Estimating the Vacuum Energy Density
- an Overview of Possible Scenarios

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1. Introduction

There are several different indications that the vacuum energy density should be non-zero, each indication being based either on laboratory experiments or on astronomical observations. These include the Planck's radiation law, the spontaneous emission of a photon by a particle in an excited state, the Casimir's effect, the van der Waals' bonds, the Lamb's shift, the Davies–Unruh's effect, the measurements of the apparent luminosity against the spectral red shift of supernovae type Ia, and more.

However, attempts to find the way to measure or to calculate the value of the vacuum energy density have all either failed or produced results incompatible with observations or other confirmed theoretical results. Some of those results are theoretically implausible because of certain unrealistic assumptions on which the calculation model is based. And some theoretical results are in conflict with observations, the conflict itself being caused by certain questionable hypotheses on which the theory is based. And the best experimental evidence (the Casimir's effect) is based on the measurement of the difference of energy density within and outside of the measuring apparatus, thus preventing in principle any numerical assessment of the actual energy density.

This article presents an overview of the most important estimation methods.

2. Planck's Theoretical Vacuum Energy Density

The energy density of the quantum vacuum fluctuations has been estimated shortly after Max Planck (1900-1901) [1] published his findings of the spectral distribution of the ideal thermodynamic black body radiation and its dependence on the temperature of the radiating black body. Although it was precisely this same law by which the quantization has been introduced in physics, Planck's calculation effectively relies on a completely classical formalism, with the additional assumption that matter emits energy in discrete (quantized) energy packets, called 'photons' by Lewis in 1927 (whilst Lewis' own theory of photons was later found to be incorrect, the name remained).

The same result has been eventually obtained by a purely quantum formalism.

Planck's radiation energy law is usually stated as a spectral function of the radiation frequency ν for every possible oscillating mode within the system, and of the radiating black body temperature T:

$$
\begin{aligned}
W(\nu,T) &= \frac{1}{2}h\nu\frac{\mathrm{e}^{\frac{h\nu}{2k_\mathrm{B}T}}+\mathrm{e}^{-\frac{h\nu}{2k_\mathrm{B}T}}}{\mathrm{e}^{\frac{h\nu}{2k_\mathrm{B}T}}-\mathrm{e}^{-\frac{h\nu}{2k_\mathrm{B}T}}} \\
&= \frac{1}{2}h\nu\coth\frac{h\nu}{k_\mathrm{B}T} \\
&= \frac{1}{2}h\nu\,\frac{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T}}+1}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T}}-1} \\
&= \frac{h\nu}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T}}-1}+\frac{1}{2}h\nu
\end{aligned}
\tag{2.1}
$$

Here the product of the Planck's constant h with the frequency ν is the photon energy:

$$W_\gamma = h\nu = \hbar\omega \tag{2.2}$$

with $\hbar$ being the reduced Planck's constant in Dirac's notation, $\hbar = h/2\pi$, and ω being the angular frequency in radians per second, $\omega = 2\pi\nu$.

The product of the Boltzmann's constant k_B with temperature T is the equivalent thermodynamic energy of the black body radiator:

$$W_\theta = k_\mathrm{B}T \tag{2.3}$$

Therefore the radiated energy (2.1) is a function of the photon energy and the exponential of the ratio of this photon energy and the thermodynamic energy of the radiating body.

This radiation energy has a spectral density (per 1 Hz bandwidth $d\nu$) of:

$$\rho_W(\nu,T)d\nu = 8\pi\frac{\nu^2}{c^3}\left(\frac{h\nu}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T}}-1}+\frac{h\nu}{2}\right)d\nu \tag{2.4}$$

From relations (2.1) and (2.4) it is clear that by lowering the temperature T of the black body radiator to absolute zero (or, equivalently, by removing the black body altogether), the exponential term goes to infinity, so the first aditive term vanishes, but the energy itself does not fall to zero, as would be expected, and neither does the energy density. What remains is the $h\nu/2$ term for each oscillating mode possible within the system.

The integration of this remainder over all possible frequencies results in the famous Einstein's *Nullspunktenergie*, the **zero-point energy**, as the term has been coined in an article published by Einstein and Stern in 1913 [2].

In most quantum theories this energy is customarily regarded as a consequence of the oscillations of all the elementary particles in the Universe in their basic ground quantum state. It is also associated with the vacuum energy content which would remain when all matter is eliminated from the Universe. However, such a view is questionable, since it is then not very clear what would be sustaining those oscillations once all the matter would have been eliminated from the Universe!

Therefore in the later development of quantum theories it is often assumed that the vacuum energy might have a cosmological origin (from the Big Bang onward) and thus represents the ground state of the quantum fluctuations within the vacuum itself. This view does not question nor attempts to explain the exact origin of quantum vacuum fluctuations, it simply assigns it to the vacuum energy content, so in a way this seems less satisfying than the previous view. It is only conjectured that the vacuum energy content should be a relic of the conditions preset at the Big Bang, and those very conditions are yet to be determined. But in this way the circulary argument of the origin of vacuum energy is avoided and the model presents a number of distinct advantages in its ability to explain a number of consequences.

It is further conjectured that elementary particles must balance their internal energy against that vacuum energy in order to remain stable (if they cannot, they will spontaneously decay sooner or later into lower energy, stable particles).

It is also conjectured that the vacuum energy density must decrease in the vicinity of elementary particles, in a similar way as it has been found to occur between the Casimir plates. Consequently, this decreasing energy density might be responsible for all the relativistic effects, including the variability of the speed of light (the bending of starlight trajectory near a massive body), and thus it might also be responsible for the very effect of gravity.

It is therefore of high theoretical and practical interest to be able to calculate the actual value of the vacuum energy density, even if in practice we can only experience energy differences, and not the absolute energy value.

