Part 2, Particle Physics

2.1 Introduction

To start the discussion on particle physics we need to first define what are the elementary particles - basic building blocks - in the nature. A reasonable definition would be to define those as the uncomposed particles. This definition, however, depends on the experimental methods available in each period of time. In the introduction to Part 1 we mentioned the idea of ancient Greeks that all matter in the nature is composed of earth, water, fire and air. In the absence of experimental methods and based on (some) observation of the nature and philosophical ideas these elements were believed to be the basic building blocks of nature.

Jumping to the present time clearly the experimental methods available are much more sophisticated. Nevertheless we must be aware of their limitations. Optical microscopes, for example, are able to distinguish details in the structure of matter the dimensions of which are at least of the order of the wavelength of the visible light ($\lambda \sim 700\ nm = 7 \cdot 10^{-7}\ m$). The light does not scatter or reflect significantly on the structure with dimensions less than this. One can overcome the limitation by using the electron microscope. In the electron microscope instead of a visible light an accelerated beam of electrons is used. Quantum mechanically such a beam can be described as a wave with the de Broglie wavelength

$\lambda = \dfrac{2\pi\hbar c}{cp}$. For electrons accelerated to kinetic energies comparable or larger than their

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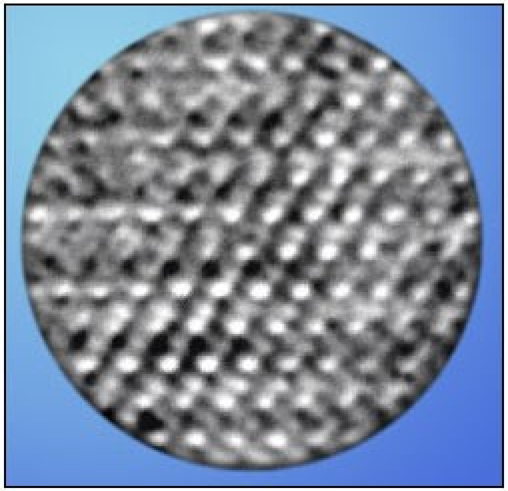
rest energy mc^2, a relativistic energy-momentum relation must be used,

$$E = T + mc^2 = \sqrt{(cp)^2 + (mc^2)^2}$$

, where T represents the kinetic and E the total energy of the particle. Electrons with $T=100\ keV$ have $cp = 300\ keV$ and hence the wavelength of $4 \cdot 10^{-12}$ m. Clearly with such a “light” much finer details in the structure of the matter can be observed than with the optical microscope. Indeed the electron microscopes enable visualization of single atoms.



Picture of lithium cobalt oxide taken by the electron transmission microscope (from www.photonics.com).

Going few steps further, one can think of today’s particle accelerators as microscopes, accelerating particles to very high energies and thus small wavelengths, to provide an insight into the smallest details of matter as observed nowadays. Modern particle accelerators provide particles with energies of the order 100 GeV. This translates into the wavelengths of 10^{-18} m which determines the size of objects for which we nowadays believe are the contemporary uncomposed particles (see also Part 1, p. 4).

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Table of elementary particles as observed nowadays:

charged leptons	e^-, electron	μ^-, muon	τ^- tau lepton (tauon)	
neutral leptons	ν_e, electron neutrino	ν_μ, muon neutrino	ν_τ tau neutrino	
quarks	u, up	c, charm	t top	
	d, down	s, strange	b bottom	
carriers of interactions	γ, photon	W^+, charged weak boson	Z^0, neutral weak boson	g gluon

All leptons and quarks are fermions (particles with half integer spin). Their spin is ½. All interaction carriers are bosons (particles with integer spin). Their spin is 1.

Beside the particles listed each particle has also its anti-particle. Anti-particle has similar properties as the particle (same mass), but opposite quantum numbers (like electric charge).

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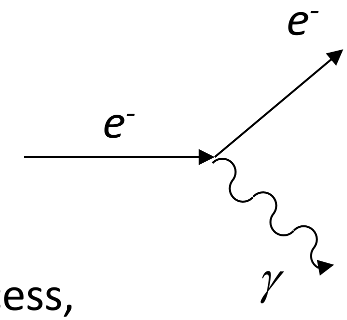
The anti-particle of an electron is a positron (e^+) with a positive basic charge.

All particles that feel the strong nuclear force are called hadrons. They are all composed of quarks. Contrary to leptons, which are all fermions, hadrons are fermions and bosons. Almost all hadrons observed so far are (in the simplest model) composed of three quarks - baryons - or a quark and an anti-quark - mesons. Baryons are fermions (well known examples are protons and neutrons) while mesons are bosons (examples are pions, composed of u and d quarks and anti-quarks).

2.2 Electromagnetic Interaction and Photons, Coupling Constants
2.2.1 EM Interaction and photons

Plot on the right represents a Feynman diagram of an electron radiating a photon. A Feynman diagram is a pictorial representation of a given process. It helps in calculation of an amplitude for the process under consideration by relating factors appearing in the amplitude to specific parts of the process, like lines of individual particles, intersections of several particle lines (called vertices), etc.

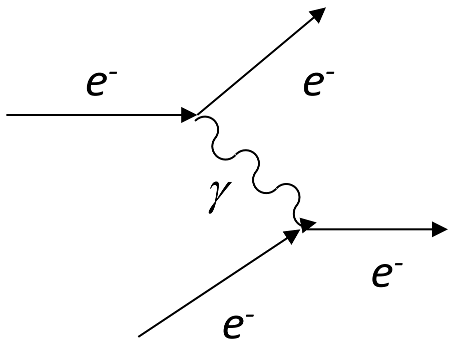


There is a problem with the process depicted in the figure. Energy-momentum is not conserved in this particular process. However, this doesn't mean that such a process cannot proceed at least as a part of a more general process. One should not forget the Heisenberg uncertainty principle, which in one of the forms reads

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Homework 1: prove that in the process $e^- \rightarrow e^- \gamma$ it is impossible to conserve energy and momentum.

For our particular example this means that the photon can exist for a short time (Δt) in which the energy may not be conserved. After this short period of time the photon is absorbed by another particle – for example another electron.



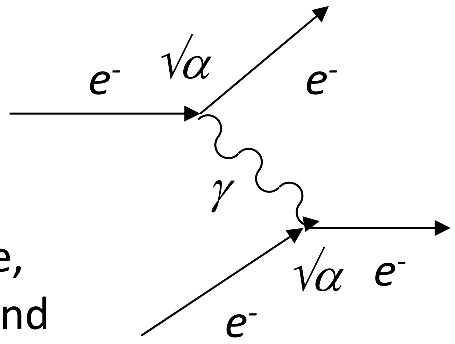
By this we get the Feynman diagram representing a different process which now all together does conserve energy and momentum. The process is the EM scattering of two electrons. The sum of energies and momenta of initial state electrons equals the sum of energies and momenta of final state electrons. The intermediate photon does not conserve energy and momentum and lives for a very short period of time, in accordance with the Heisenberg uncertainty principle. Such a photon is called a virtual photon. The EM interaction between the two electrons is mediated by the exchange of the photon.

The matrix element for EM scattering of two electrons will be proportional to $e^2/4\pi\varepsilon_0$ (the Coulomb potential between two particles with an elementary charge e),

$$\left|V_{fi}\right| \propto \left|\frac{e^2}{4\pi\varepsilon_0}\right| = \left|\alpha\hbar c\right|$$

where α is the fine structure constant.

This factor entering the matrix element can now be assigned to vertices of electrons and photon in the Feynman diagram (there is no reason to prefer either of the $e^-\gamma$ vertices, and hence $\sqrt{\alpha}$ is assigned to each of them). Each $e^-\gamma$ interaction (vertex) contributes $\sqrt{\alpha}$ to the amplitude (matrix element) for the process; probability for the process per unit of time (Fermi golden rule, see part 1, p. ??) is proportional to matrix element squared and hence to α^2.



The dimensionless factor determining the probability of a process which is a consequence of a specific interaction is called the coupling constant of the interaction. For the EM interaction the coupling constant is α.

In the above description we started from the description of the EM scattering through the Coulomb potential. How can one quantitative describe the same process through the exchange of a photon?

The relativistic relation between energy and momentum reads

$$E^2 = c^2 p^2 + m^2 c^4$$

Replacing the observables by operators in quantum mechanics leads to

$$\hat{E}^2 = c^2 \hat{p}^2 + \hat{m}^2 c^4$$

The mass operator $\hat{m}$ is just multiplication by m. On the other hand the energy and momentum operators are not trivial. The easiest way to check the form of those is to consider a plane wave $\psi = \frac{1}{\sqrt{V}} e^{-ikx}$

Here, k and x are the four vectors $x = (ct, \vec{x}), k = \frac{cp}{\hbar c} = \frac{1}{\hbar c}(E, c\vec{p})$.

The product of the two is $kx = ct\frac{E}{\hbar c} - \vec{x}\frac{c\vec{p}}{\hbar c} = \frac{1}{\hbar}(Et - \vec{p}\vec{x})$.

Hence $\psi = \frac{1}{\sqrt{V}} e^{-\frac{i}{\hbar}(Et - \vec{p}\vec{x})}$.

If we operate on ψ with the operator of the form $i\hbar\frac{\partial}{\partial t}$ we get

$i\hbar\frac{\partial}{\partial t}\psi = i\hbar(-\frac{i}{\hbar}E)\psi = E\psi$, and hence $\hat{E} = i\hbar\frac{\partial}{\partial t}$.

Similarly, using the operator $-i\hbar\vec{\nabla}$ we get $-i\hbar\vec{\nabla}\psi = -i\hbar(\frac{i}{\hbar}\vec{p})\psi = \vec{p}\psi$

from which it's obvious that $\hat{\vec{p}} = -i\hbar\vec{\nabla}$. By inserting operators $\hat{E}$ and $\hat{\vec{p}}$ into the operator relation on the previous page we arrive at

$$\hat{E}\hat{E}\psi = c^2\hat{\vec{p}}\hat{\vec{p}}\psi + m^2c^4\psi$$

$$(i\hbar\frac{\partial}{\partial t})(i\hbar\frac{\partial}{\partial t})\psi = c^2(-i\hbar\vec{\nabla})(-i\hbar\vec{\nabla})\psi + m^2c^4\psi$$

$$-\hbar^2\frac{\partial^2}{\partial t^2}\psi = -c^2\hbar^2\nabla^2\psi + m^2c^4\psi$$

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi = 0$$

The derived equation is called the Klein-Gordon equation. It is an analogy of the Schrödinger equation in the sense that it represents a quantum mechanical description of a system with the wave function ψ, but with the distinction that while the Schrödinger equation describes non-relativistic systems the Klein-Gordon equation describes relativistic particles (since it was derived from the relativistic energy-momentum relation).

If for the moment we neglect the mass term (i.e. *m=0*) the Klein-Gordon eq. reduces to

$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$, which is just the wave equation describing wave (e.g. electromagnetic wave) propagation. This is an example of the so called wave-particle (wave-corpuscular) duality; the equation describes a relativistic particle of energy *E* and zero mass (photon) or a propagation of an EM wave.

In case of a stationary (time independent) field the solution of the equation

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = 0$$

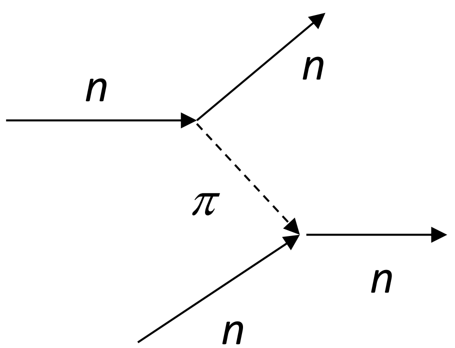
is $U = \frac{g}{r}$. The constant *g*, multiplied by an appriopriate factor to be dimensionless, is the coupling constant of the interaction. For $g=e/4\pi\varepsilon_0$, *U* is just the electrostatic potential of a point like charge *e*. The particle described by the Klein-Gordon equation (photon in this particular case) represents the EM potential of the particle which emitted it (electron).

The idea can be evolved further by considering massive particles. Considering the mass term the solution of the (time independent) Klein-Gordon eq. is

$$U = \frac{g}{r} e^{-r/R} ,$$

with $R=\hbar/mc$. This can be interpreted in a similar manner as the massless photon being the carrier of the electromagnetic interaction. A massive particle carries an interaction with a finite reach, the latter being determined by *R*.

This lead Hideki Yukawa in 1935 to propose the idea of the strong interaction (which holds nucleons bound inside the nuclei) being mediated by a particle he called a meson (the name follows from Greek *mesos* meaning middle, intermediate; it relates to the mass of such a particle). The Feynman diagram of the strongly interacting particles could thus look like



π denotes the meson mediating the interaction, and *n* is any nucleon. Like in the case of EM interaction the meson can only live for the time interval Δt in accordance with the Heisenberg uncertainty principle:

$$\Delta E \Delta t \geq \frac{\hbar}{2}; \quad \Delta E \sim m_\pi c^2 \sim \frac{\hbar c}{2\Delta t c}$$

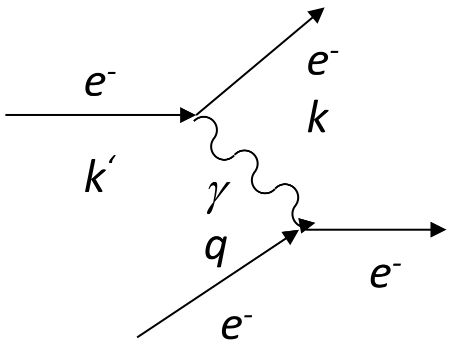
Inserting for Δtc a typical nuclear distance (few fm) one arrives at the order of magnitude estimate for the mass of the meson

$$m_\pi c^2 \sim \frac{\hbar c}{2\Delta tc} \sim O(100\,MeV)$$

The mesons called pions (π) are nowadays of course well known, their mass being ~139 MeV/c^2. We also know today that the strong interaction among quarks inside the nucleons is mediated by particles called gluons. However, at the energies achievable in the first half of the 20th century the description of the effective interaction among the nucleons as being mediated by pions was successful and, moreover, represented an important breakthrough in quantum mechanical interpretation of individual interactions.

Hideki Yukawa was the first Japanese to receive the Nobel prize, in 1949 (following the experimental discovery of pions in 1947)

If the incoming and outgoing particles are described as plane waves the cross section of a specific process (it may be helpful to think about $e^- e^-$ EM scattering, for example) is proportional to $|f|^2$ where



$$f = \int U(r) e^{iqr} d^3 r \ ,$$

q is the wave vector of the exchanged particle ($q=k-k'$) (verify this with the expression for calculation of the matrix element for Coulomb scattering of a projectile on a charge distribution, Part 1, p. ??).

If for $U(r)$ we now use the solution of the Klein-Gordon eq. with $m \neq 0$:

$$f = \int \frac{g}{r} e^{r(-1/R+iq)} r^2 dr \propto -\frac{g}{(-1/R+iq)^2} \Rightarrow |f|^2 \propto \frac{g^2}{(m^2c^4 + c^2p^2)^2} \ .$$

In the last step we used $R=\hbar/mc$ and $q=p/\hbar$ with p the momentum of the exchanged particle (sometimes also called the momentum transfer).

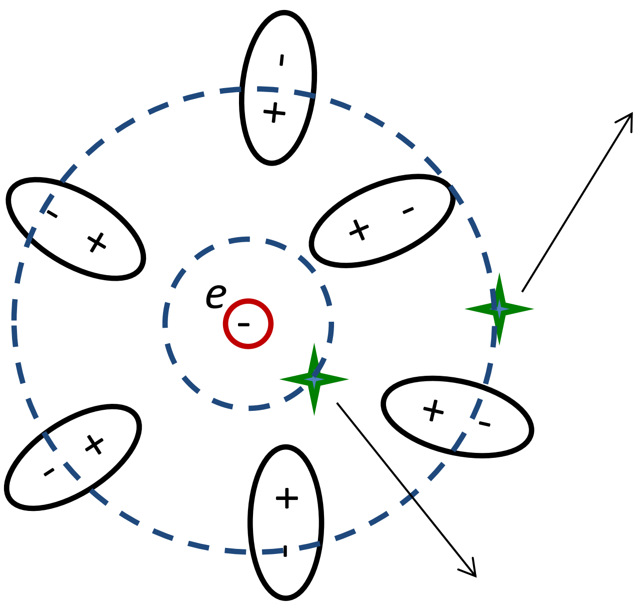
The result tells us that for the EM scattering, the cross section is $\sigma \propto \frac{g_{EM}^2}{(cp)^4} = \frac{\alpha^2}{(cp)^4}$,

in accordance with eq. Part 1, ??: $$\frac{d\sigma}{d\Omega} = \left[\frac{me}{8\pi\varepsilon_0 p^2}\right]^2 \frac{1}{\sin^4\frac{\vartheta}{2}} |F(\vec{q})|^2$$

In case of a massive particle mediating the interaction, and if $mc^2 >> cp$, one should be aware that the measured cross section reflects not the „bare“ coupling constant g of the corresponding interaction, but rather $g^2/(m^2c^4 + c^2p^2)^2 \sim g^2/m^4c^8$. This becomes evident especially in the case of weak interaction, as explained below.

