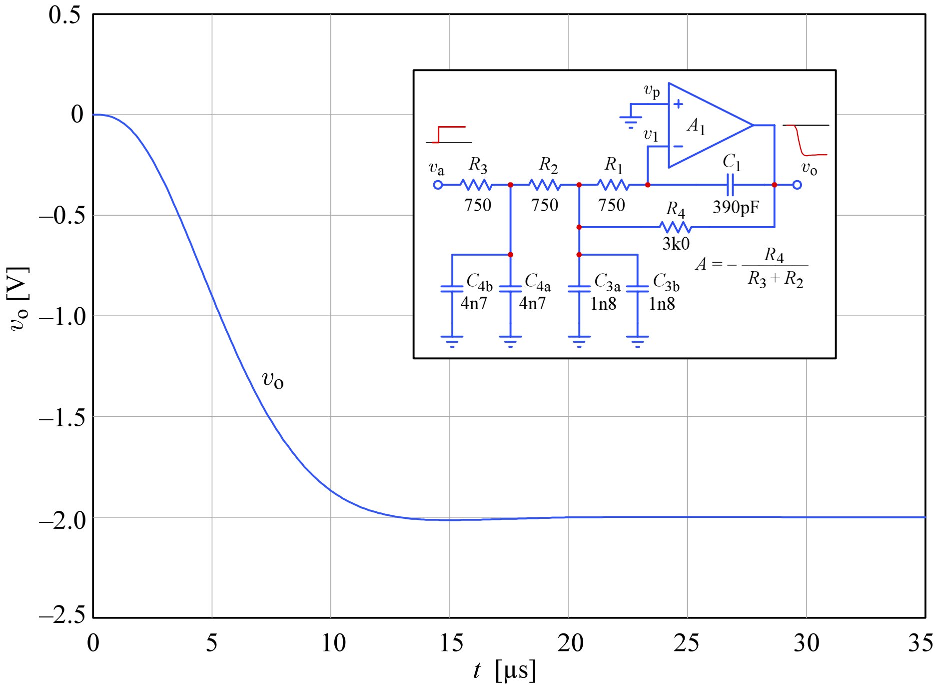

Fig.7: Time domain response to a unit step (going from $0\,\mathrm{V}$ to $1\,\mathrm{V}$ at $t = 0$). Note the gain of -2, the 50% level delay of ~$6\,\mu\mathrm{s}$ (as indicated by the envelope delay), the rise time (from 10% to 90%, or from $-0.2\,\mathrm{V}$ to $-1.8\,\mathrm{V}$) of ~$7\,\mu\mathrm{s}$, and the overshoot at ~$15\,\mu\mathrm{s}$ of $< 1\%$ above the final value.

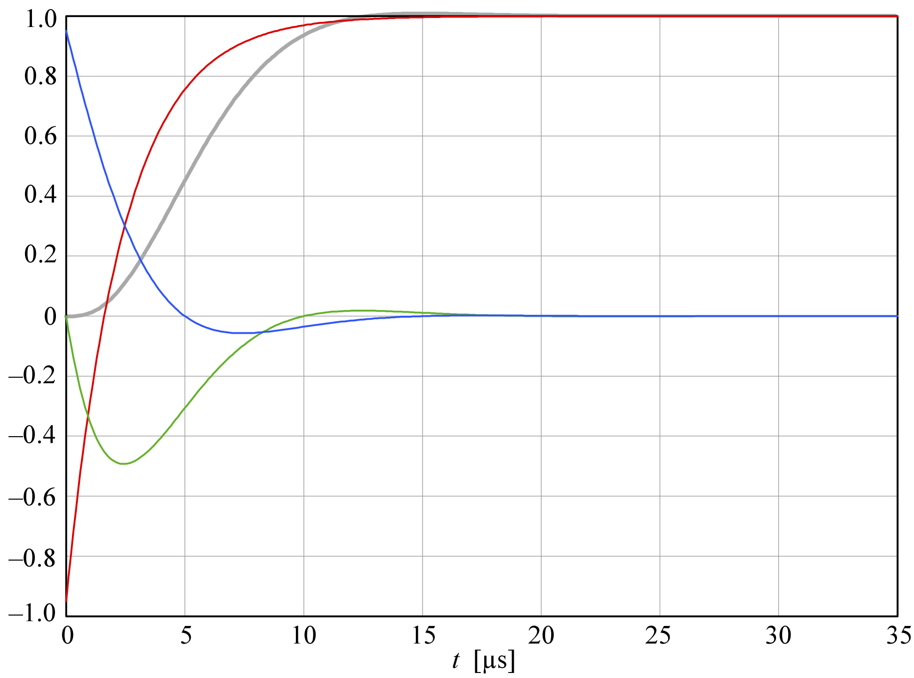

Fig.8: Plot of the row by row components of equation (130) in the red-green-blue order. The gray plot is their sum (as in Fig.7, but without A_0). Note the purely exponential red line (first row), and the heavily damped sine and cosine (green and blue) components (second and third row).

This completes our filter analysis.

I hope that the analysis has been presented with enough detail and without gaps, so that it should be easy to follow, and the general design principles shown are clear and easily applicable to other similar circuits. Those readers who will encounter some difficulties in the required theoretical or mathematical background should consult the relevant literature, some good examples are indicated in the references, but of course the list of available literature is much much broader.

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