In his later work Planck tried to develop a suitable system of units which would not be based on any material artifacts, such as the meter or the kilogram, but only on the values of natural fundamental constants. By starting from the speed of light c, the energy quantum action h (which we now call in his honor the Planck's constant), and the Newton's universal gravitational constant G, he developed what is now known as the Planckian metric. It determines the maximum energy W_P, the maximum frequency, ω_P, and the minimum length (the quantum of space) l_P, equal to the wavelength $\lambda_\mathrm{P} = 2\pi c/\omega_\mathrm{P}$.

In order to understand the reasons behind such assumptions, we start with a semi-classical argument derived from orbital mechanics, and impose on it the necessary constraints of relativity and of quantum mechanics.

A body within a circular orbit experiences a centripetal acceleration of v^2/r, derived from a gravitational force per unit mass, Gm/r^2. Now allow the velocity v to become the largest velocity possible, $v \to c$. Then the maximum acceleration possible within a system is $c^2/r = Gm/r^2$. From this equality the Schwarzschild's radius for a given mass m is readily obtained:

$$r_\mathrm{S} = \frac{Gm}{c^2} \tag{2.5}$$

Now the Heisenberg's Uncertainty Principle requires that the position x of an object and its instantaneous momentum p cannot be known at the same time to a precision better than the amount imposed by the quantum of action, $\Delta x \cdot \Delta p \geq \hbar$. By expressing Δp as mc, we obtain in the limiting case the Compton's radius:

$$r_\mathrm{C} = \frac{\hbar}{mc} \tag{2.6}$$

which is understood as the minimum possible size of a quantum object of mass m.

The laws of physics should be the same, regardless of the size of an object, so it makes sense to assume that the limits r_C and r_S must apply for all limiting cases, whether we are dealing with black holes or elementary particles. By equating this minimum quantum size for an object of mass m with the Schwarzschild's radius for that same object, $Gm/c^2 = \hbar/mc$, we readily obtain the Planck's mass limit:

$$m_\mathrm{P} = \sqrt{\frac{c\hbar}{G}} \approx 2.2{\times}10^{-8}\,\mathrm{kg} \tag{2.7}$$

The Compton's radius of the Planck's mass is then the shortest possible length in space, the Planck's length:

$$l_\mathrm{P} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6{\times}10^{-35}\,\mathrm{m} \tag{2.8}$$

This means that, because of the uncertainty relation, the Planck's mass cannot be compressed into a volume smaller than the cube of the Planck's length. A Planck's mass m_P contained within a Planck's volume l_P^3 therefore represents the maximum density of matter that can possibly exist:

$$\rho_\mathrm{P}(m) = \frac{c^5}{\hbar G^2} \tag{2.9}$$

Following the Einstein's most popular equation:

$$W = mc^2 \tag{2.10}$$

this mass density is equivalent to an energy density of:

$$\rho_{\mathrm{P}}(W) = \frac{c^7}{\hbar G^2} \tag{2.11}$$

Likewise, the maximum energy that can be contained within a volume l_{P}^3 is:

$$W_{\mathrm{P}} = m_{\mathrm{P}} c^2 = \sqrt{\frac{c^5 \hbar}{G}} \approx 1.98 \times 10^9 \,\mathrm{J} \tag{2.12}$$

or, if divided by the elementary charge $q_{\mathrm{e}} \approx 1.602 \times 10^{-19}$ As, about 1.23×10^{28} eV (instead of the electron charge some may prefer to see here the Planck charge $q_{\mathrm{P}} = \sqrt{\hbar c 4\pi\varepsilon_0} \approx 1.876 \times 10^{-18}$ As).

Consequently the speed of light limit, c, together with the minimum quantized space l_{P}, or equivalently the maximum energy density ρ_{P}, constrain the highest oscillations that can be sustained in space:

$$\omega_{\mathrm{P}} = \frac{W_{\mathrm{P}}}{\hbar} = \sqrt{\frac{c^5}{\hbar G}} \approx 1.9 \times 10^{43} \,\mathrm{rad/s} \tag{2.13}$$

All these relations represent the fundamental limits of nature, and thus the natural units of measure. By using these units it is possible to calculate the total average volumetric energy density, owed to all the frequency modes possible within the visible size of the Universe. Because we are averaging over all possible oscillating modes, it is proper to express it as the square root of the squared energy density given by (2.11):

$$\rho_{\mathrm{P}} = \sqrt{\frac{c^{14}}{\hbar^2 G^4}} \approx 4.6 \times 10^{113} \,\mathrm{J/m^3} \tag{2.14}$$

This result is unbelievably high, but there is nothing in any cosmological theory to prevent such, or any other particular value from being realized in nature.

Could there be a rational explanation for such a huge energy density? Following the Standard Model Big Bang scenario we expect that the Universe must have undergone a thermodynamic expansion, cooling down from the initial Planck's temperature of:

$$T_{\mathrm{P}} = \frac{m_{\mathrm{P}} c^2}{k_{\mathrm{B}}} = \sqrt{\frac{\hbar c^5}{G k_{\mathrm{B}}^2}} = 1.41679 \times 10^{32} \,\mathrm{K} \tag{2.15}$$

down to the present epoch temperature of ~2.7 K (the effective temperature of the cosmic microwave background radiation, first measured by Penzias and Willson in 1965). During this cooling process there must have been a number of epochs within which the radiation energy was balanced by particle pair formations, and inversely, annihilations of particle pairs into radiation. Likewise, those epochs must have been separated by well defined phase transitions, with lower energy particles becoming dominant over the previous higher energy particles, since the available energy density was becoming lower (owed to the expansion), so higher energy particle pairs could not be formed any more once they have mutually annihilated.