2.2.2 Charge Screening and Vacuum Polarization

Any electric charge in media is the source of polarization of the latter:

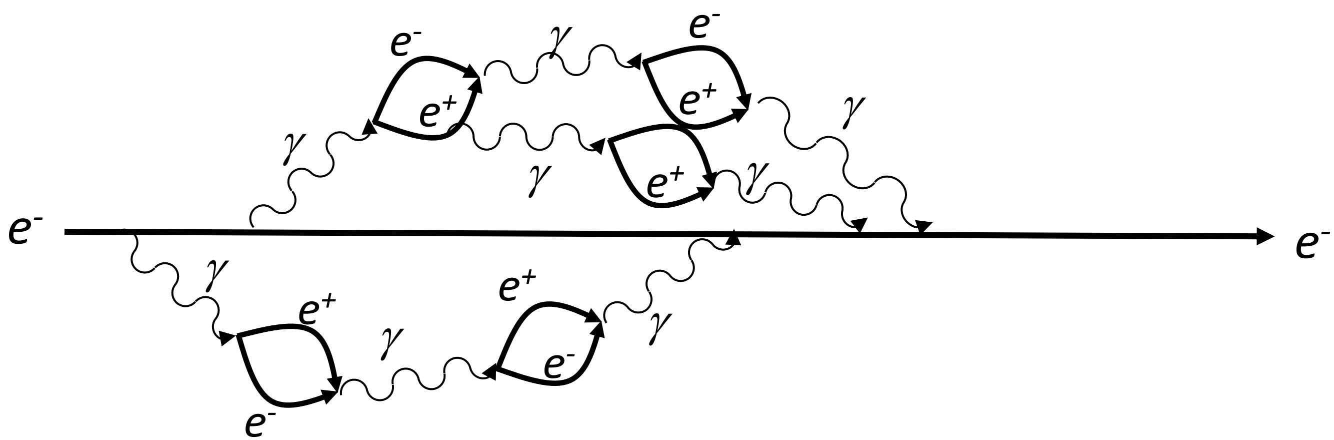


at this point in media one observes the screened charge e

at this point in media one observes the bare charge e

The size of the observed charge thus depends on how close to the charge the probe reaches, for example – if one probes the charge through EM interaction – how close to the charge the projectile can penetrate. The size of the charge depends on the energy of the projectile. The phenomena is called the charge screening.

A similar thing happens also in vacuum. The EM interaction is described by radiation of γ's, which in turn can yield new e^-e^+ pairs. An electron, travelling through the vacuum, could thus be represented as

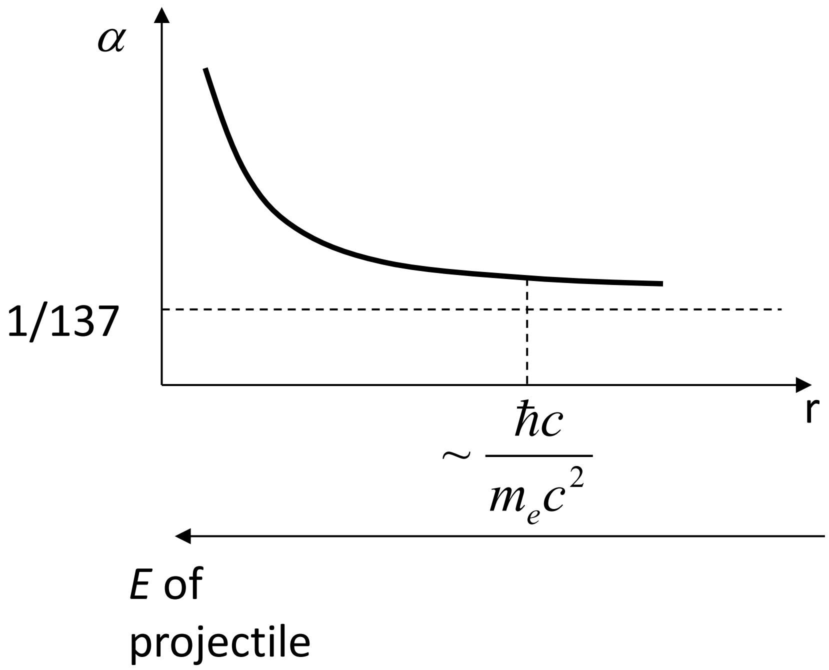


Similarly as in some media positrons in e^-e^+ pairs tend to be closer to the original e^- than electrons. Such a cloud of photons and e^-e^+ pairs is of course subject to the Heisenberg uncertainty principle and extends on the average

$$c\Delta t \sim \frac{\hbar c}{\Delta E} \sim \frac{\hbar c}{m_e c^2} \sim 10^3 \ fm$$

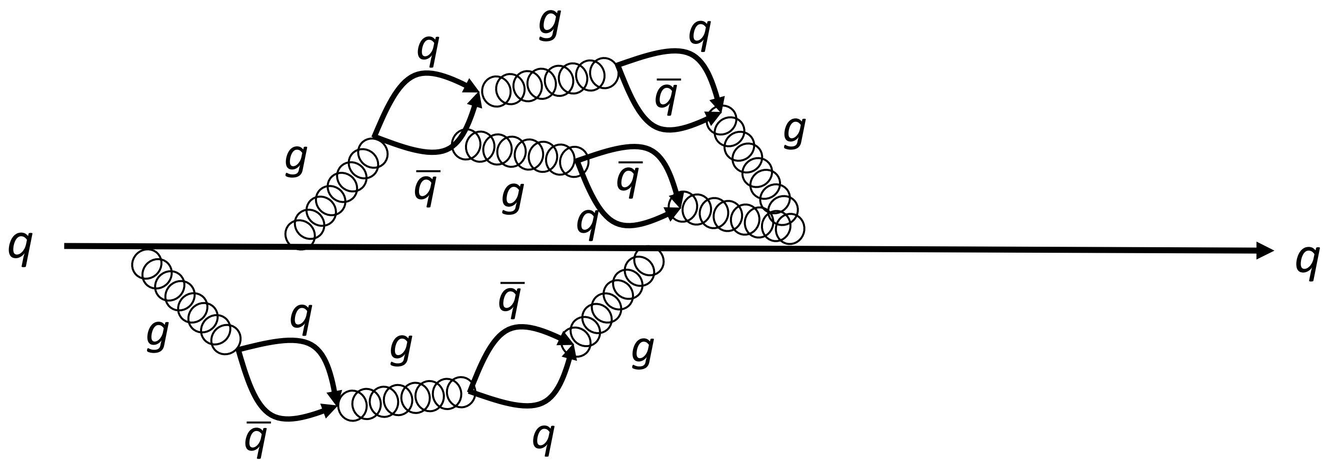
away from the original electron. If one observes the electron charge at distances $\geq 10^3$ fm the measured value would be the „usual" electron charge, $-e_0=-1.6\text{x}10^{-19}$ As. Closer to the charge its value is larger. This phenomena is called the vacuum polarization.

The vacuum polarization causes the elementary charge and by this also the EM coupling constant α to be energy dependent (in terms of the energy of a projectile used to probe the charge or in other words in terms of the energy at which an EM scattering takes place).

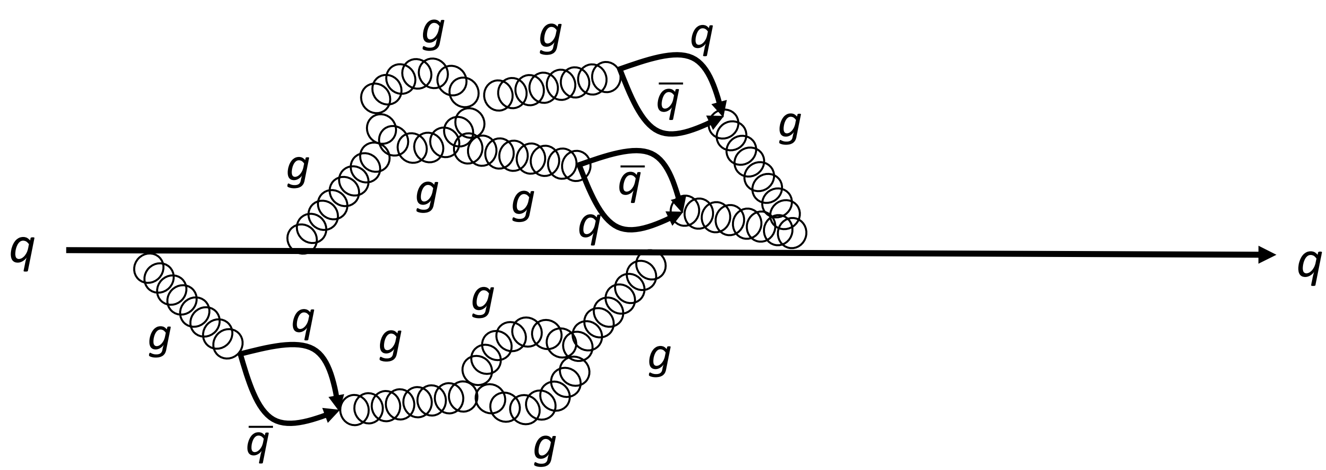


A similar vacuum polarization also takes place in weak and strong interaction. In these cases the coupling constants of these interactions depend on the energy (because of the „screening“ of appropriate „charges“ rersponsible for the two interactions – analogies of the electric charge in case of EM interaction). However, in the vacuum polarization related to the strong and weak interaction there is an important difference with respect to the EM interaction: while photon itself does not carry an electric charge, gluons and weak bosons do carry the corresponding strong and weak charges. The consequence is that the picture drawn for en electron travelling through the vacuum is sligthly different in the case, for example, of a quark.

A complete analogy of the EM vacuum polarization for the case of strongly interacting quark is:



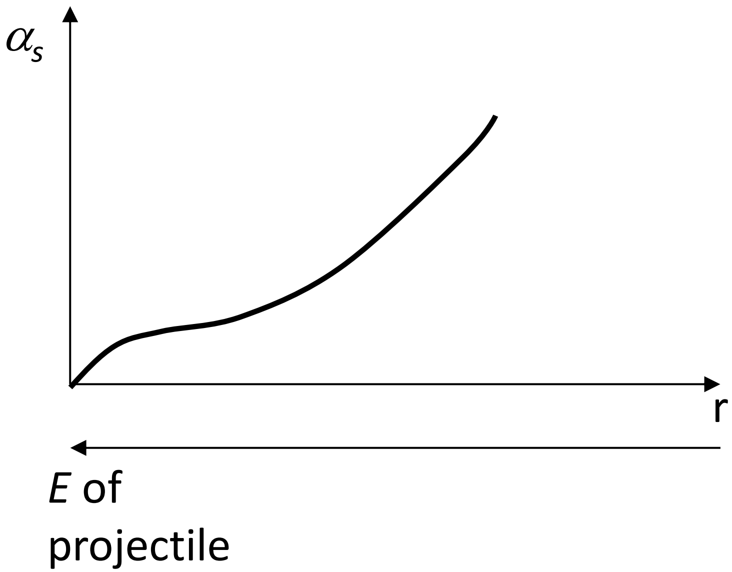
where *q* represents quarks and *g* gluons.
But due to the fact that gluons (*g*) itself carry the strong charge, also gluon loops are possible:



Despite the fact that graphically this doesn‘t seem to be a large difference it carries far reaching consequences.

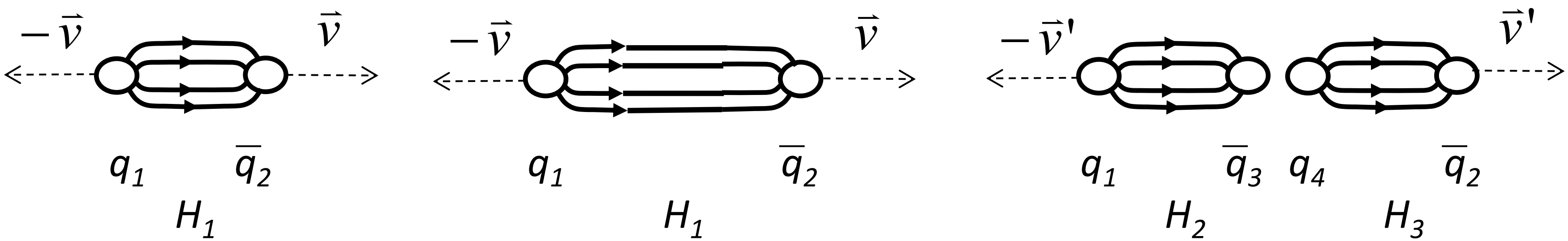
In 1973 D.J. Gross, H.D. Politzer and F. Wilczek have shown that a consequence of the posibility shown in the last figure (gluon-gluon interaction) is an „anti-screening“; the coupling constant of the strong interaction increases with the distance (decreases with energy), rather than decreases as in the case of EM interaction. This fact is called asymptotic freedom since quarks at high enough energies behave as free particles (this is not to be confused by possible observation of free quarks; the latter does not happen since quarks are always bound inside hadrons).

The three above mentioned physicists received Nobel prize for their discovery in 2004.



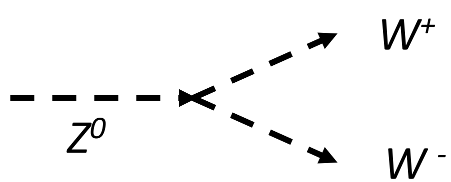
Increase of the strong interaction coupling as the energy decreases is the source of an important problem in particle physics: at low enough energies (typically at energies involved in processes among quarks bound inside hadrons) α_s becomes too large in order to use a common approach of calculating variables using the perturbation theory (based on Taylor series in coupling constant). Hence other approaches must be used leading to significant uncertainties in calculations of processes of strong interaction at low energies.

Increase of α_s at large distance also means that the quarks always remain bound inside hadrons (as the distance between two bound quarks increases also the strong potential increases). At large enough distance it becomes energetically favourable to produce a new quark – antiquark pair instead of enlarging the distance further. Schematically:



In the above sketch arrows denote the strong field lines, q_i are quarks and H_i hadrons. This results in another property of strong interaction: despite the fact that gluons are massless (and hence one would, in accordance with p. ?? expect an infinite range of the interaction) the interaction has a finite range.

Coupling constant of the weak interaction qualitatively depends on the energy in the same manner as the strong coupling constant. Also for the weak interaction interactions of the type



are possible, also leading to the decrease of the coupling constant of weak interaction, α_w, with energy.

Coupling constants of various interactions are thus not really constant but depend on the energy at which the process takes place.

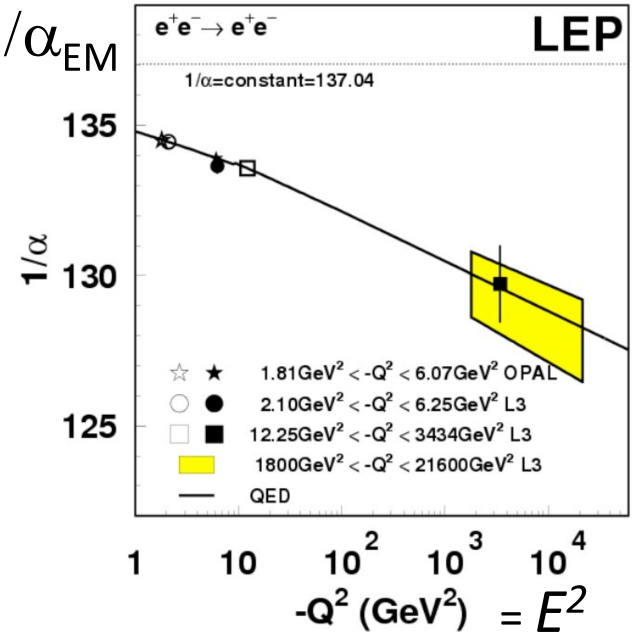
At energies E~O(100 GeV) the value of α_{EM} ~ 1/128 (note that at E~O(1 MeV) α_{EM} ~ 1/137). The range of the interaction is infinite (photon is massless).