The last such transition has occurred some 3×10^5 years after the Big Bang, as the radiation energy decreased below a couple of MeV, thus making it impossible to form new electrons and positrons with enough kinetic energy to get them apart. The positrons thus eventually annihilated with electrons, leaving only a small amount of electrons which coupled to the protons that remained from a previous phase transition to form the first atoms. Thus radiation decoupled from matter, the average lifetime (free path) of a photon increased dramatically, and the Universe became transparent.

As we know from many particle interactions, certain creation and annihilation processes are slightly non-symmetrical. There are cases in which the probability of matter being created is higher than for antimatter, and this non-symmetry is responsible for all the matter in the Universe. Such non-symmetrical processes are known in theory as symmetry violations of quantum number conservation laws. Discovered were parity (P), charge (C), and even charge-parity (CP) symmetry violations. One such well studied case is the decay of kaons (K-mesons), which on rare occasions decays into three pions, instead of a $\pi^{\pm}$ pair. And since all the antimatter annihilated with the same amount of matter back into radiation, the number of photons remaining from all the annihilation transitions should be very high indeed, possibly many orders of magnitude higher than the number of matter particles which remained unpaired.

This process has been beautifully described in *The First Three Minutes*, a book published in 1973 by the Nobel laureate Steven Weinberg.

However, it must be noted that the result (2.14) has been obtained by assuming that the laws of the electromagnetic phenomena apply down to the wavelength comparable to the Planck's scale. This might be questionable. From the results of the DELPHI experiment of the CERN's LEP collider we know that the electromagnetic and the weak nuclear forces merge at an energy level of ~100 GeV, and a similar unification is foreseen also for the electroweak and strong nuclear force at ~10^{16} GeV (the Grand Unification Theory, GUT); finally, the electro-nuclear force is expected to merge with gravity at or somewhere below the $W_{\mathrm{P}} \approx 10^{19}$ GeV of the Planck's scale. Thus it may well be that the electromagnetic phenomena (as mediated by photons) cease to play a role at an energy density considerably lower than Planck's (2.14).

Such a high energy density (2.14) is unrealistic also for a number of other reasons. It would take us too far from the subject to discuss all those reasons here.

Because the photon energy is proportional to its frequency, the low frequency limit (with a wavelength comparable to the size of the visible Universe, or possibly beyond) is probably unimportant, as it contributes negligible energy in comparison with the energies within the scale of elementary particles. But what would be a plausible 'natural' high frequency limit, if we disregard the obvious Planck limit?

One possibility would be to assume a limit at or slightly above the energy necessary for the creation of the heaviest known elementary particle pair, a top quark pair, $t\bar{t}$. The top quark energy is currently known to be within the interval:

$$W_t = 172.0 \pm 2.2\,\mathrm{GeV} \tag{2.16}$$

or around $2.7554{\times}10^{-8}$ J. It is associated with a wavelength of $\lambda_t = 2.4{\times}10^{-26}$ m, or $\lambda_{tt} = 1.2{\times}10^{-26}$ m, as needed to create a $t\bar{t}$ pair. The appropriate frequency is then:

$$\nu_{tt} = \frac{c}{\lambda_{tt}} \approx \frac{3{\times}10^{8}}{1.2{\times}10^{-26}} \approx 2.49{\times}10^{34}\,\mathrm{Hz} \tag{2.17}$$

By taking the lower limit of integration as $\nu_{\min} = c/\lambda$, with $\lambda = 1\,\mathrm{m}$, we obtain the total volumetric energy density:

$$\rho_{tt} = \int_{\nu_{\min}}^{\nu_{tt}} \frac{8\pi\nu^2}{c^3}\frac{h\nu}{2}d\nu \tag{2.18}$$

$$= \frac{4\pi h}{c^3}\int_{\nu_{\min}}^{\nu_{tt}} \nu^3 d\nu$$

$$= \frac{4\pi h}{c^3}\cdot\frac{\nu^4}{4}\Bigg|_{\nu_{\min}}^{\nu_{tt}}$$

$$= \frac{\pi h}{c^3}\left(\nu_{tt}^4 - \nu_{\min}^4\right)$$

$$= \frac{3.14\cdot 6.6210^{-34}}{(3\times10^8)^3}\left[\left(2.49\times10^{34}\right)^4 - \left(3\times10^8\right)^4\right]$$

$$= 2.5676\times10^{-67}\cdot 3.8441\times10^{137} = 9.87\times10^{70}\,\mathrm{J/m^3} \tag{2.19}$$

Another possibility, as mentioned above, would be to allow for the super-symmetric extension of the Standard Model, and the appropriate high frequency limit would then be at the point where the coupling constants of the strong and the electroweak interactions meet, which is known as the 'grand unification ponit' from the Grand Unification Theory (GUT); this unification is associated with an energy of $W_{\mathrm{GUT}} \approx 10^{16}$ GeV, or ~1.6 MJ. In such a case, the associated wavelength would be $\lambda_{\mathrm{GUT}} = 1.24\times10^{-31}$ m, and the frequency:

$$\nu_{\mathrm{GUT}} \approx 2.4\times10^{39}\,\mathrm{Hz} \tag{2.20}$$

and in this case the energy density would be:

$$\rho_{\mathrm{GUT}} = \int_{\nu_{\min}}^{\nu_{\mathrm{GUT}}} \frac{8\pi\nu^2}{c^3}\frac{h\nu}{2}d\nu = 2.56\times10^{99}\,\mathrm{J/m^3} \tag{2.21}$$

So, those should be the 'more realistic' values of the vacuum energy density.

Well, are they?

Any figure obtained in such a way would only reflect the volumetric density of linearly superimposed photons, with no other particles involved. However, we also know that at certain energy density the vacuum becomes nonlinear; beyond that threshold the linear superposition of photons does not apply any more, since beyond that limit electron-positron pairs start to appear. And we also know that any field would be subject to quantum fluctuations because of the Heisenberg's Uncertainty Principle. This means that other factors may set the actual vacuum energy density limit, as will be discussed later.

Another limitation may arise from elementary statistics. By assuming a Gaussian distribution of random quantum fluctuations, a finite probability always exists for the local energy to briefly become high enough to create a certain particle pair. The higher the energy density, the higher will be the probability for the production of a more massive particle pair type.

This means that the photon density determines the particle production rate. Also, the average lifetime of the created particles will be influenced by the photon energy density, determining in turn the annihilation rate of the created pairs. In spite of their brief life time, within a unit volume a large number of particle pairs must exist at any time. However, during that short life time the pair's energy is unavailable to the rest of the field for the creation of other pairs. So the effective energy density depends heavily also on the number and types of particle pairs created.

If only these statistical processes would influence the vacuum energy density, and if the electromagnetic interaction would indeed apply up to the Planck's frequency, then it would be reasonable to estimate the vacuum energy density as proportional to the square root of the Planck's energy density:

$$\rho_{\mathrm{a}} = \sqrt{\rho_{\mathrm{P}}} \approx 10^{56}\ \mathrm{J/m^3} \tag{2.22}$$

and, of course, suitably lower in the case of ν_{GUT} or ν_{tt} as the respective upper frequency limit.

In spite of being substantially lower, the result (2.22) seems still too high to be plausible. But we will not pursuit this path any further.

Instead, let us see now the lowest non-zero value known so far.

3. Observational Cosmological Vacuum Energy Density

We have seen that the theoretical vacuum energy density expectations, based on either semi-classical or quantum calculations, are very high indeed.

However, recent measurements [3] of the relative brightness of supernovae type Ia against their distance (inferred from the amount of the red shift of their light spectrum, as well as the light spectrum of other objects within the same galaxy) indicates that the Universe is undergoing an accelerated rate of expansion, and not slowing down, as it was thought before. This expansion is being attributed to a certain small vacuum energy density. There are several ways (five at least!) of interpreting the cosmological vacuum energy, and the most simple one is in the form of Einstein's cosmological constant Λ, which is just an additive term in the Einstein–Friedmann equation.

By assuming a homogeneous and isotropic universe at a very large scale (>100Mpc), and starting from the Einstein's field equations of gravitation, Friedmann derived two independent equations for modeling the expanding universe, both based on a time-dependent scale parameter $x(t)$, such that $H = \frac{1}{x} \cdot \frac{dx}{dt}$ is the Hubble parameter. The first equation is:

$$H^2 = \left(\frac{1}{x} \cdot \frac{dx}{dt} \right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{x^2} + \frac{\Lambda c^2}{3} \tag{3.1}$$

and the second equation is:

$$\frac{dH}{dt} + H^2 = \frac{1}{x} \cdot \frac{d^2x}{dt^2} = -\frac{4\pi G}{3}\left(\rho - \frac{3p}{c^2} \right) + \frac{\Lambda c^2}{3} \tag{3.2}$$

In these equations x (the scale parameter), H (the Hubble parameter), ρ (the mass density), and p (the pressure) depend on the age of the Universe, whilst k is a constant determined by the shape of the universe ($+1$ for a closed 3–sphere, 0 for a flat Euclidean space, and -1 for an open 3–hyperboloid).

By solving the Einstein–Friedmann equation it is possible to obtain the scale factor x of the Friedmann–Lemaître–Robertson–Walker metric and its dependence of the expansion of the Universe with time, which must be proportional to the exponential function of the product of the Hubble parameter and time:

$$x(t) \propto \mathrm{e}^{Ht} \tag{3.3}$$

The Hubble parameter is then given by:

$$H = \sqrt{\frac{8\pi G \rho_{\mathrm{F}}}{3}} = \sqrt{\frac{\Lambda}{3}} \tag{3.4}$$

The value of the Hubble parameter, as inferred from the red shift and the apparent size (luminosity) of known stellar objects, is currently estimated to be about:

$$H \approx 70.88 \,\mathrm{km\,s^{-1}Mpc^{-1}} \tag{3.5}$$

and the currently estimated age of the Universe is 13.7 billion years, or:

$$t = 4.32339\times10^{17}\,\text{s} \tag{3.6}$$

Then the scale factor x is presently:

$$x = 2.699738 \tag{3.7}$$

The average mass density is about one hydrogen atom per cubic meter, or:

$$\rho_\text{F}(m) = 1.674\times10^{-27}\,\text{kg/m}^3 \tag{3.8}$$

and the value of the cosmological constant is :

$$\Lambda \approx 2\times10^{-35}\,\text{s}^{-2} \tag{3.9}$$

The value of Λ can also be expressed in other units as either ~10^{-47} GeV^4, or ~10^{-9} J/m^3, or ~10^{-120} in Planck units ($\hbar = c = G = 1$). In contrast, the Planck's energy density in Planck units is of course itself equal to 1.

This discrepancy of 120 orders of magnitude between the indirectly measured cosmological energy density and the theoretically expected energy density at the Planck scale is undeniably the most puzzling and embarrassing problem in modern physics, so it is only natural that many researchers devote their efforts to find a reasonable solution.

There can be a number of reasons why we have arrived at such a small result. One problem is that our measurements are indirect and depend heavily on the red shift, which is modeled as either a pure Doppler effect, or as an expansion of the spacetime itself (in accordance with the requirements of general relativity). This of course need not be the case, the red shift may be caused by a number of other mechanisms (say, gravitational, and many others), and only a part of it may be owed to the expansion of the universe.

But then the observed value of Λ should be even lower, ideally zero! This need not be in conflict with those high energy density estimations, because that would only mean that the vacuum energy is indeed Lorentz invariant because of its dependence on the 3$^{\text{rd}}$ power of frequency, as in (2.4) (any other power law would make the inertial movement impossible, since a moving object would then either loose or gain kinetic energy because of its interaction with the vacum energy), so essentially no effects can be measured within an inertial frame of reference (or a weak acceleration), regardless of the actual value of the vacuum energy density.