At this energy $\alpha_s \sim 20\ \alpha_{EM}$. Hence indeed one can say that the strong interaction is „stronger" than the electromagnetic one. The range of the interaction is limited despite the fact that gluons are massless because of the reasons explained on the previous page.

On the other hand $\alpha_w \sim \alpha_{EM}$, and thus bare coupling constant of weak interaction is not smaller than the electromagentic one. However, bearring in mind that the probability of weak interaction processes $\propto \alpha_w^2/m^4$(see p. ??) the weak interaction appears „weak" because of the high mass of weak bosons (~ 80 GeV/c^2). The range of the interaction is of the order of $c\Delta t \sim 2\pi \hbar c/m_W c^2 \sim 0.01$ fm.

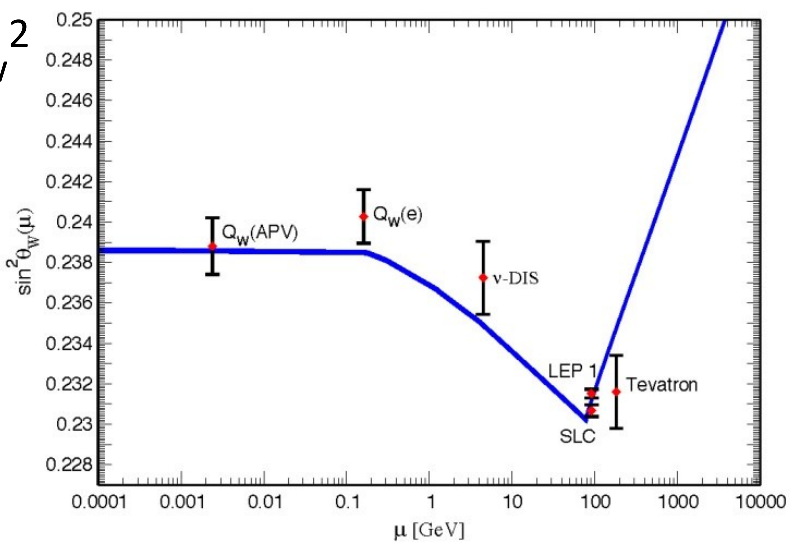
Energy dependence of coupling constants as measured by various experiments:

$1/\alpha_{EM}$



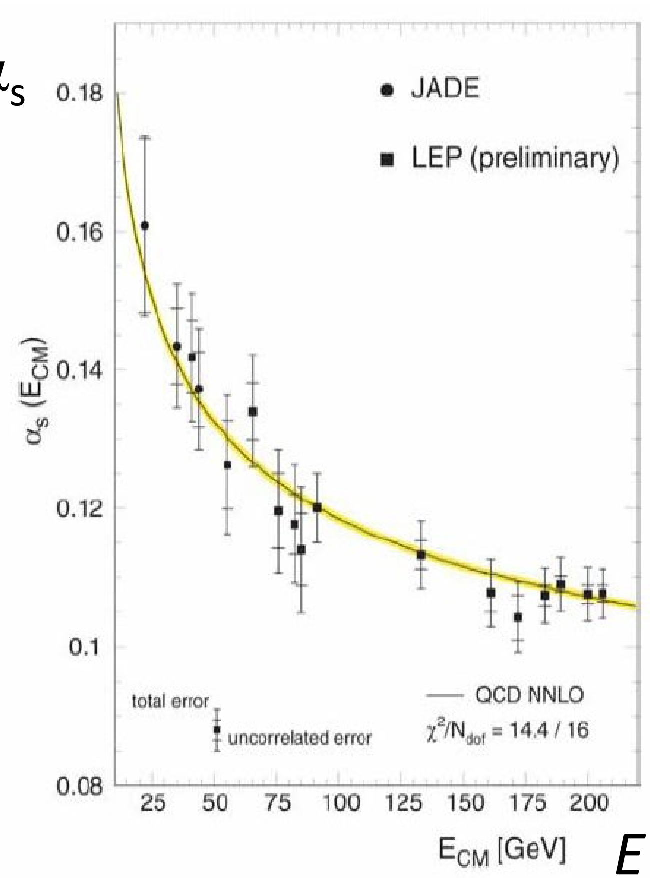
$= E^2$

$\propto 1/\alpha_w^{\,2}$



E

α_s



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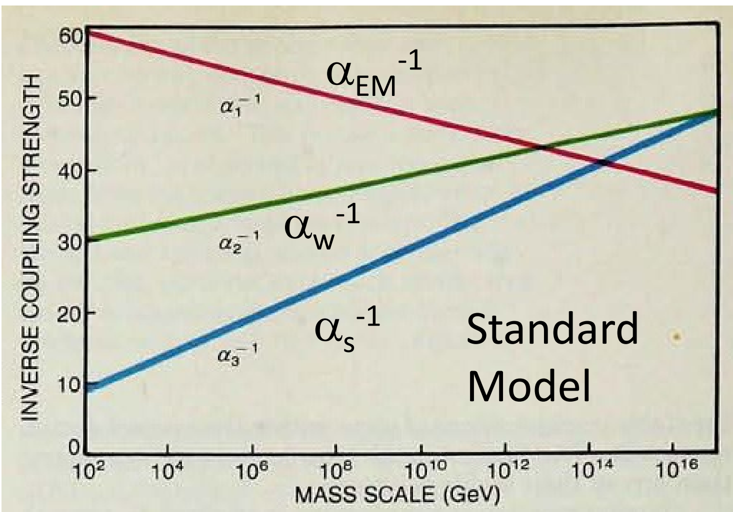
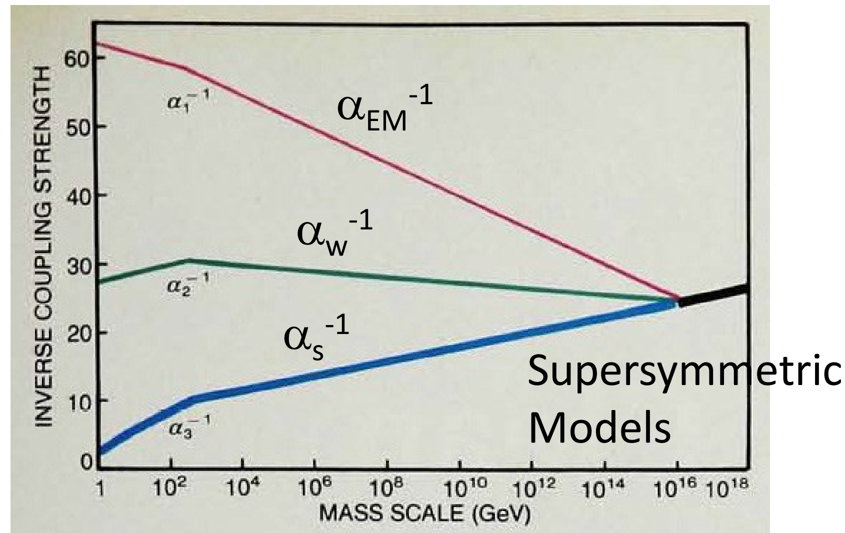
Based on the energy dependence of the coupling constants it is not difficult to understand the source of the idea about the „unification of interactions“. The idea states that all interactions observed at the processes observed so far (i.e. at presently available energies) are just a low-energy manifestations of a single interaction. Computation of the energy eviolution of the coupling constants within the Standard Model of interactions predicts:





It is thus easy to imagine that perhaps at some higher energy scale all the coupling constants become equal. Detailed calculations, however, show that this is not exactly true (as seen in the above figure left). Extensions of the Standard Model theory, specifically the so called Supersymmetric models, predict further (yet unobserved) elementary particles. Detailed calculations of the energy evolution in such models yield the right figure above, where all the coupling constants do reach exactly the same value at a certain energy. This represents one of the strongest motivations for Supersymmetric models.

2.3. Symmetries and Conservation Laws

2.3.1 Constant observables

Consider a state (wave function) described at an initial time *t*=0 by $\left|\psi(t=0)\right\rangle$.

Time evolution of the system is governed by the time dependent Schrödinger equation:

$i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \hat{H}\left|\psi(t)\right\rangle$, where $\hat{H}$ is the Hamilton operator. Formally, the solution can be written as $\left|\psi(t)\right\rangle = e^{-i\hat{H}t/\hbar}\left|\psi(t=0)\right\rangle$, or,

$$\left|\psi(t)\right\rangle = \hat{U}(t)\left|\psi(t=0)\right\rangle, \quad \hat{U}(t) = e^{-i\hat{H}t/\hbar}$$

. We can write the expectation value of an observable *x*, which is conserved (i.e. it is time independent):

$$\left\langle\psi(t)\right|x\left|\psi(t)\right\rangle = \left\langle\psi(t=0)\right|\hat{U}^+ x\hat{U}\left|\psi(t=0)\right\rangle = \left\langle\psi(t=0)\right|x_0\left|\psi(t=0)\right\rangle,$$

where in the last step we took into account the fact that *x* is constant and denoted it's value by x_0. Hence $\hat{U}^+ x\hat{U} = x_0$, and since $\hat{U}$ is a unitary operator $(\hat{U}^+\hat{U} = I)$ we get

$x = \hat{U}x_0\hat{U}^+$. Derivation over *t* yields

$$\frac{\partial x}{\partial t} = \frac{\partial \hat{U}}{\partial t}x_0\hat{U}^+ + \hat{U}x_0\frac{\partial \hat{U}^+}{\partial t}$$

Since $\frac{\partial \hat{U}}{\partial t} = -\frac{i\hat{H}}{\hbar}\hat{U}$ and $\frac{\partial \hat{U}^+}{\partial t} = \frac{i\hat{H}}{\hbar}\hat{U}^+$ we arrive at

$$\frac{\partial x}{\partial t} = -i\frac{\hat{H}}{\hbar}\underbrace{\hat{U}x_0\hat{U}^+}_{x} + \hat{U}x_0 \underset{\uparrow}{} i\frac{\hat{H}}{\hbar}\hat{U}^+ = 0$$ (which has to be zero since *x* is a constant).

$\hat{U}^+\hat{U}$

at this place
we insert identity

The second term thus reads $x\hat{U}i\frac{\hat{H}}{\hbar}\hat{U}^+ = 0$.

$$\hat{U}\hat{H}\hat{U}^+ = e^{-i\hat{H}t/\hbar}\hat{H}e^{i\hat{H}t/\hbar} =$$

$$(1 - \frac{it}{\hbar}\hat{H} + \frac{(it\hat{H})^2}{\hbar^2 2!} - \frac{(it\hat{H})^3}{\hbar^3 3!} + \ldots)\hat{H}(1 + \frac{it}{\hbar}\hat{H} + \frac{(it\hat{H})^2}{\hbar^2 2!} + \frac{(it\hat{H})^3}{\hbar^3 3!} + \ldots) =$$

$$\hat{H} + \frac{it}{\hbar}\hat{H}^2 + \frac{(it)^2\hat{H}^3}{\hbar^2 2!} + \ldots - \frac{it}{\hbar}\hat{H}^2 - \frac{(it)^2\hat{H}^3}{\hbar^2}\ldots + \frac{(it)^2\hat{H}^3}{\hbar^2 2!} + \ldots = \hat{H}$$

Hence $$\frac{\partial x}{\partial t} = -i\frac{\hat{H}}{\hbar}x + xi\frac{\hat{H}}{\hbar} = -\frac{i}{\hbar}\left[\hat{H}, x\right] = 0$$

The last equation tells us that in the case that an operator (x) commutes with the Hamiltonian (H) then the expectation value of this operator is constant.

Example: the operator of the third component of angular momentum commutes with the Hamiltonian. Hence the third component of the angular momentum is conserved.

We can do one step further in exploring the relation between the Hamiltonian and the conservation of specific observables. The third component of the angular momentum actually represents the operator of the rotation around the z-axis.

Homework 2: prove that the operator of rotation around the z-axis can be written in terms of the z-component angular momentum oprator.

Hence any system which is rotationally symmetric around the z-axis (i.e. it‘s Hamiltonian is invariant to the rotations around the z-axis) will preserve the third component of the angular momentum.

The rather familiar example of the angular momentum and rotations is actually just a specific example of a more general law: any symmetry of Hamiltonian reflects in a conservation law (i.e. in conservation of some observable).

We will meet operators performing rotations in other than the usual 3-dimensional space (for example in the space of spin) and see that their expectation values are conserved.

2.3.2 Baryon and Lepton Number Conservation

Let us define a new quantum number, the baryon number. All baryons (composed of three quarks) carry the baryon quantum number B = +1, all anti-baryons (composed of three anti-quarks) carry the baryon number B = -1. All other hadrons and leptons have B = 0.
For example

particle	symbol	quark composition	B
proton	p	uud	+1
neutron	n	udd	+1
lambda	Λ	uds	+1
anti-proton	$\overline{p}$	$\overline{u}\overline{u}\overline{d}$	-1
anti-lambda	$\overline{\Lambda}$	$\overline{u}\overline{d}\overline{s}$	-1
pion	π^+	$u\overline{d}$	0

All interactions conserve the baryon number. Some examples of processes:

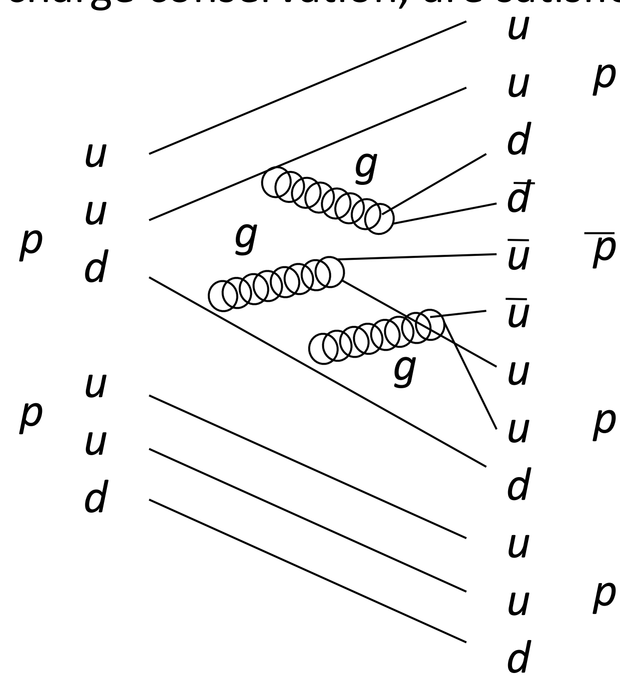
$$p \quad p \rightarrow p \quad p \quad p \quad \overline{p}$$
$$B: +1 \; +1 \qquad +1 \; +1 \; +1 \; -1$$

allowed process (if all other conservation laws, for example energy conservation, are satisfied)

$$p \quad p \rightarrow p \quad \overline{p} \quad \pi^+ \quad \pi^+$$
$$B: +1 \; +1 \qquad +1 \; -1 \quad 0 \quad 0$$

forbidden process (despite the fact that other conservation laws, for example charge conservation, are satisfied)

If we consider baryons as being composed of three quarks (a picture which proves to be too naive in some cases, but for our purpose works well) the conservation of the baryon number is in principle just the conservation of the number of quarks, or in other words, it is only possible to produce the same number of quarks and anti-quarks. This can also be seen from the sketch on the right for the first process listed above.



In a similar manner as the baryon number we define also the lepton number *L*. All leptons have +1, their anti-particles have *L* = -1, and all the hadrons have *L* = 0. All interactions conserve the lepton number. Some examples of processes:

	e^+	e^-	$\rightarrow$	τ^+	τ^-	
L:	-1	+1		-1	+1	allowed process

	p	p	$\rightarrow$	e^+	e^+	
L:	0	0		-1	-1	forbideen process (violates both, conservation of baryon
B:	+1	+1		0	0	and lepton number)

Homework 3*:* Determine whether $\pi^+ \rightarrow \mu^+ \ \nu_\mu$ is an allowed process.