Another possibility is that Einstein's field equations are not completely correct and therefore the Friedmann equations, or at least one of its starting assumptions, do not apply to this type of universe. Whilst observations and experiments which confirm the predictions of the General Theory of Relativity pose a very high constraint for any alternative explanation, they do not exclude them entirely. Indeed, a number of physicists consider such solutions (such as modified laws of gravity at very large distances) to be more natural, and therefore preferable to solutions based on new fields, exotic new particles, extra dimensions, etc.

So we are left with a dilemma: either we are making some big mistake somewhere in our cosmological theories, or the vacuum is exactly Lorentz invariant and therefore no influence of the actual vacuum energy density can manifest in inertial or weak acceleration conditions, thus we cannot measure the energy density directly or indirectly. In both cases the very low or zero vacuum energy density is excluded.

4. Experimental Vacuum Energy Density

We have seen the two extremes of energy density calculations, a very high theoretical value, and a very low macroscopically observable value. Whilst the low value of the cosmological constant is appealing to many theorists, others would prefer a high value, because it would then be much easier to explain a number of otherwise incomprehensible experimental results.

Two such results are most indicative: one is obtained from the strength of the electric field of the atomic nucleus, and the other from the experiments producing fermion–antifermion particle pairs. The electric field strength at the nucleus of a completely ionized uranium atom $^{91}U^{+}$ (stripped off of all its electrons) is:

$$E_{\mathrm{U^+}} \approx 10^{18}\,\mathrm{V/m} \tag{4.1}$$

Because the uranium nucleus is the heaviest relatively stable one, it is reasonable to assume that it is very close to the maximum energy tolerable by the quantum vacuum fluctuations. Indeed, the experiments in which particle–antiparticle pairs are being produced at a very high field strength indicate that the vacuum starts to behave nonlinearly at about:

$$E_{\mathrm{b}} = \frac{m_{\mathrm{e}}^2 c^3}{q_{\mathrm{e}} \hbar} \tag{4.2}$$

$$= \frac{(9.1095\times10^{-31}\ [\mathrm{kg}])^2 (299792458\ [\mathrm{m/s}])^3}{1.602\times10^{-19}\ [\mathrm{As}] \times 1.0536\times10^{-34}\ [\mathrm{Js}]}$$

$$\approx 1.3\times10^{18}\,\mathrm{V/m} \tag{4.3}$$

At this value of the electric field the vacuum starts to break down: additional photons do not superimpose linearly on the existing field, but will provoke the generation of electron–positron pairs (e^-, e^+).

From the classical Maxwell's electromagnetic theory we know that the electric and magnetic field components of an E wave must provide equal energy density. This can be derived by starting from the vector expressions **E** and **H** of the electric and magnetic field, and using the vector identity:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \tag{4.4}$$

The curls of the electric and magnetic field strength can be expressed by the first two Maxwell equations:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{J} \tag{4.5}$$

In vacuum, without the presence of free elementary particles there can be no charge transfer, and thus no electric current, so $\mathbf{J} = 0$. Also, since vacuum is a simple medium (linear, isotropic, and homogeneous, LIH), the relations between **D** and **E**, as well as between **B** and **H**, are simple linear proportionalities governed by the magnetic permeability and dielectric permittivity constants, which are of equal value in all spatial directions. This is expressed by $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$. We can then write:

$$\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} = \mathbf{H} \cdot \frac{\partial(\mu_0 \mathbf{H})}{\partial t} = \frac{1}{2} \frac{\partial(\mu_0 \mathbf{H} \cdot \mathbf{H})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0 H^2\right) \tag{4.6}$$

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \mathbf{E} \cdot \frac{\partial(\varepsilon_0 \mathbf{E})}{\partial t} = \frac{1}{2} \frac{\partial(\varepsilon_0 \mathbf{E} \cdot \mathbf{E})}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2\right) \tag{4.7}$$

and we arrive at the Poynting vector theorem:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\mu_0 H^2\right) \tag{4.8}$$

where the vector product $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is the Poynting vector. Its physical interpretation is the instantaneous power density flow in the direction of propagation of the electromagnetic wave. Thus the energy density of the EM wave is:

$$\rho_{\mathrm{EM}} = \frac{1}{2}\left(\varepsilon_0 E^2 + \mu_0 H^2\right) \tag{4.9}$$

which implies $\varepsilon_0 E^2 = \mu_0 B^2$. From this we find that the relation between the electric and magnetic field component represents the vacuum electromagnetic impedance:

$$\mathbf{E} = \sqrt{\frac{\mu_0}{\varepsilon_0}}\,\mathbf{B} = Z\mathbf{B} \tag{4.10}$$

where $Z = \sqrt{\mu_0/\varepsilon_0} \approx 377\,\Omega$ is the free space impedance. Incidentally, by solving the Helmholtz wave equation it is possible to obtain the propagation speed of the electromagnetic wave in free space as $c = 1/\sqrt{\mu_0 \varepsilon_0}$, which is of course also the propagation speed of light. By assuming that the wave oscillates at a frequency ω, and therefore has a phase number β (the number of radians per meter of propagation):

$$\beta = \frac{\omega}{c} \tag{4.11}$$

we can write the general solution for the propagation of a homogeneous plane wave at a spatial coordinate **r**:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \mathrm{e}^{-j\boldsymbol{\beta}\cdot\mathbf{r}} \tag{4.12}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 \mathrm{e}^{-j\boldsymbol{\beta}\cdot\mathbf{r}} \tag{4.13}$$

It is of interest here to observe the interference of two waves of equal power and frequency travelling in opposite directions. The following Poynting relation applies:

$$\mathbf{S} = \frac{\mathbf{E} \times (\mathbf{H}^{+*} + \mathbf{H}^{-*})}{2} = \frac{1}{2Z}(\mathbf{E} \times \mathbf{1}_{\boldsymbol{v}} \times \mathbf{E}^{+*} - \mathbf{E} \times \mathbf{1}_{\boldsymbol{v}} \times \mathbf{E}^{-*}) \tag{4.14}$$

where the asterisk (*) marks the complex conjugate value of the **H** and **E** vectors, and $\mathbf{1}_{\boldsymbol{v}}$ is the unit vector of propagation in the forward direction, whilst $-\mathbf{1}_{\boldsymbol{v}}$ represents the same quantity in the reverse direction.

By applying the following vector identity:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \times \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$$

we arrive at:

$$\mathbf{S} = \frac{1}{2Z} \left[\left(\mathbf{E}_0^+ \mathrm{e}^{-j\boldsymbol{\beta}\cdot\mathbf{r}} + \mathbf{E}_0^- \mathrm{e}^{j\boldsymbol{\beta}\cdot\mathbf{r}} \right) \cdot \mathbf{E}_0^{+*} \mathrm{e}^{j\boldsymbol{\beta}\cdot\mathbf{r}} - \left(\mathbf{E}_0^+ \mathrm{e}^{-j\boldsymbol{\beta}\cdot\mathbf{r}} + \mathbf{E}_0^- \mathrm{e}^{j\boldsymbol{\beta}\cdot\mathbf{r}} \right) \cdot \mathbf{E}_0^{-*} \mathrm{e}^{-j\boldsymbol{\beta}\cdot\mathbf{r}} \right] \mathbf{1}_{\boldsymbol{v}}$$

$$= \frac{1}{2Z} \left[|\mathbf{E}_0^+|^2 - |\mathbf{E}_0^-|^2 + \mathbf{E}_0^- \cdot \mathbf{E}_0^{+*} \mathrm{e}^{2j\boldsymbol{\beta}\cdot\mathbf{r}} - \mathbf{E}_0^+ \cdot \mathbf{E}_0^{-*} \mathrm{e}^{-2j\boldsymbol{\beta}\cdot\mathbf{r}} \right] \mathbf{1}_{\boldsymbol{v}} \tag{4.15}$$

Since the sum of the last two terms in backets is imaginary, the real part of the Poynting vector, which is the real power density in an ideal dielectric, is:

$$\mathbf{P} = \Re\{\mathbf{S}\} = \frac{1}{2Z} \left[|\mathbf{E}_0^+|^2 - |\mathbf{E}_0^-|^2 \right] \mathbf{1}_{\boldsymbol{v}} = \mathbf{P}^+ + \mathbf{P}^- \tag{4.16}$$

This means that the flow of power in both directions is mutually independent, the waves are not influencing one another in any way, and we have a situation in which a linear superposition applies. Now in the electro–quasi–static limit (when the wave frequency $\omega \to 0$) we have the so called ‘electrostatic’ field.

It is therefore possible to calculate the energy density at the vacuum breakdown simply by taking a double value of the breakdown electric field:

$$\rho_{\mathrm{EM}} = 2\,\varepsilon_0 E_{\mathfrak{b}}^2$$

$$= 2 \cdot 8.8542{\times}10^{-12}\ [\mathrm{As/Vm}] \cdot \left(1.3247{\times}10^{18}\right)^2 [\mathrm{V/m}]^2$$

$$= 3.1075{\times}10^{25}\ \mathrm{J/m^3} \tag{4.17}$$

But there is yet another way to calculate the energy density required by the $(\mathrm{e}^-, \mathrm{e}^+)$ pair production: the minimum energy necessary for a production of a fermion–antifermion pair must be equivalent to the rest mass of the pair produced, or $W = 2mc^2$. So with the electron mass $m_{\mathrm{e}} = 9.1{\times}10^{-31}$ kg, the pair’s rest energy is $W = 1.64{\times}10^{-13}$ J (or 1.022 MeV).

Now in order to arrive at the spatial energy density, we need to know the volume affected by the pair creation and the average concentration of such pairs within a unit volume. First, the average life time of a pair can be inferred from the Heisenberg’s Uncertainty Principle, if instead of the usual momentum and the space coordinate product, $\Delta p \cdot \Delta x$, we consider the product of the energy difference with the time interval, that product being always greater than the Planck’s constant, $\Delta W \cdot \Delta t \geq h$. So the average $\mathrm{e}^-\mathrm{e}^+$ pair lifetime should be:

$$\tau \approx \frac{h}{W} \approx 4.0{\times}10^{-21}\ \mathrm{s} \tag{4.18}$$

after which time the pair annihilates into a pair of photons, thus restoring the energy to the vacuum field.

To estimate the volumetric energy we must somehow deduce how many such pairs are present within the unit of volume at every moment. The maximum possible density of particle pairs, assuming that all pairs are of the same kind, should be limited by the Pauli Exclusion Principle. By allowing the minimum distance between pairs to be of the order of their Compton wavelength:

$$\lambda_C = \frac{h}{mc} = 2.4\times10^{-12}\,\mathrm{m} \tag{4.19}$$

we can calculate the maximum possible number of e^-e^+ pairs per unit volume:

$$\rho_m = \frac{2}{\lambda_C^3} \approx 1.45\times10^{35}\,\mathrm{m}^{-3} \tag{4.20}$$

From this particle volumetric density and from the known rest energy of the e^-e^+ pair, $W = 2m_e c^2 \approx 1.6372\times10^{-13}\,\mathrm{J}$ (or $1.022\,\mathrm{MeV}$), we can calculate that the minimum necessary vacuum energy density must be:

$$\begin{aligned}\rho_e &= \rho_m W \\ &\approx 1.45\times10^{35} \cdot 1.6372\times10^{-13} \\ &\approx 2.374\times10^{22}\,\mathrm{J/m^3}\end{aligned} \tag{4.21}$$

The two results (4.17, 4.21) are 3 orders of magnitude apart. Which value is more probable, more plausible, more suitable?