In 1960's experiments used neutrinos produced in and collide those with neutrons in various targets:

	ν_μ	n	$\rightarrow$	p	μ^-
L:	+1	0		0	+1
B:	0	+1		+1	0

This is an allowed, experimentally confirmed process. What is interesting is that – from the point of view of baryon and lepton number conservation – also allowed process $\nu_\mu \ n \rightarrow p \ e^-$ was never observed. Such and similar experiments confirmed that a muon neutrino in the initial state always leads to a muon in the final state, never to an electron or a tau lepton.

Based on these experimental facts one concludes that each generation of leptons can be assigned its own lepton number (denoted by L_e, L_μ and L_τ) which is also always conserved. For example,

$$\mu^+ \to e^+ \gamma$$

is not an allowed process because it does not conserve separately L_e and L_μ (although it conserves the general lepton number L). The above conservation is frequently referred to as the lepton flavor conservation (to be distinguished from the general lepton number conservation).

Homework 4: determine whether the following process are allowed or forbidden:

$\pi^0 \to e^+ e^-$ $\quad p \to n\, e^+ \nu_e$ $\quad K^+ n \to \Sigma^+ \pi^0$ $\quad K^- p \to \Sigma^0 \pi^0$

Quark composition of some particles appearing above:

π^0 : $u\bar{u}$; K^+: $u\bar{s}$; Σ^+ : uus

2.3.1 Wave function symmetry

Wave function describing a system of two indistinguishable particles should satisfy

$$\left|\psi(1,2)\right|^2 = \left|\psi(2,1)\right|^2$$

since all the experimental facts one can tell about the system depend on the probability density (i.e. $|\psi|^2$) and since the two praticles can not be distringuished this can not depend on the order of particles denoted above by arguments 1 and 2. It follows that

$$\psi(1,2) = \pm\psi(2,1)$$

The wave function of the two particle system can be expressed as the product of one particle states

$$\psi(1,2) = \phi(1)\phi(2)$$

Let's assume that either of the particles can be found in only two states, denoted by *a* and *b*. In this case the two-particle wave function satisfying the condition above can be written in two (and only two) ways (denoted as ψ_A and ψ_S):

$$\psi_A(1,2) = \frac{1}{\sqrt{2}}\left[\phi_a(1)\phi_b(2) - \phi_a(2)\phi_b(1)\right]$$

$$\psi_S(1,2) = \frac{1}{\sqrt{2}}\left[\phi_a(1)\phi_b(2) + \phi_a(2)\phi_b(1)\right]$$

ψ_A is anti-symmetric upon the exchange of the two particles, while ψ_S is symmetric.

If the two possible states are equal, *a=b*, then

$$\psi_A(1,2) = 0$$
$$\psi_S(1,2) \neq 0$$

Pauli exclusion principle tells us that two identical fermions can not occupy the same state. Hence the wave function for a system of identical fermions must be anti-symmetric (ψ_A). On the other hand bosons do not fulfill the Pauli exclusion principle and hence the wave function for a system of identical bosons must be symmetric (ψ_S).

2.4 Quark Model of Hadrons
2.4.1 Isospin

The quark model explains the „periodic" system of experimentally observed hadrons based on their quark content. The full system of hadrons composed of 6 quarks is complicated but we can start with three quark flavors that were known in the 1960's at the time when Gell-Mann and Zweig proposed the quarks.

To begin with we will start with baryons which are fermions composed of three quarks. In order to compose a fermion from quarks (more than one quark, that is) the lowest number of ingredients is three (taking into account the fact that a *p* has the electric charge of $+e_0$ this also means that the quarks must carry third(s) of the elementary charge; furthermore since there are baryons with zero electric charge one needs quarks with charge +1/3 and quarks -2/3 of the elementary charge). From three quarks with three different flavors one can construct $3^3 = 27$ possible combinations.

We start with the completely symmetric combination composed of only *u* or *d* quarks:

$$\psi_{S1} = |uuu\rangle, \psi_{S2} = |ddd\rangle$$

Before proceeding we should discuss a new quantum number called isospin. *p* and *n* in nuclei are bound together by the strong interaction. While *p* and *n* carry a different electric charge there is no distinction between them in terms of the strong interaction.

This lead Werner Heisenberg in 1932 to the idea that as far as the strong interaction is concerned, protons and neutrons are identical particles (in a similar manner as two fermions are identical in having the spin value = ½, for example). He introduced the quantum number called isospin which is the same for *p* and *n*. They both have the isospin value of I = ½. They differ only in the third component of the isospin (like do the before mentioned fermions): I_3 = +1/2 for *p* and I_3 = -1/2 for *n*.

Since *p* and *n* experience the strong interaction in exactly the same manner this means that the strong interaction (Hamiltonian) is invariant to the rotations in the isospin space (transforming the I_3 = +1/2 component, that is a *p*, into the I_3 = -1/2 component, that is a *n*, and vice-versa).
Remembering about the relation between the symmetry of the Hamiltonian and conservation laws (p. ??) this means that the isospin value is conserved in the processes of strong interaction.

One can define operators of increasing ($\hat{I}_+$)and decreasing ($\hat{I}_-$) the third component of the isospin through:

$$\hat{I}_+\left|p\right\rangle = 0, \hat{I}_-\left|p\right\rangle = \left|n\right\rangle$$
$$\hat{I}_+\left|n\right\rangle = \left|p\right\rangle, \hat{I}_-\left|n\right\rangle = 0$$

Coming back to the quark composition of baryons it is rather easy to conclude that a proton must be composed of two quarks with the charge +2/3 e_0 (*u* quark) and one quark with the charge -1/3 e_0 (*d* quark). Requiring a usual summation of the third component of the isospin (like the summation of the third component of spin) we arrive to the conclusion that the *u* quarks have I_3 = +1/2 and *d* quarks have I_3 = -1/2. The effect of the operators $\hat{I}_+$ and $\hat{I}_-$ is thus $\hat{I}_-|u\rangle = |d\rangle, \hat{I}_+|d\rangle = |u\rangle$.

From the basic wave functions $|uuu\rangle, |ddd\rangle$ one can obtain some other states with symmetric wave function by applying $\hat{I}_+$ and $\hat{I}_-$ to those:

$$\hat{I}_-|uuu\rangle = \frac{1}{\sqrt{3}}\left[|duu\rangle + |udu\rangle + |uud\rangle\right] = \psi_{S3},$$

$$\hat{I}_+|ddd\rangle = \frac{1}{\sqrt{3}}\left[|udd\rangle + |dud\rangle + |ddu\rangle\right] = \psi_{S4}$$

Note that in the above equation operators affect all quarks in a row, i.e. $\hat{I}_-|uuu\rangle$ actually represents $\sum_{i=1}^{3} \hat{I}_-^i |q_1 q_2 q_3\rangle$, with the factor 1/ √ 3 being an approriate normalization of the state.

2.4.2 Strangeness

The strange (s) quark was discovered through the studies of hadrons exhibiting „strange" properties. These hadrons (like neutral kaons, K^0, or Lambda baryons, Λ^0) are always produced in pairs, for example in

$\pi^- p \rightarrow K^0 \Lambda^0$

Their lifetimes are much longer than the lifetimes of hadrons decaying through the strong interaction

$\Lambda^0 \rightarrow \pi^- p$ ($\tau(\Lambda^0) = 10^{-10}$ s), as compared to

$\Delta^0 \rightarrow \pi^- n$ ($\tau(\Delta^0) = 10^{-23}$ s).

Nowadays we know that the first decay above proceeds through the weak interaction which does not conserve a new quantum number strangeness (S) assigned to hadrons composed of s quarks. Strangeness is an analogy of isospin carried by u and d quarks. s quarks have S= -1 and $\bar{s}$ quarks have S=+1. The Hamiltonian of the strong interaction (but not of weak interaction) is invariant to the rotations in the space of isospin and strangeness and hence the two quantum nimbers are conserved in the strong interaction.

What happens if we enlarge the set of four completely symmetric baryon states composed of *u* and *d* quarks with an addition of an *s* quark? If we take care to preserve the symmetry of the wave function we get :

$$|uuu\rangle \to \frac{1}{\sqrt{3}}\left[|suu\rangle + |usu\rangle + |uus\rangle\right] = \psi_{S5}$$

$$\frac{1}{\sqrt{3}}\left[|udd\rangle + |dud\rangle + |ddu\rangle\right] \to \frac{1}{\sqrt{3}}\left[|sdd\rangle + |dsd\rangle + |dds\rangle\right] = \psi_{S6}$$

$$\frac{1}{\sqrt{3}}\left[|duu\rangle + |udu\rangle + |uud\rangle\right] \to$$

$$\frac{1}{\sqrt{6}}\left[|dus\rangle + |dsu\rangle + |sdu\rangle + |uds\rangle + |sud\rangle + |usd\rangle\right] = \psi_{S7}$$

We can continue by replacing the remaining *u* quarks in the resulting states by an *s* quark to get

$$\frac{1}{\sqrt{3}}\left[|suu\rangle+|usu\rangle+|uus\rangle\right]\rightarrow\frac{1}{\sqrt{3}}\left[|sus\rangle+|uss\rangle+|ssu\rangle\right]=\psi_{S8}$$

$$\frac{1}{\sqrt{6}}\left[|dus\rangle+|dsu\rangle+|sdu\rangle+|uds\rangle+|sud\rangle+|usd\rangle\right]\rightarrow$$

$$\frac{1}{\sqrt{3}}\left[|sds\rangle+|dss\rangle+|ssd\rangle\right]=\psi_{S9}$$

If we replace the last *u* quark in the above states we of course get $\psi_{S10}=|sss\rangle$.

This rounds up the set of symmetric states composed of *u*, *d* and *s* quarks to 10 states (decuplet) denoted by ψ_{Si}. From the rest of 27-10 = 17 combinations there is only one wave function (ψ_{A1}) which is completely anti-symmetric. It is composed of an anti-symmetric *u* and *d* combination, to which we add an *s* quark in a symmetric manner:

$$|ud\rangle-|du\rangle\rightarrow$$

$$\psi_{A1}=\frac{1}{\sqrt{6}}\left[|uds\rangle-|dus\rangle+|usd\rangle-|dsu\rangle+|sud\rangle-|sdu\rangle\right]$$

The remaining 16 wave functions do not have a well defined symmetry, they are neither symmetric or anti-symmetric with respect to the interchange of particles. An example of such a function is

$$\psi_{MA1} = \frac{1}{\sqrt{2}}\left[\left|udu\right\rangle - \left|duu\right\rangle\right].$$

It has a mixed symmetry, but it is anti-symmetric w.r.t. the interchange of the first two quarks (hence the notation MA). There exists also a wave function composed of the same quarks which is symmetric w.r.t. the interchange of the first two quarks. It has to be orthogonal to ψ_{MA1} as well as to all other wave functions composed of two *u* and one *d* quark. We can write

$$\psi_{MS1} = a\left|uud\right\rangle + b\left|udu\right\rangle + c\left|duu\right\rangle$$

and determine *a, b* and *c* from the requirements

$$\left\langle\psi_{MA1}\middle|\psi_{MS1}\right\rangle = 0, \left\langle\psi_{MA1}\middle|\psi_{S3}\right\rangle = 0, \left\langle\psi_{MA1}\middle|\psi_{MA1}\right\rangle = 1$$

We get

$$\psi_{MS1} = \frac{1}{\sqrt{6}}\left[\left|udu\right\rangle + \left|duu\right\rangle - 2\left|uud\right\rangle\right].$$

In summary for baryons composed of *u*, *d* and *s* quarks we get 10 symmetric combinations of quark flavors (ψ_{S1}-ψ_{S10}), 1 anti-symmetric combination (ψ_{A1}), 8 combinations of mixed symmetry which are antisymmetric to the exchange of the first two particles (ψ_{MA1}-ψ_{MA8}) and

8 mixed symmetry combinations which are symmetryc w.r.t. the exchange of the first two particles (ψ_{MS1}-ψ_{MS8}).

Upon the inspection of the wave functions that we wrote so far we can observe a relation among the electric charge of baryons (Q) and other quantum numbers preserved by the strong interaction (I_3, S, B):

$Q = I_3 + (B+S)/2$

In the above clasification we have only considered the quark structure of the baryons, or what is usually called the flavor part of the wave function. In order to describe a baryon state we next need to consider its spin. Since each of the quarks in the baryon carries a spin ½, and hence the 3rd component of the spin of ±½, we have $2^3 = 8$ possibilities for the spin part of the wave function. One of the spin parts is completely symmetric: $|\uparrow\uparrow\uparrow\rangle$, where the notation $\uparrow$ represents a quark with the 3rd component of spin +1/2, and the corresponding notation $\downarrow$ will represent quarks with the 3rd component of spin of -1/2. The written symmetric spin part of the wave function represents a baryon with the 3rd component of spin of J_3=+3/2. It is not difficult to write down other spin parts of the wave function for J=3/2 and $|J_3| \leq 3/2$. We can take the flavor parts of the wave functions composed of u and d quarks and change u $\rightarrow \uparrow$ and d $\rightarrow \downarrow$. $\psi_{S1} - \psi_{S4}$ represent a quadruplet of states with J=3/2. From the flavor parts of mixed symmetry there are four composed only of u and d quarks. ψ_{MS1}-ψ_{MS2} represent one dublet with J = ½ and ψ_{MA1}-ψ_{MA2} another other dublet. The summary of the spin part of the wave function is thus one quadruplet with J=3/2 which is symmetric and two dublets with J=1/2 with mixed symmetry.

Are the flavor and the spin part of the wave function a complete description? Let's take the Δ^{++} baryon with J=3/2 as an example. Considering the charge it has to be composed of 3 u quarks and hence its flavor part is symmetric. Furthermore since we know experimentally its spin is 3/2 also the spin part of the wave function is symmetric. The product of two symmetric parts of the wave function is also a symmetric wave function and in accordance with the discussion on p. ?? this is not possible (since Δ^{++} is a fermion). There is a need for another quantum number that provides the overall antisymmetric wave function. This quantum number is called the color (or color charge, or strong charge). The quark color can take three values, R – red, G – green and B – blue.

All hadrons are colorless, i.e. they don't carry the color charge. Only quarks inside the hadrons carry color. This implies that the color part of the wave function of any hadron must be a singlet. If it wouldn't be a singlet, a rotation in the color space would transfrom this particular color state into another – distinguishalble - one, which would obvioulsy not be colorless.

A singlet of three quarks carrying three possible values of quantum number is already composed: ψ_{A1} for three quark flavors. Hence the color part of the wave function for baryons is ψ_{A1} with the replacement $u \rightarrow$ R, $d \rightarrow$ G and $s \rightarrow$ B:

$$\psi_{A1} = \frac{1}{\sqrt{6}}\left[\left|RGB\right\rangle - \left|GRB\right\rangle + \left|RBG\right\rangle - \left|GBR\right\rangle + \left|BRG\right\rangle - \left|BGR\right\rangle \right]$$

The color part of the wave function is antisymmetric and hence the whole Δ^{++} baryon wave function composed of the flavor, spin and color part is antisymmetric, as it should be for fermions.
We have to remember that there is still one part of the wave function missing – the one describing the spatial coordinates of the three quarks or the spatial part. It turns out that the symmetry of the spatial part is $(-1)^{\ell}(-1)^{\ell'}$, where ℓ and ℓ' are the angular momentum quantum numbers of the two pairs of quarks inside the baryon. This follows from the spatial dependence being described by the spherical harmonics $Y_{\ell m}(\theta,\phi)$ and $Y_{\ell' m'}(\theta,\phi)$. The ground states of all baryons have ℓ , ℓ' =0 and hence the spatial part of the wave function is symmetric, again leading to the overall antisymmetric wave function.

We argued that the wave function of the Δ^{++} baryon composed of the symmetric flavor, spin and spatial parts and of an antisymmetric color part is antisymmeric. How about possibilities for other baryons? Clearly all products of the spin quadruplet and symmetric flavor decuplet are symmetric and the color part takes care of the global antisymmetry. This is called a decuplet of ground baryons with J = 3/2, of which Δ^{++} is one of the members, with the wave functions

$$\psi_{S1-10}(flavor)\psi_{S1-4}(spin)\psi_{A1}(color)\psi_{S}(space)$$

There is another possibility for the antisymmetric wave function: product of flavor and spin parts with mixed symmetry yields a symmetric product if we take both spin and flavor parts to be ψ_{MSi} or both to be ψ_{MAi} . We can check this by writing out one example explicitly:

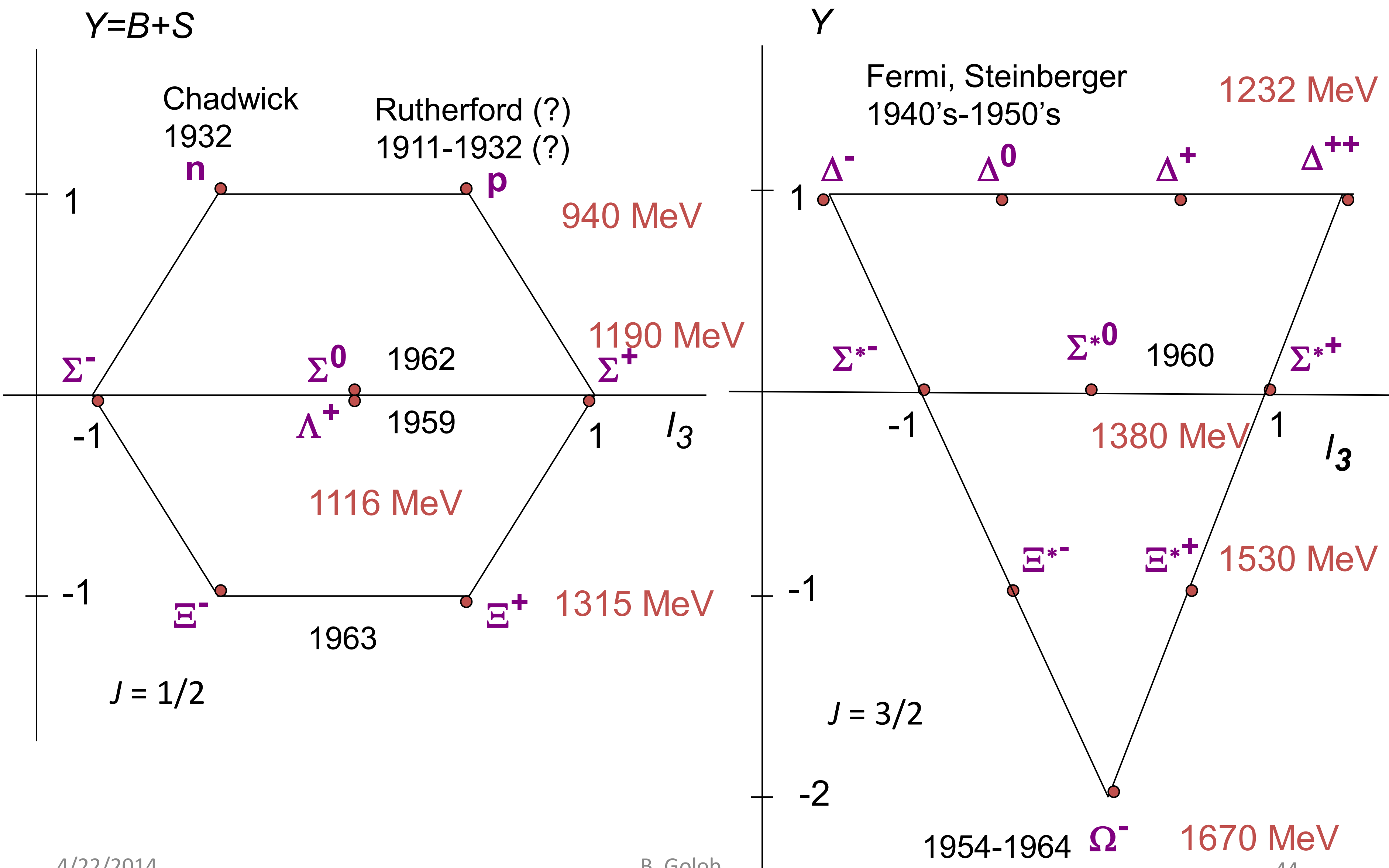
$$\psi_{MA1}(flavor)\psi_{MA1}(spin) = \frac{1}{2}\left[|udu\rangle - |duu\rangle\right]\left[|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\right] =$$
$$\frac{1}{2}\left[|u\uparrow d\downarrow u\uparrow\rangle - |u\downarrow d\uparrow u\uparrow\rangle - |d\uparrow u\downarrow u\uparrow\rangle + |d\downarrow u\uparrow u\uparrow\rangle\right]$$

One can check that the above wave function is indeed symmetric on the interchange of any two particles. Together with the symmetric spatial and antisymmetric color part it yields an antisymmetric overall wave function.

These possibilities represent an octet of ground baryon states with J = ½, with the wave function of the form

$$a\,\psi_{MS1-8}(flavor)\psi_{MS1-4}(spin)\psi_{A1}(color)\psi_{S}(space) +$$
$$b\,\psi_{MA1-8}(flavor)\psi_{MA1-4}(spin)\psi_{A1}(color)\psi_{S}(space)$$

The decuplet and the octet of ground baryons can be nicely gathered into a „periodic" system considering the quantum numbers (B+S) and I_3:

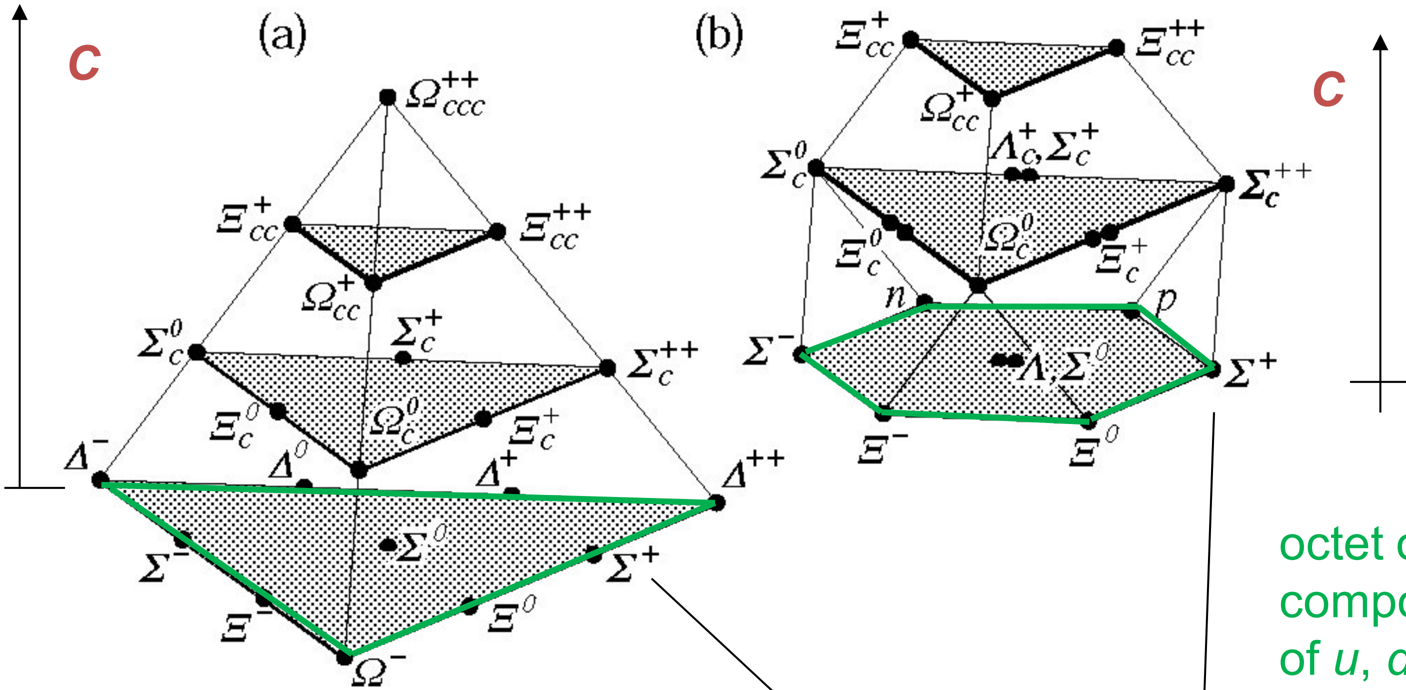


$Y = B + S$ is called the hypercharge. In the figure on the previous page also masses (mc^2) of some of the baryons are given together with the discovery year and names of scientists most credited for their discovery.

Homework 5: based on the hypercharge and the 3rd component of the isospin determine the quark composition of the following baryons: Σ^-, Ξ^-, Δ^-, Ω^-

Baryons composed of *u, d* and *s* quarks do not represent the full palette of baryons. Adding also a possibility of charm quarks content, additional quantum number (charm, *C*) must be added. *c* quark has C=+1, $\bar{c}$ quark C=-1, and all other quarks have C=0. By this the „periodic" system of baryons aquires additional axis (beside the I_3 and the *S* - or *Y*) and now expands into 3 dimensions as shown in the next figure.

$J = 3/2$ $J = 1/2$



octet of baryons composed of *u*, *d* and *s* quarks

decuplet of baryons composed of *u*, *d* and *s* quarks

Naming scheme of barions not completely unified, these two Σ^+ barions are not the same; one has J=1/2, the other J=3/2; in Particle Data Group listings the former is denoted as Σ^+, the latter as $\Sigma(1385)\ 3/2^+$; sometimes also Σ^{*+}

As an example of the quark model we can estimate the dipole magnetic moment of a proton. Protons have $J=1/2$ and hence the wave function of the form

$$\psi_{MS1-8}(flavor)\psi_{MS1-4}(spin)+\psi_{MA1-8}(flavor)\psi_{MA1-4}(spin).$$

Since a proton is composed from *u, u* and *d* quarks, the wave function is (for a proton with $J_3=½$)

$$\psi_p=\frac{1}{\sqrt{2}}\left[|udu\rangle-|duu\rangle\right]\left[|\uparrow\downarrow\uparrow\rangle-|\downarrow\uparrow\uparrow\rangle\right]+$$
$$+\frac{1}{\sqrt{6}}\left[|udu\rangle+|duu\rangle-2|uud\rangle\right]\left[|\uparrow\downarrow\uparrow\rangle+|\downarrow\uparrow\uparrow\rangle-2|\uparrow\uparrow\downarrow\rangle\right]=\dots$$
$$\dots=\frac{1}{\sqrt{18}}\Big[2|u\uparrow u\uparrow d\downarrow\rangle-|u\uparrow u\downarrow d\uparrow\rangle-|u\downarrow u\uparrow d\uparrow\rangle+$$
$$2|d\downarrow u\uparrow u\uparrow\rangle-|d\uparrow u\downarrow u\uparrow\rangle-|d\downarrow u\uparrow u\downarrow\rangle+$$
$$2|u\uparrow d\downarrow u\uparrow\rangle-|u\downarrow d\uparrow u\uparrow\rangle-|u\uparrow d\uparrow u\downarrow\rangle\Big]$$

Magnetic dipole moment operator of the i-th quark in the proton is $\hat{\vec{\mu}}_i=g_s\frac{e_0Q_i\hat{\vec{s}}_i}{2m_i}$, with $g_s=2$ (fermionic giromagnetic ratio, p. ??), or, for its third component (which as we said is usualy quoted as the expectation value) $\hat{\mu}_{3i}=g_s\frac{e_0Q_i\hat{s}_{3i}}{2m_i}$

The expectation value of the proton magnetic moment is thus $\mu_p=\left\langle\psi_p\left|\sum_{i=1}^{3}\hat{\mu}_{3i}\right|\psi_p\right\rangle$

Inserting the above ψ_p into this expression, assuming $m_u \cong m_d = m_q$, and taking into account the orthonormality of individual terms in ψ_p, we arrive at

$$\mu_p = \frac{1}{18}\frac{e_0}{m_q}\left[4\left(\frac{2}{3}+\frac{2}{3}+\frac{1}{3}\right)+\left(\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right)+\left(-\frac{2}{3}+\frac{2}{3}-\frac{1}{3}\right)+\ldots\right]=$$

$$\ldots = \frac{e_0}{2m_q}$$

In order to compare this calculated value to the experimental measurement we must insert m_q. Naively taking $m_q = m_p/3$ we get $\mu_p = 3e_0/2m_p$, to be compared to $\mu_p = g_{s,p}\, e_0\, s \,/\, 2\, m_p$ $= 5.6/2\;\; e_0\; / \,2\, m_p\;\; = 2.8\, e_0\; / \,2\, m_p$ alculated.

Because of the unknown actual value of m_q it is more reasonable to compare the ratio of the proton and neutron magnetic moments.

Homework 6: calculate the dipole magnetic moment of a neutron within the quark model.

$$\mu_n = -\frac{2}{3}\frac{e_0}{2m_q} \Rightarrow \frac{\mu_n}{\mu_p} = -\frac{2}{3}$$, to be compared to the experimental value of -0.685.

So far we have encountered isospin and strangness, quantum numbers assigned to individual quark flavors. Every quark flavor has its associated quntum number given in the Table below:

flavor	spin	baryon number *B*	Charge *Q*	3rd component of isospin I_3	strangness *S*	charm *C*	beauty *B*	topness *T*
u	½	1/3	2/3	+1/2	0	0	0	0
d	½	1/3	2/3	-1/2	0	0	0	0
s	½	1/3	2/3	0	-1	0	0	0
c	½	1/3	2/3	0	0	+1	0	0
b	½	1/3	2/3	0	0	0	-1	0
t	½	1/3	2/3	0	0	0	0	+1

Note that top quarks do not form hadrons (their mass is so high that before hadronization they decay through the weak interaction).

After discussing baryons we can move to description of mesons within the quark model. Mesons are composed of a quark and an anti-quark. Considering again only mesons composed of u, d and s (anti)quarks we are left with nine possible combinations.
Before proceeding to the wave function composition we must discuss the transformation $|q\rangle \to |\bar{q}\rangle$, i.e. the transformation of a particle into its anti-particle. The transformation is called the charge conjugation, and the coresponding operator is denoted by $\hat{C}$

$$\hat{C}|q\rangle = e^{i\varphi}|\bar{q}\rangle$$
$$\hat{C}^2|q\rangle = \hat{C}e^{i\varphi}|\bar{q}\rangle = e^{i\varphi}e^{-i\varphi}|q\rangle = |q\rangle$$

In the above equation we intorduced an unobservable phase ϕ (since it can't be measured it's arbitrary). For simplicity we take

$$\hat{C}|u\rangle = -|\bar{u}\rangle$$
$$\hat{C}|d\rangle = |\bar{d}\rangle$$

This prescription influences the effect of the isospin 3rd component increase and decrease operators (see p. ??) on the $|\bar{u}\rangle$ and $|\bar{d}\rangle$ states:

$$\hat{I}_-|u\rangle = |d\rangle, \hat{I}_+|d\rangle = |u\rangle$$

$$\hat{I}_-|\bar{d}\rangle = -|\bar{u}\rangle, \hat{I}_+|\bar{u}\rangle = -|\bar{d}\rangle$$

$$\hat{I}_-\left[|d\bar{d}\rangle - |u\bar{u}\rangle\right] = -|d\bar{u}\rangle - |d\bar{u}\rangle$$

$$|\pi^+\rangle = |u\bar{d}\rangle \; (I_3 = +1)$$

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}\left[|d\bar{d}\rangle - |u\bar{u}\rangle\right] \; (I_3 = 0)$$

$$|\pi^-\rangle = -|d\bar{u}\rangle \; (I_3 = -1)$$

$$\hat{I}_-|u\rangle = |d\rangle, \hat{I}_+|d\rangle = |u\rangle$$

$$\hat{I}_-|\bar{d}\rangle = -|\bar{u}\rangle, \hat{I}_+|\bar{u}\rangle = -|\bar{d}\rangle$$

In construction of mesons wave function we can start with the $|u\bar{d}\rangle$ state (I_3 = +1) and operate on it with the I_- operator: $\hat{I}_-|u\bar{d}\rangle = |d\bar{d}\rangle - |u\bar{u}\rangle$

Operating once again to the resulting state yields

$$\hat{I}_-\left[|d\bar{d}\rangle - |u\bar{u}\rangle\right] = -|d\bar{u}\rangle - |d\bar{u}\rangle .$$

We end up with a triplet of states (I=1, I_3=0, ±1) composed of *u* and *d* (anti)quarks, called pions.

$$|\pi^+\rangle = |u\bar{d}\rangle \; (I_3 = +1)$$

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}\left[|d\bar{d}\rangle - |u\bar{u}\rangle\right] \; (I_3 = 0)$$

$$|\pi^-\rangle = -|d\bar{u}\rangle \; (I_3 = -1)$$

What about the symmetry of the flavor part of the wave function? In the case of mesons the symmetry of the wave function doesn't play a significant role (like it does in the case of baryons), since in mesons one encounters a particle and an antiparticle and hence the two particles in the system are not undistinguishable. This leads to all 9 possible combinations of mesons (actually 18, considering the possibility of J=0 and J=1), as opposed to the case of baryons where out of 27 possible combinations we saw only 18 represent the actual baryon ground sates).