Well, the second result is the minimum required, and is lower, so it would seem that the first one (4.17) should fit in well. However, it must be realized that the second result has been calculated by assuming that the vacuum is closely packed with e^-e^+ pairs (within a Compton wavelength from each other). Such a vacuum would represent an electromagnetic energy cutoff already at the relatively low Compton frequency, $\nu_C = c/\lambda_C = 1.25\times10^{20}\,\mathrm{Hz}$, equivalent to a photon energy of 511 keV.

But high-energy gamma rays (from 80 to 500 GeV) arriving from distant quasars have been measured, making both results (4.17, 4.21) far too low. This then requires a much higher vacuum energy density, producing fewer but more massive pairs.

So, the energy density necessary for the top quark pair production seems to be too high, and the energy for the e^-e^+ pair production is certainly too low. Let us find a suitable particle candidate for a more probable scenario.

5. Pion Pair as a Possible Average Product of Quantum Fluctuations of the Vacuum Energy

The mass of the top quark is $m_\mathrm{t} \approx 172\,\mathrm{GeV}/c^2$, and that the mass of an electron is $m_\mathrm{e} \approx 511\,\mathrm{keV}/c^2$. Curiously (or possibly in line with the Dirac's *Large Number Hypothesis*, [4]), at the geometric mean of those two figures we find a value which is strikingly close to a mass of a pion pair, $\pi^{\pm}$:

$$\sqrt{m_\mathrm{t} m_\mathrm{e}} = \sqrt{172{\times}10^9 \cdot 511{\times}10^3} = 296.5{\times}10^6\,\mathrm{eV}/c^2 \tag{5.1}$$

The mass of a pion, according to the Particle Data Group pages is:

$$m_\pi = 139.6\,\mathrm{MeV}/c^2 \tag{5.2}$$

What if our candidate for the average produced particle pair from quantum vacuum fluctuations is indeed the pion pair?

Hajduković (2008, 2009) [5] has published two articles in which he finds a number of strong "coincidences" (his own modest term) for exactly such a scenario. He starts from a long known Dirac's similarity:

$$m_\pi^3 \sim \frac{\hbar}{cG} H_0 \tag{5.3}$$

Here the present day value of the Hubble parameter H_0 actually varies with the age of the Universe, forcing Dirac to conclude that G should also vary accordingly. By considering some relations from modern cosmology, Hajduković then derives this equation:

$$m_\pi^3 = \frac{\hbar}{cG} H \left\{ \frac{\Omega_\mathrm{v}}{\Omega - 1} \right\} \tag{5.4}$$

in which G is a true constant, and the Ω ratio within the braces compensates the variability of the Hubble parameter H, providing also the necessary scale factor for a true equality.

The Ω term represents the total energy of the Universe, and the Ω_v term is the vacuum energy parameter, which, if constant, can be attributed to the cosmological constant as Ω_Λ, but in the general case it may vary with the age of the universe.

It is important to note that here a geometrically closed model of the Universe is assumed (in the sense of the Friedmann–Lemaître–Robertson–Walker metric), in which Ω is composed of four terms: Ω_r represents the content of relativistic particles, Ω_m represents the content of pressureless matter (dust), Ω_Λ is the cosmological constant, and the remaining part $\Omega - 1$ represents the variability of the density parameter with time. Note also that Ω is defined as the ratio of the actual energy content to the critical value for which the geometry of the Universe is still of a closed form, as required by the assumed model, thus justifying the $\Omega - 1$ expression in (5.4).

So, if relation (5.4) is more than just a coincidence, it is reasonable to assume that the pion pair is the dominant particle pair produced by vacuum quantum fluctuations. Of course, statistically, other particle pairs should also be produced, with their own number density and lifetime, but the center of mass of the volumetric average of all those processes should be very close to the pion mass.

We can therefore start from this assumption and calculate the value of the vacuum energy density in the same way as in the examples before. We take the value of the charged pion mass from the pages of the Particle Data Group, where we read: Pion mass (π^+ or π^-): 139.6 MeV/c^2, or in kg:

$$m_\pi = \frac{139.6{\times}10^6\,[\mathrm{eV}] \times 1.602{\times}10^{-19}\,[\mathrm{C}]}{(299792458\,[\mathrm{m/s}])^2} = 2.4883{\times}10^{-28}\,\mathrm{kg} \tag{5.5}$$

Thus by taking twice the pion mass, $2m_\pi$, in the same way as before, we obtain respectively the energy necessary for the production of a pion pair, the associated lifetime, and the associated Compton wavelength. These values are:

$$W_\pi = 2m_\pi c^2 = 2.097{\times}10^{-11}\,\mathrm{J} \tag{5.6}$$

$$\tau_\pi = \frac{h}{W_\pi} = 3.16{\times}10^{-23}\,\mathrm{s} \tag{5.7}$$

$$\lambda_\pi = \frac{h}{m_\pi c} = 1.89{\times}10^{-14}\,\mathrm{m} \tag{5.8}$$

Then we can find the maximum possible number of pion pairs per unit volume:

$$\rho_\mathrm{m} = \frac{2}{\lambda_\pi^3} \approx 2.95{\times}10^{41}\,\mathrm{m}^{-3} \tag{5.9}$$

From this, the required vacuum energy density would be:

$$\rho_\pi = \rho_\mathrm{m} W_\pi = 6.18{\times}10^{30}\,\mathrm{J/m^3} \tag{5.10}$$

The Compton wavelength for pions would represent an electromagnetic cutoff about 4 orders of magnitude lower than what we have observed so far in nature (if the already mentioned cosmic rays of ~500 GeV are taken as the upper limit, this would be equivalent to a wavelength of $2.5{\times}10^{-18}$ m).