We can now proceed by adding s quarks (i.e. replacing u or d quarks by s quark):

$$\left|\pi^+\right\rangle \underbrace{\rightarrow}_{d\rightarrow s} \left|K^+\right\rangle = \left|u\bar{s}\right\rangle \; (I_3 = +1/2)$$

$$\left|\pi^+\right\rangle \underbrace{\rightarrow}_{u\rightarrow s} \left|\bar{K}^0\right\rangle = \left|s\bar{d}\right\rangle \; (I_3 = -1/2)$$

$$\left|\pi^-\right\rangle \underbrace{\rightarrow}_{d\rightarrow s} \left|K^-\right\rangle = -\left|s\bar{u}\right\rangle \; (I_3 = -1/2)$$

$$\left|\pi^-\right\rangle \underbrace{\rightarrow}_{u\rightarrow s} \left|K^0\right\rangle = -\left|d\bar{s}\right\rangle \; (I_3 = +1/2)$$

This results in two isospin dublets (I=1/2, I_3 = ±½) with S= ±1, called kaons.
All together we now have 4 kaons and three pions. Under the flavor transformations $u \leftrightarrow d$, $u \leftrightarrow s$ or $d \leftrightarrow s$ these states transform from one into another (such transformations are also called SU(3) transformations, or rotations in the SU(3) flavor space, where SU denotes the properties of the group, and 3 the number of flavors).

We can construct another combination with symmetric flavor part of the wave function, which is untransformed under the flavor transformations (and is hence called the flavor singlet): $\left|\eta_0\right\rangle = \frac{1}{\sqrt{3}}\left[\left|u\bar{u}\right\rangle + \left|d\bar{d}\right\rangle + \left|s\bar{s}\right\rangle\right]$

The subscript 0 denotes that this state is a flavor singlet. Last out of 9 mesons composed of u, d and s quarks is constructed as a cobmination of $\left|u\bar{u}\right\rangle, \left|d\bar{d}\right\rangle, \left|s\bar{s}\right\rangle$, but orthogonal to $\left|\eta_0\right\rangle$ (and all other states, for example $\left|\pi^0\right\rangle$):

$$\left|\eta_8\right\rangle = a\left|u\bar{u}\right\rangle + b\left|d\bar{d}\right\rangle + c\left|s\bar{s}\right\rangle, \quad \left\langle\eta_8\middle|\eta_0\right\rangle = 0$$

We get $\left|\eta_8\right\rangle = \frac{1}{\sqrt{6}}\left[\left|u\bar{u}\right\rangle + \left|d\bar{d}\right\rangle - 2\left|s\bar{s}\right\rangle\right]$.

In this case the subscript 8 reminds us that under the flavor transformations this state is transformed into pions or kaons, and is hence a member of the flavor octet (together with

7 pions and kaons).

The spin part of the meson wave function must encompass the spin 1 and spin 0 states. For J=1, J_3 = +1 the only possibility is $|\uparrow\uparrow\rangle$. Applying an anology of isospin operators I_+ and I_-,

spin raising and lowering operators S_+ and S_-, we get

$$\hat{S}_-|\uparrow\uparrow\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$$
$$\hat{S}_-\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right] = 2|\downarrow\downarrow\rangle$$

Hence the mesons with spin 1 belong to the spin triplet:

$$|\uparrow\uparrow\rangle$$
$$\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right]$$
$$|\downarrow\downarrow\rangle$$

The spin 0 part can be obtained as a linear combination of $a|\uparrow\downarrow\rangle + b|\downarrow\uparrow\rangle$ which is orthogonal to the spin 1 wave functions. We get the spin singlet $\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$.

We have 3 pions, 4 kaons, η_0 and η_8 with spin 0 as the ground state mesons, that can be grouped into the flavor octet and flavor singlet. This are the mesons with *J*=0 (also called pseudoscalar mesons).
These states can, similarly as baryons, be presented in a „periodic“ system depending on the 3rd component of isospin and strangeness, as shown in the next page. As suggested in the figure, the states η_0 and η_8 appear in nature as linear combionations,

$$\left|\eta\right\rangle = \sin\theta\left|\eta_0\right\rangle + \cos\theta\left|\eta_8\right\rangle$$
$$\left|\eta'\right\rangle = \cos\theta\left|\eta_0\right\rangle - \sin\theta\left|\eta_8\right\rangle$$

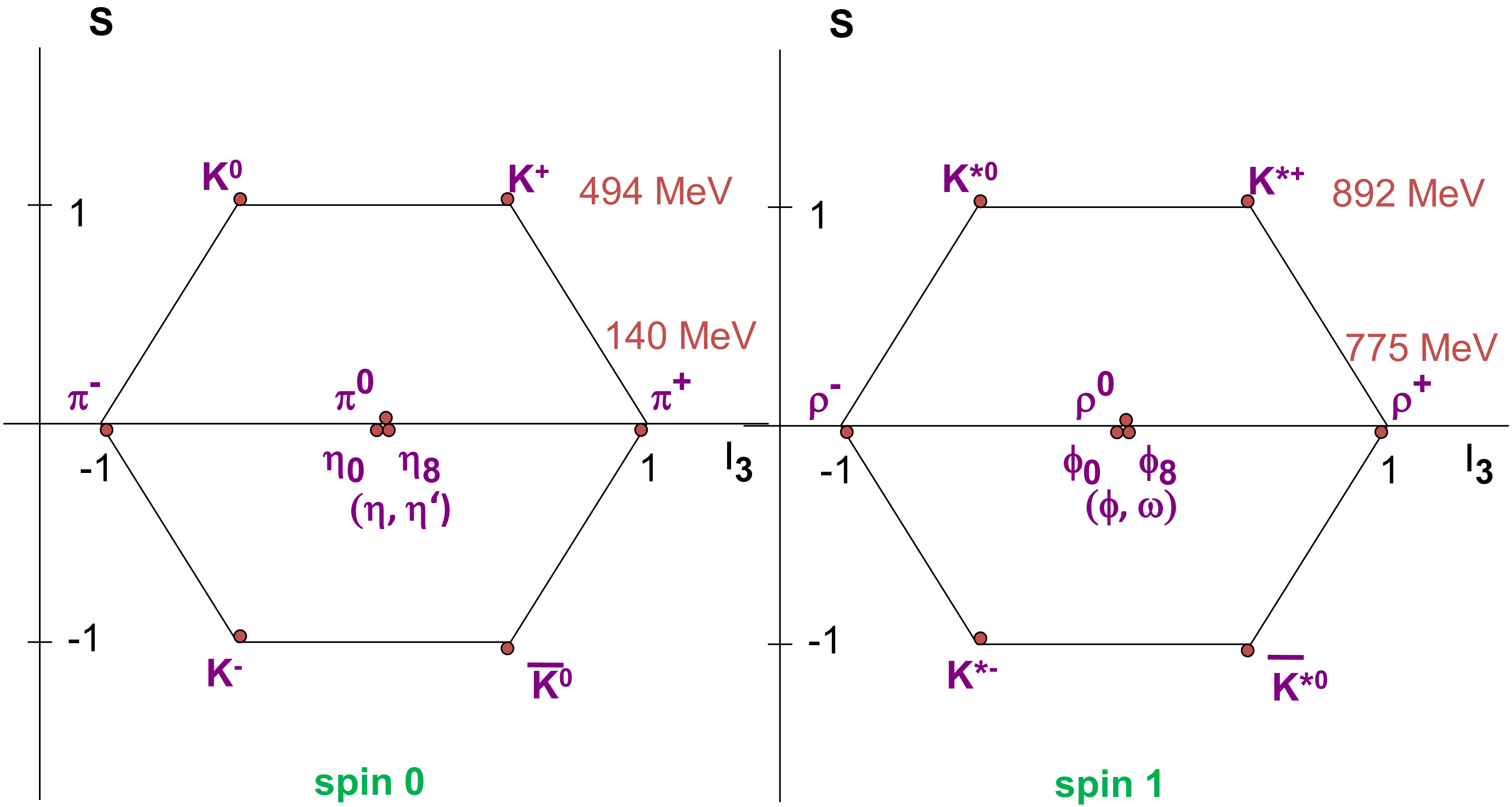
The same flavor pattern is repeated for *J*=1 (vector mesons), where we have 3 ρ mesons (analogy of pions) and 4 *K** mesons (analogy of kaons). One also has the states corresponding to η_0 and η_8 with spin 1 (denoted by ϕ_0 and ϕ_8) and the two linear combinations

$$\left|\phi\right\rangle = \sin\theta'\left|\phi_0\right\rangle + \cos\theta'\left|\phi_8\right\rangle$$

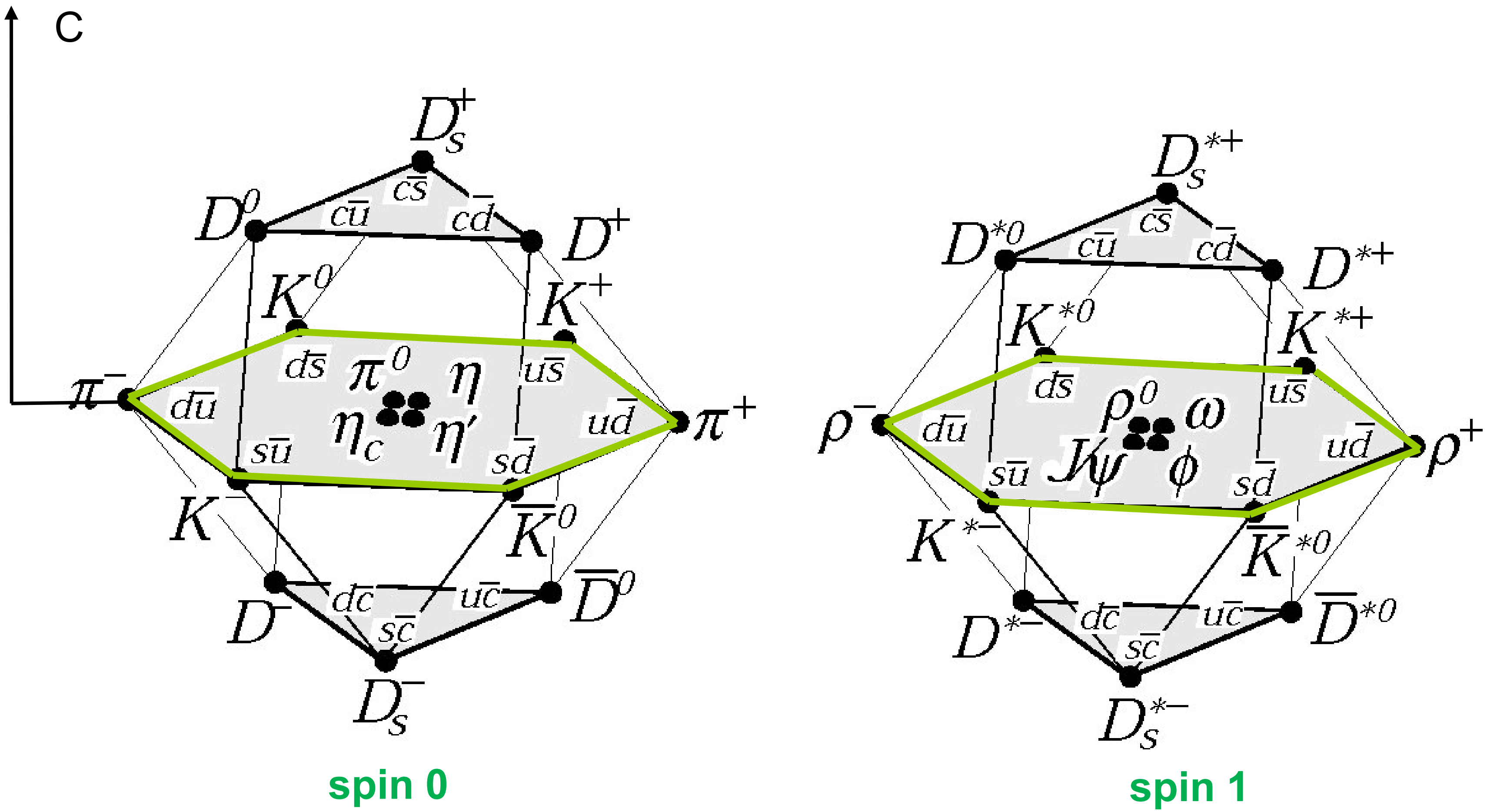
$$\left|\omega\right\rangle = \cos\theta'\left|\phi_0\right\rangle - \sin\theta'\left|\phi_8\right\rangle$$

(for the *J*=1 mesons, the mixing angle θ' is such that ϕ is almost exactly the $s\bar{s}$ combination and ω is composed of $u\bar{u}$ and $d\bar{d}$ only, i.e. $\theta' \approx -0.615$ rad).

These states can, similarly as baryons, be presented in a „periodic“ system depending on the 3rd component of isospin and strangeness:



Inclusion of charm quarks again requires additional axis (*C*) in the periodic system:



Homework 7: Determine the relative rate of decays to e^+e^- for ω, ρ, ϕ and J/ψ mesons neutral vector (i.e. J=1) mesons.

2.5 Probability density and current, antiparticles
2.5.1 Probability density and current

Based on the classical relation between the energy and momentum, $E = p^2/2m$, and replacing the observables by operators,

$$\hat{E} = i\hbar\frac{\partial}{\partial t}, \ \hat{\vec{p}} = -i\hbar\vec{\nabla}$$

we arrive to the Schrödinger equation,

$$i\frac{\partial\psi}{\partial t} + \frac{\hbar}{2m}\nabla^2\psi = 0.$$

The square of the absolute value of the wave function, $\rho = |\psi|^2$, is interpretted as the probability density, i.e. $|\psi|^2 dV$ represents the probability to find the particle described by ψ in a space volume element dV.

We can derive the current density of particle flow $\vec{j}$ (needed in evaluation of the cross section for a specific process, see p. ??) from the continuity equation:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{j} = 0 \ .$$

First we write the complex conjugate of the Schrödinger equation

$$-i\frac{\partial\psi^*}{\partial t}+\frac{\hbar}{2m}\nabla^2\psi^*=0$$

and multiply it by $i\psi$ from the right:

$$\frac{\partial\psi^*}{\partial t}\psi+\frac{i\hbar}{2m}(\nabla^2\psi^*)\psi=0.$$

We multiply the original Schrödinger equation by $-i\psi^*$ from the right:

$$\frac{\partial\psi}{\partial t}\psi^*-\frac{i\hbar}{2m}(\nabla^2\psi)\psi^*=0\cdot$$

We sum the two equations,

$$\frac{\partial\psi^*}{\partial t}\psi+\frac{\partial\psi}{\partial t}\psi^*+\frac{i\hbar}{2m}\left[(\nabla^2\psi^*)\psi-(\nabla^2\psi)\psi^*\right]=0,$$

and after some rearrangement obtain

$$\frac{\partial}{\partial t}(\psi\psi^*)+\frac{i\hbar}{2m}\left[\psi\nabla^2\psi^*-\psi^*\nabla^2\psi\right]=0$$

$$\frac{\partial}{\partial t}(\psi\psi^*)+\frac{i\hbar}{2m}\vec{\nabla}\left[\psi\vec{\nabla}\psi^*-\psi^*\vec{\nabla}\psi\right]=0$$

Taking into account that $\rho = \psi\psi^*$ and comparing the last equation with the continuity equation we see

$$\vec{j} = \frac{i\hbar}{2m}\left[\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right]$$

In order to simplify the notation to some extent in the following, it is very common to introduce the so called natural units. Writing out, for example, the relativistic energy – momentum relation, $E^2 = m^2 c^4 + c^2 p^2$, one notices it is easier to write if one simply takes $c=1$. Similarly, it is less bothering to write some wave vector instead of $k = p/\hbar$ rather in a form $k = p$, i.e. taking $\hbar = 1$.
Writing out equations in a such a simplified form is definitely easier, but at the end of course one needs to take care that the derived quantities have correct units. This task is easier than it may look at the first sight. Assuming we know what units any quantity we are interested in should have, it consists merely of adding an appropriate power of the conversion constants $\hbar c$ = 197 MeV fm and $c = 3\cdot 10^8$ m/s to the result, derived using the natural units.

In natural units a plane wave can be written as

$$\psi = \frac{1}{\sqrt{V}} e^{i\vec{p}\vec{r} - iEt}.$$

For a plane wave the probability density is $1/V$ (not surprisingly, we have one particle in the normalization volume V), and the current density of the particle flow is

$$\vec{j} = \frac{\vec{p}}{m}\frac{1}{V}$$

The latter equation is also not really surprising, clasically any current density is just

$$\vec{j} = \vec{v}\rho .$$

In special theory of relativity we start with the appropriate energy – momentum relation and in a similar manner as for the Schrödinger equation we get the Klein-Gordon equation (p. ???):

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi \cdot$$

Proceeding the same way as with the Schrödinger equation to obtain the current density,

$$\left.\begin{aligned} -\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi \;/\cdot(-i\phi^*) \\ -\frac{\partial^2 \phi^*}{\partial t^2} + \nabla^2 \phi^* = m^2 \phi^* \;/\cdot(-i\phi) \end{aligned}\right\} subtraction$$

$$\frac{\partial}{\partial t}\underbrace{\left[i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) \right]}_{\rho} + \vec{\nabla}\underbrace{\left[i\left(\phi \vec{\nabla}\phi^* - \phi^* \vec{\nabla}\phi \right) \right]}_{\vec{j}} = 0$$

By comparison to continuity equation we again identify the probability density and the particle flow current density as

$$\rho = i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)$$
$$\vec{j} = i\left(\phi \vec{\nabla} \phi^* - \phi^* \vec{\nabla} \phi\right) \quad .$$

For a plane wave this corresponds to

$$\rho = \frac{2E}{V}, \vec{j} = \frac{2\vec{p}}{V}$$

We learn that for relativistic particles instead of normalizing to a single particle in the normalization volume *V*, we need to normalize to *2E* particles in the normalization volume. One should not that in this case ρ transforms under a Lorentz transformation as the time component of the Lorentz vector (i.e. as *E*),

$$\rho \underbrace{\rightarrow}_{\substack{Lorentz \\ transform.}} \frac{\rho}{\sqrt{1-(v^2/c^2)}}$$

A volume element d^3x transforms like

$$d^3x \underbrace{\rightarrow}_{\substack{Lorentz \\ transform.}} d^3x\sqrt{1-(v^2/c^2)}$$

Hence the number of particles, $\rho\, d^3x$, is invariant to the Lorentz transformation,

$$\rho d^3x \underbrace{\rightarrow}_{\substack{Lorentz\\ transform.}} \frac{\rho}{\sqrt{1-(v^2/c^2)}} d^3x\sqrt{1-(v^2/c^2)} = \rho d^3x \quad .$$

As we will see later this means that also the density of final states remains unchanged under any Lorentz transformation.

Since we made a notation simplification using the natural units it is appropriate to mention that also the Klein-Gordon equation can be written in a more compact form, using the four-vectors. Defining the derivatives four-vector, $\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right)$,

and $\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$

the equation can be written as $\left(\partial_\mu \partial^\mu + m^2 \right) \phi = 0$.

Once can also define the current four-vector $j^\mu = \left(\rho, \vec{j} \right)$, and the contnuity equation is then written simply as $\partial_\mu j^\mu = 0$

Klein-Gordon equation is named after Oskar Klein and Walter Gordon. The former was a Swedish physicist also known for his contribution to the Kaluza-Klein theories. Walter Gordon was a German physicist working for some time with Max Planck and W.L. Bragg. It seems that Schrödinger was already aware of the equation since it has been found in his notes but never used it.

A comment regarding the four-vector notation is in place. For a general four-vector a^μ, the notation is

$$a^\mu = \left(a^0, \vec{a}\right), a_\mu = \left(a^0, -\vec{a}\right)$$
$$a^\mu a_\mu = (a^0)^2 - (\vec{a})^2 \quad ,$$

for example $$p^\mu p_\mu = E^2 - (\vec{p})^2 \; .$$

An exception in the notation is the derivative four-vector ∂^μ (because of its properties under the Lorentz transformation):

$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \quad .$$

2.5.2 Antiparticles

Coming back to the Klein-Gordon equation, inserting a plane wave $\phi = \frac{1}{\sqrt{V}} e^{ip^\mu x_\mu}$,

where p^μ is the momentum four-vector and x^μ the coordinates four-vector, $x_\mu = (t, -\vec{r})$,

one of course gets the relation $E^2 = p^2 + m^2$, the solution of which is

$E = \pm\sqrt{p^2 + m^2}$. An obvious question arises what the solutions with the negative energy are.

We can write out the particle flow density of an electron with the energy E, momentum p and the charge $-e_0$: $j^\mu = \frac{2}{V}(E, \vec{p})$.

This particle current can be re-interpreted as electromafgnetic current by inclusion of the particle's charge: $j^\mu = -\frac{2}{V} e_0 (E, \vec{p})$

The reason for this interpretation (without even bothering to use a different notation for such a current) will become obvious later when discussing the electromagnetic interaction in the context of the Dirac equation (p. ??), where such a current appears in the amplitude of a given process.

How about an analogous current for a positron with the same energy and momentum (and of course charge + e_0)? We can write it as

$$j^{\mu} = \frac{2}{V} e_0 \left(E, \vec{p}\right) = -\frac{2}{V} e_0 \left(-E, -\vec{p}\right)$$

In the last step we emphasized that such an electromagnetic current for a positron can be written in exactly the same form like for an electron (using the charge of the latter, i.e. - e_0), but with a negative energy and momentum. This is the basis of the Feynman – Stückelberg interpretation of solutions (of the Klein-Gordon, or to that matter of the Dirac equation, to be discussed later) with negative energy. Solutions for particles (electron) with negative energy can be interpreted as solution for antiparticles (positron) that have a positive energy.

Ernst Stückelberg was a Swiss physicist and mathematician. In 1941 (working at the University of Geneva and University of Lausanne) he proposed the interpretation of the antiparticles which is closely related to the methods of Feynman diagrams proposed later.

Interestingly enough, already in 1938 Stückelberg realizes that electrodynamics with a massive propagator would require an additional scalar boson which later became known as the Higgs boson.

Richard Feynman is probably one of best known physicists, not only due to his important discoveries but also for his character. He is known to public by his autobiographic books „Surely You're Joking, Mr. Feynman!“ and „What Do You Care What Other People Think?“.

During the 2nd World War he participated in the Manhattan Project (USA nuclear bomb project) under the leadership of Robert Oppenheimer, but due to his youth he was not a key person there.

He did most of his scientific work at the California Institute of Technology (Caltech). There he developed his theory of quantum electrodynamics for which he received the Nobel prize for physics in 1965 (together with Sin-Itiro Tomonaga and Julian Schwinger).

He also developed the method of Ferynman diagrams, a pictorial representation of the perturbative claculations in particle physics.

2.6 Dirac Equation

Paul Dirac tried to write an equation for relativistic particles which (contrary to Klein-Gordon equation) would include a first derivative over time. It is the second derivative over time that causes the probability density of a plane wave to be $\rho = 2E/V$, which could be negative for the solutions with $E<0$ (which we know now actually represent the solutions for antiparticles).

Of course the equation would have to satisfy also

$$\hat{H}^2\psi = (p^2 + m^2)\psi$$

to reproduce the relativistic energy-momentum relation. We know that the Hamiltonian includes the first derivative over time:

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t}$$

(note that any ψ can be written as a superposition of plane waves; for the latter the energy is always obtained by applying $i\hbar\frac{\partial}{\partial t}$ to the ψ)

Hence Dirac tried to derive an equation linear in $\hat{H}$ but satisfying the relativistic energy-momentum relation. While this can not be achieved using a scalar form of the wave function it turns out this is possible one assumes a more dimensional form of ψ:

$$\hat{H}\psi = \left[\vec{\alpha}\,\hat{\vec{p}} + \beta m\right]\psi,$$

where α and β are matrices, and ψ is a more dimensional vector. The notation $\vec{\alpha}$ represents a vector of matrices, $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$

and hence $$\vec{\alpha}\,\hat{\vec{p}} = \alpha_1 \hat{p}_1 + \alpha_2 \hat{p}_2 + \alpha_3 \hat{p}_3$$

It truns out that the requirements can be satisfied by 4 x 4 matrices α and β (and ψ is thus a vector with 4 compnents):

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

where each of the matrices σ_i, called Pauli's matrices, is a 2 x 2 matrix,

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

and *I* in the β matrix also represents a 2 x 2 identical matrix:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rewriting the equation

$$\hat{H}\psi = \left[\vec{\alpha}\hat{\vec{p}} + \beta m\right]\psi$$

in the form

$$i\frac{\partial}{\partial t}\psi = \left[\vec{\alpha}\hat{\vec{p}} + \beta m\right]\psi\ ,$$

inserting the momentum operator

$$i\frac{\partial}{\partial t}\psi = \left[-i\vec{\alpha}\vec{\nabla} + \beta m\right]\psi$$

and multiplying it with β from the right, we get

$$i\beta\frac{\partial}{\partial t}\psi = \left[-i\beta\vec{\alpha}\vec{\nabla} + \beta^2 m\right]\psi\ .$$

Taking into account $\beta^2 = 1$ and we can write it in the form

$$i\beta\frac{\partial}{\partial t}\psi + i\beta\vec{\alpha}\vec{\nabla}\psi = m\psi$$

We can now define a four-vector of matrices γ^μ, called the Dirac gamma matrices:

$$\gamma^\mu = (\beta, \beta\vec{\alpha})$$

Using the four-vector of derivatives ∂^μ we can write the resulting equation in a very compact form:

$$[i\gamma^\mu \partial_\mu - m]\psi = 0$$

which is called a covariant form of the Dirac equation. After all the definitions one of course would like to know if the Dirac equation indeed satisfies

$$\hat{H}^2\psi = (p^2 + m^2)\psi$$

We can check this by the explicit calculation

$$\hat{H}^2\psi = (\vec{\alpha}\vec{p} + \beta m)^2\psi = (\vec{\alpha}\vec{p})^2\psi + \beta^2 m^2\psi + \vec{\alpha}\beta\vec{p}m\psi + \beta\vec{\alpha}\vec{p}m\psi$$

By inspection of the properties of α and β matrices we see $\alpha_i \beta = - \beta \alpha_i$, and hence

$$\vec{\alpha}\beta = -\beta\vec{\alpha}.$$

So $\hat{H}^2\psi = (\vec{\alpha}\vec{p})^2\psi + m^2\psi$

$$(\vec{\alpha}\vec{p})^2 = (\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3)^2 = (\alpha_1^2 p_1^2 + \ldots + \alpha_1 p_1 \alpha_2 p_2 + \alpha_2 p_2 \alpha_1 p_1 + \ldots)$$

$$\sigma_i^2 = 1 \Rightarrow \alpha_i^2 = 1$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \Rightarrow (\vec{\alpha}\,\vec{p})^2 = p^2$$

We see that the Dirac equation indeed satisfies the relativistic energy-momentum relation.

From the properties of α and β matrices we also see that for the Dirac gamma matrices the anti-commutation relation holds,

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

with $g^{\mu\nu}$ denoting the antisymmetric tensor

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

2.6.1 Solutions of Dirac Equation

In the Dirac equation

$$\left[i\gamma^{\mu}\partial_{\mu} - m\right]\psi = 0$$

the operator $i\partial^{\mu}$ represents the operator of the momentum four-vector, and hence

$$\left[\gamma^{\mu}p_{\mu} - m\right]\psi = 0$$

Gamma matrices are 4x4 matrices and clearly the solution ψ must be a vector with four components. The solution ansatz is

$$\psi = u(\vec{p})e^{-ip^{\mu}x_{\mu}}$$

where $u(\vec{p})$ is called a bispinor. The equation becomes

$$\left[\gamma^{\mu}p_{\mu} - m\right]u(\vec{p}) = 0.$$

Remembering that this eq. was obtained from $\hat{H}\psi = \left(\vec{\alpha}\,\vec{p} + \beta m\right)\psi$,

the equation can actually be more obviously solved in its original form,

$$\hat{H}u(\vec{p}) = \left(\vec{\alpha}\,\vec{p} + \beta m\right)u(\vec{p}) = Eu(\vec{p})$$

We rewrite the equation in the form

$$\begin{bmatrix} 0 & \vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & 0 \end{bmatrix} u(\vec{p}) + \begin{bmatrix} m & 0 \\ 0 & -m \end{bmatrix} u(\vec{p}) = E u(\vec{p})$$

The bispinor can be written in a form of two components,

$$u(\vec{p}) = \begin{bmatrix} u_A \\ u_B \end{bmatrix}$$

, each of which (u_A, u_B) is called a spinor. Matrix quation

$$\begin{bmatrix} 0 & \vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & 0 \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} + \begin{bmatrix} m & 0 \\ 0 & -m \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} = E \begin{bmatrix} u_A \\ u_B \end{bmatrix}$$

yields two equations for spinors:

$$\vec{\sigma}\vec{p} u_B = (E - m) u_A$$
$$\vec{\sigma}\vec{p} u_A = (E + m) u_B$$

We need to be careful when dividing by *(E+m)* or *(E-m)* since these expression may equal 0. For example, in the rest frame of the particle *E=m* (obviously a solution with *E*>0). In this case we can take as two linearly independent solutions for u_A:

$$u_A^{(1)} = \chi^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_A^{(2)} = \chi^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and u_B is expressed from the second equation above:

$$u_B^{(s)} = \frac{\vec{\sigma}\vec{p}}{E+m}\chi^{(s)}$$

Linearly independent solutions for the bispinor with positive energy are thus

$$u^{(s)} = N\begin{bmatrix} \chi^{(s)} \\ \dfrac{\vec{\sigma}\vec{p}}{E+m}\chi^{(s)} \end{bmatrix}, s = 1,2, \quad \chi^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \chi^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with *N* denoting a normalization constant.

For soultions with *E*<0 one needs to be careful about (*E+m*), which in the rest frame of the particle is 0. Hence we take

$$u_B^{(s)} = \chi^{(s)}$$

and express u_A from the first equation,

$$u_A^{(s)} = \frac{\vec{\sigma}\vec{p}}{E-m}\chi^{(s)} = -\frac{\vec{\sigma}\vec{p}}{|E|+m}\chi^{(s)}$$

Solutions for the bispinor with $E<0$ are thus

$$u^{(s+2)} = N\begin{bmatrix} -\frac{\vec{\sigma}\vec{p}}{|E|+m}\chi^{(s)} \\ \chi^{(s)} \end{bmatrix}$$

In summary, the Dirac equation provides solutions for a particle in terms of a bispinors (4-component vectors). As in case of the Klein-Gordon equation we get solutions with $E>0$ ($u^{(1,2)}$) and solutions with $E<0$ ($u^{(3,4)}$), however, for each of the energy signs we get two linearly independent solutions.
Next question to be resolved is why there appears an additional two-fold degeneracy for each solution with given energy E.