However, the vacuum need not be so densely packed with particle pairs, because when a pair has been created it is highly improbable that the local energy fluctuations would remain as high anywhere near the spot for the creation of another pair. Statistically, from the energy density sufficient for the creation of N particles within a unit volume, only about $\sqrt{N}$ particles will actually be created, the rest of the vacuum energy remains in the form of photons. Only when a particular pair is annihilated again, thus restoring its energy to the environment, the conditions for another pair being created at the same location (or in close vicinity) will appear on average after at least one characteristic particle pair lifetime.

This statistics explains the necessary reduction in the volumetric particle density and allows for a considerably higher electromagnetic cutoff.

We may therefore conclude that our vacuum at the present epoch has a total energy density equal to (5.10), with the actual electromagnetic contribution of about

$\sqrt{\rho_\pi}$, whilst the remaining part is in form of particle pairs, $\pi^-\pi^+$ on average, but ranging from e^-e^+ pairs up to $t\bar{t}$ quark pairs.

6. Vacuum Energy Density and Particle Mass

In order to broaden the perspective of the interconnection between the vacuum energy density and particle mass, the formalism of Stochastic Electrodynamics is invoked. In contrast with the Higgs field, by which the properties of mass are being explained within the Standard Model, Stochastic Electrodynamics does not require any additional exotic or yet to be discovered particles.

In 1994, Haisch, Rueda, and Puthoff [6] published an article in which they derived the inertial property of an accelerated particle from the interaction of its own charge field in the accelerated frame and the vacuum energy. The Newtonian equation $F = ma$, and of course its relativistic version, $\mathcal{F} = d\mathcal{P}/d\tau$, are shown to be connected with the long known spectral distortion of the vacuum energy observed from a rectilinear accelerated reference frame (the Davies–Unruh effect [7]).

However, the authors also show that there is another so far neglected effect of the Lorentz electromagnetic force (actually its magnetic component), which is felt in the direction opposite to the accelerating force, thus exhibiting a property identical to an ordinary inertial resistance to acceleration.

The arguments by which the authors support their idea are briefly summarized as follows.

Based on the equation (2.4) representing the spectral density of the vacuum energy, written in a different way, accounting for $\hbar = h/2\pi$ and $\omega = 2\pi\nu$:

$$\rho_W(\omega)d\omega = \frac{\hbar\omega^3}{2\pi^2 c^3}d\omega \tag{6.1}$$

it is possible to prove that the spectrum is covariant in the Lorentz sense (because of its dependence on the third power of frequency, as required). Therefore any movement through space with a constant speed does not have any measurable consequences, and in turn it is impossible to detect the presence and the exact amount of the vacuum energy.

On the other hand, the movement with a constant acceleration a puts in evidence the Davies–Unruh effect (also derived by Boyer using the SED formalism), which is manifested as a pseudo-Planckian spectrum with an equivalent temperature:

$$T = a\frac{\hbar}{2\pi c k_\mathrm{B}} \tag{6.2}$$

In the accelerated reference frame the vacuum energy density has a slightly different form, which is acceleration dependent:

$$\rho_W(\omega)d\omega = \left(\frac{\omega^2}{\pi^2 c^3}\right)\left(1 + \frac{a^2}{\omega^2 c^2}\right)\left[\frac{\hbar\omega}{\mathrm{e}^{2\pi c\omega/a} - 1} + \frac{\hbar\omega}{2}\right]d\omega \tag{6.3}$$

Haisch, Rueda, and Puthoff find out that such a modified vacuum energy density spectrum leads to a new result. By analysing the reaction force **F** exerted by the vacuum energy on any accelerated particle, it appears that this force must be proportional in size, and in the opposite direction of the acceleration vector **a**.

This means that the vacuum energy is resisting acceleration, and this resistance is a function of the damping radiation reaction constant Γ (determining the interaction of the particle with the radiation field), and the acceleration $\mathbf{a}$. Such a behaviour can be interpreted as if the particle under acceleration exhibits the property of inertial mass. This is equivalent to the statement that the Newton's law of motion $\mathbf{F} = m\mathbf{a}$ can be derived directly from classical electrodynamics, assuming only the existence of the vacuum energy, in the sense that the electrodynamic dependence $\mathbf{F}(\mathbf{a})$ predicts the existence of the perticle's inertial mass of the form:

$$m_{\mathrm{i}} = \frac{\hbar\omega_{\mathrm{P}}}{c^2}\left(\Gamma\omega_{\mathrm{P}}\right) \tag{6.4}$$

Here Γ is the Abraham-Lorentz damping constant of an oscillating particle:

$$\Gamma = \frac{2q_{\mathrm{e}}^2}{3m_0c^2} \tag{6.5}$$

and ω_{P} is the Planck's frequency, whilst m_0 is the rest (non-relativistic) mass of the particle. This rest mass has been derived in the next article by Rueda and Haisch [8] in the following form:

$$m_0 = \frac{V_0}{c^2}\int \eta(\omega)\frac{\hbar\omega^3}{2\pi^2c^3}d\omega \tag{6.6}$$

The explanation is simple: the expression $\hbar\omega^3 d\omega/2\pi^2c^3 = \rho_W d\omega$ is the volumetric spectral energy density (6.1), whilst the dimensionless parameter $\eta(\omega)$ represents the frequency dependent part of the scattering of the energy flux (the gauge factor).

Thus the presence of a particle with a volume V_0 expels from the vacuum energy within this volume exactly the same amount of energy as is the particle's internal energy (the energy equivalent of the rest mass).

As an example, for an electron we can assume $V_0 \approx \lambda_{\mathrm{C}}^3$, a volume determined by Compton's wavelength, as in (4.19).

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