2.6.2 Commutators of Hamiltonian and angular momentum

In order to shed light on the additional two-fold degeneracy of the Dirac equation solutions we should first investigate the commutator between the Hamiltonian and the orbital angular momentum operators:

$$\hat{H} = \vec{\alpha}\,\hat{\vec{p}} + \beta m, \quad \hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}, \quad \hat{p}_i = -i\frac{\partial}{\partial x_i}$$

$$\left[\hat{H}, \hat{L}_1\right] = \left[\vec{\alpha}\,\hat{\vec{p}} + \beta m, x_2\hat{p}_3 - x_3\hat{p}_2\right] = \left[\vec{\alpha}\,\hat{\vec{p}}, x_2\hat{p}_3 - x_3\hat{p}_2\right] + \left[\beta m, x_2\hat{p}_3 - x_3\hat{p}_2\right]$$

Since β is a constant (i.e. idependent of x_i or derivatives of thereof):

$$\left[\beta m, x_2\hat{p}_3 - x_3\hat{p}_2\right] = 0.$$

Furthermore

$$\left[x_i, \hat{p}_j\right]\psi = -i\left(x_i\frac{\partial\psi}{\partial x_i} - \frac{\partial}{\partial x_j}x_i\psi\right) = -i\left(x_i\frac{\partial\psi}{\partial x_i} - \left(\frac{\partial x_i}{\partial x_j}\right)\psi - \left(\frac{\partial\psi}{\partial x_j}\right)x_i\right) =$$

$$= i\psi\frac{\partial x_i}{\partial x_j} = i\psi\delta_{ij}$$

$$\left[\vec{\alpha}\hat{\vec{p}}, x_2\hat{p}_3 - x_3\hat{p}_2\right] = \left[\alpha_1\hat{p}_1 + \alpha_2\hat{p}_2 + \alpha_3\hat{p}_3, x_2\hat{p}_3\right] -$$
$$-\left[\alpha_1\hat{p}_1 + \alpha_2\hat{p}_2 + \alpha_3\hat{p}_3, x_3\hat{p}_2\right]$$

α_i are independent of x_i. p_1 and p_3 operators include derivatives over x_1 and x_3 and hence commute with x_2p_3. Hence all there is left from the fiorst commutator is

$$\alpha_2\left[\hat{p}_2, x_2\hat{p}_3\right] = \alpha_2\left(\hat{p}_2 x_2\hat{p}_3 - x_2\hat{p}_3\hat{p}_2\right)$$

p_3 commutes with p_2 and x_2 and hence

$$\alpha_2\left[\hat{p}_2, x_2\hat{p}_3\right] = -i\alpha_2\hat{p}_3$$

The second commutator yields $-\alpha_3\left[\hat{p}_3, x_3\hat{p}_2\right] = i\alpha_3\hat{p}_2$

and hence $\left[\hat{H}, \hat{L}_1\right] = -i\left(\vec{\alpha}\times\hat{\vec{p}}\right)_1$

In an analogous way other components can be checked leading to

$$\left[\hat{H}, \hat{\vec{L}}\right] = -i\left(\vec{\alpha}\times\hat{\vec{p}}\right).$$

The conclusion is that the Hamiltonian does not commute with the orbital angular momentum operator. The latter is thus not a good quantum number (see p. ???). This is a clear difference

between the Dirac equation on one and Schrödinger or Klein-Gordon equation on the other hand.
In the next step let‘s check the following commutator

$$\left[\hat{H},\vec{\Sigma}\right], \quad \vec{\Sigma}=\begin{bmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{bmatrix}.$$

Calculation for one of the components yields

$$\left[\vec{\alpha}\,\hat{\vec{p}},\Sigma_1\right]=\left[\alpha_1\hat{p}_1+\alpha_2\hat{p}_2+\alpha_3\hat{p}_3,\Sigma_1\right]$$

$$\left[\alpha_i,\Sigma_1\right]=\begin{bmatrix}0 & \left[\sigma_i,\sigma_1\right]\\ \left[\sigma_i,\sigma_1\right] & 0\end{bmatrix}$$

One can readily check the Pauli matrices commutation rules:

$$\left[\sigma_1,\sigma_2\right]=2i\sigma_3, \quad \left[\sigma_2,\sigma_3\right]=2i\sigma_1, \quad \left[\sigma_3,\sigma_1\right]=2i\sigma_2, \quad \sigma_i^2=1$$

From these we get

$$\left[\vec{\alpha}\,\hat{\vec{p}},\Sigma_1\right]=\left[\alpha_1\hat{p}_1+\alpha_2\hat{p}_2+\alpha_3\hat{p}_3,\Sigma_1\right]=0\cdot\hat{p}_1+$$

$$+\hat{p}_2\begin{bmatrix}0 & -2i\sigma_3\\ -2i\sigma_3 & 0\end{bmatrix}+\hat{p}_3\begin{bmatrix}0 & 2i\sigma_2\\ 2i\sigma_2 & 0\end{bmatrix}=$$

$$=2i(\vec{\alpha}\times\vec{p})_1$$

Other components yield $\left[\vec{\alpha}\,\hat{\vec{p}},\vec{\Sigma}\right]=2i(\vec{\alpha}\times\vec{p})$

In addition by explicit multiplication of matrices it can be shown $\left[\beta m,\vec{\Sigma}\right]=0$
and hence

$$\left[\hat{H},\vec{\Sigma}\right]=2i(\vec{\alpha}\times\vec{p})$$

Combining this result with the commutator for the orbital angular momentum we see

$$\left[\hat{H},\hat{\vec{J}}\right]=0 \quad \text{for} \quad \vec{J}=\vec{L}+\frac{1}{2}\vec{\Sigma}$$

A good quantum number is thus *J*, the sum of orbital angular momentum, and additional angular momentum, spin of the particle.

We can also check that another good quantum number is $\vec{\Sigma}\dfrac{\vec{p}}{p}$. It represents the projection of particle's spin to the direction of its momentum, called helicity.

Solutions of the Dirac equation have a positive or negative helicity,

$$\vec{\Sigma}\frac{\vec{p}}{p}\psi^{(1)} = +\psi^{(1)}, \;\; \vec{\Sigma}\frac{\vec{p}}{p}\psi^{(2)} = -\psi^{(2)}.$$

This can be checked by choosing the z-axis in the direction of particle's momentum,

$$\vec{p} = (0,0,p)$$

Then

$$\vec{\Sigma}\vec{p}\,\psi^{(1)} = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} pN \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{\vec{\sigma}\vec{p}}{E+m}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \sigma_3 & 0 \\ 0_3 & \sigma_3 \end{bmatrix} pN \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{\sigma_3 p}{E+m}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} =$$

$$= Np \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{p}{E+m}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = p\,\psi^{(1)}$$

This explains the two-fold solutions appearing in the Dirac equation: they differ by their helicity, for one solution particle's spin points in the direction of flight and for the second in the opposite direction.

2.6.3 Probability density and current in Dirac equation

Writing the covariant form of the Dirac equation and following similar steps as in Sec. ???? we can derive expressions for the probability density and current:

$$\left[i\gamma^{\mu}\partial_{\mu} - m\right]\psi = 0$$

$$i\gamma^{0}\frac{\partial\psi}{\partial t} + i\gamma^{k}\frac{\partial\psi}{\partial x_k} - m\psi = 0,\ k = 1,2,3$$

Hermitian conjugated (complex conjugated and transponsed) equation is

$$-i\frac{\partial\psi^{+}}{\partial t}\gamma^{0} - i\frac{\partial\psi^{+}}{\partial x_k}(-\gamma^{k}) - m\psi^{+} = 0,$$

where we used the property of gamma matrices $(\gamma^{0})^{+} = \gamma^{0}, (\gamma^{k})^{+} = -\gamma^{k}$.

Multiplying the above equation by γ^0 from the right we get

$$-i\frac{\partial\psi^{+}}{\partial t}\gamma^{0}\gamma^{0} + i\frac{\partial\psi^{+}}{\partial x_k}(-\gamma^{k}\gamma^{0}) - m\psi^{+}\gamma^{0} = 0$$

$$i\frac{\partial\psi^{+}}{\partial t}\gamma^{0}\gamma^{0} + i\frac{\partial\psi^{+}}{\partial x_k}(\gamma^{0}\gamma^{k}) + m\psi^{+}\gamma^{0} = 0$$

We now define an adjugated bispinor:

$$\overline{\psi} = \psi^{+}\gamma^{0}$$

and write the equation in the form

$$i\frac{\partial\overline{\psi}}{\partial t}\gamma^{0} + i\frac{\partial\overline{\psi}}{\partial x_{k}}\gamma^{k} + m\overline{\psi} = 0$$

$$i\partial_{\mu}\overline{\psi}\gamma^{\mu} + m\overline{\psi} = 0 \;/\cdot\psi \;\; from\ right$$

$$i(\partial_{\mu}\overline{\psi})\gamma^{\mu}\psi + m\overline{\psi}\psi = 0$$

The original Dirac equation we multiply by $\overline{\psi}$ from the left:

$$i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi = 0$$

Upon summation of the two equations we get

$$\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi + (\partial_{\mu}\overline{\psi})\gamma^{\mu}\psi = 0$$

$$\partial_{\mu}(\overline{\psi}\gamma^{\mu}\psi) = 0$$

The last equation has the form of the continuity equation (p. ???) and thus $j^{\mu} = \overline{\psi}\gamma^{\mu}\psi$.
The electromagnetic current (as discussed on p. ???) for a particle with charge $-e_0$ (electron) is

$$j^{\mu} = -e_{0}\overline{\psi}\gamma^{\mu}\psi$$

2.6.4 Interaction of a Dirac particle with electromagnetic field

Electromagnetic potential is included into the Dirac equation by replacing the four-vector of derivatives ∂_μ by the covariant derivative :

$$D_\mu = \partial_\mu - ieA_\mu \ ,$$

where A_μ is the four-vector of potential, $A^\mu = (A_0, \vec{A})$, with Maxwell equations

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A_0, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

The reason for the inclusion of the potential by replacing $\partial_\mu \to D_\mu$ is non-trivial and is beyond the scope of these lectures. Let us only say that the Lagrangian (which through the Euler-Lagrange equations leads to the Dirac equation) must be invariant to the so called local gauge transformations of the type $\psi \to \psi e^{i\alpha(x)}$. $\alpha(x)$ is an unobservable phase and hence the Lagrangian must be invariant to such transformations. It turns out this is only possible if one includes the electromagnetic potential in the way described above.
Inclusion of the covariant derivative leads to

$$\left[i\gamma^\mu (\partial_\mu - ieA_\mu) - m \right] \psi = 0$$

$$\left[i\gamma^{\mu}\partial_{\mu} - m\right]\psi = -e\gamma^{\mu}A_{\mu}\psi \equiv \gamma^{0}V\psi$$

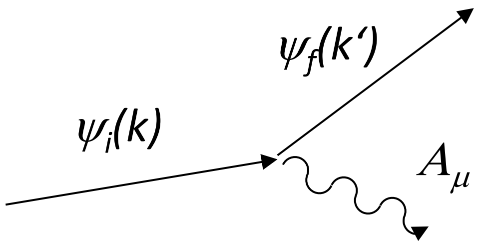
How do we know that the potential *V* should be defined as shown above? The first term on the left hand side of the equation, i.e. $i\gamma^0 \partial_0$, is just $\gamma^0 E$. Hence also the right hand equation should (beside the kinetic energy term) include the potential with the same sign and multiplied by γ^0.

The potential is
$$\gamma^0 V = -e\gamma^{\mu}A_{\mu}$$
$$V = -e\gamma^0\gamma^{\mu}A_{\mu}$$

After inclusion of the electromagentic potential we can now see how the particles, described by the Dirac equation, interact through the electromagnetic interaction. The matrix element appearing in the Fermi golden rule (see p. ???) for the transition from an initial state ψ_i to a final state ψ_f is written as

$$T_{fi} = -i\int \psi_f^{+}(k',x)V\psi_i(k,x)d^4x = ie\int \psi_f^{+}\gamma^0\gamma^{\mu}A_{\mu}\psi_i d^4x =$$
$$= -i\int\left(-e\overline{\psi}_f\gamma^{\mu}\psi_i\right)A_{\mu}d^4x =$$
$$= -i\int j_{fi}^{\mu}A_{\mu}d^4x$$

In the last line we used the electromagnetic current following from the Dirac equations as derived on p. ???. The elctromagnetic current thus appears in the matrix element.
Note that the inclusion of *-i* factor in the definition od T_{fi} is conventional. The above matrix element represents the following process depicted by a Feynman diagram:



Straight lines represent a Dirac particle, and the wiggly line is the electromagnetic field (a photon). The latter must originate from a source. Source can be for example another charged Dirac particle. It also has to obey the Maxwell equations. The latter can be written in a covariant form as

$\partial^{\nu}\partial_{\nu}A_{\mu} = j_{\mu}$ (see App. A), where j_{μ} is the electromagnetic current representing the source of the electromagnetic field. In case this is another charged particle

$$j_{\mu} = -e\overline{\psi}(p')\gamma_{\mu}\psi(p) = -e\overline{u}(p')e^{ip'x}\gamma_{\mu}u(p)e^{-ipx} =$$
$$= -e\overline{u}(p')\gamma_{\mu}u(p)e^{iqx}$$
$$q = p' - p$$

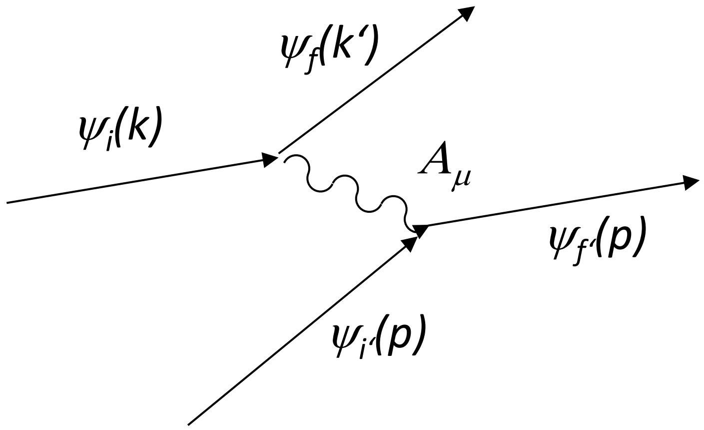
We see that the Maxwell equation is satisfied if $A_{\mu} = -\dfrac{1}{q^2}j_{\mu}$

Homework 8: prove the above statement.

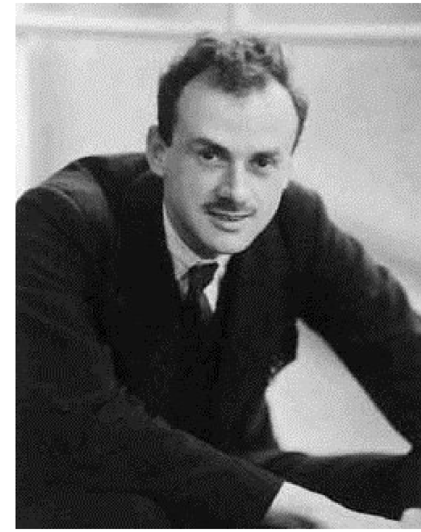
Inserting A_μ into the expression for the matrix element we get

$$T_{fi} = -i \int j_{fi}^{\mu} \left(-\frac{1}{q^2} \right) j_{\mu}^{f'i'} d^4 x \ ,$$

which describes electromagnetic scattering of two Dirac particles



Paul Dirac (1902 – 1984)



The Quantum Theory of the Electron.

By P. A. M. Dirac, St. John's College, Cambridge.

(Communicated by R. H. Fowler, F.R.S.—Received January 2, 1928.)

The new quantum mechanics, when applied to the problem of the structure of the atom with point-charge electrons, does not give results in agreement

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the wave function ψ being a function of x_1, x_2, x_3, t. This gives rise to two difficulties.

The first is in connection with the physical interpretation of ψ. Gordon, and also independently Klein,† from considerations of the conservation theorems,

~

and p_3. Our wave equation is therefore of the form

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\,\psi = 0, \qquad (4)$$

Nobel prize for physics in 1933 with E. Schrödinger „for the discovery of new productive forms of atomic theory„.

P.A.M. Dirac, Proceedings of the Royal Society A, 117, 610 (1928)

Next we will check how does the electromagnetic interaction described by the Dirac equation reflects in the low energy (non-relativistic) limit.

The equation including the EM potential si $\left[\gamma^{\mu} p_{\mu} + e\gamma^{\mu} A_{\mu} - m\right]\psi = 0$

Inserting $\gamma^{\mu} = \left(\beta, \beta\vec{\alpha}\right), \quad A_{\mu} = (A_0, \vec{A})$ we obtain

$$\left[\beta E - \beta\vec{\alpha}\,\vec{p} + e\beta A^0 - e\beta\vec{\alpha}\,\vec{A} - m\right]u = 0 \,/\cdot\, \beta \ from\ left$$

$$\left[E - \vec{\alpha}\,\vec{p} + eA^0 - e\vec{\alpha}\,\vec{A} - \beta m\right]u = 0$$

$$\left[\vec{\alpha}\left(\vec{p} + e\vec{A}\right) - eA^0 + \beta m\right]u = Eu$$

$$\begin{bmatrix} m - eA^0 & \vec{\sigma}\left(\vec{p} + e\vec{A}\right) \\ \vec{\sigma}\left(\vec{p} + e\vec{A}\right) & -m - eA^0 \end{bmatrix}\begin{bmatrix} u_A \\ u_B \end{bmatrix} = E\begin{bmatrix} u_A \\ u_B \end{bmatrix}$$

We get similar equations for spinors u_A and u_B as on p. ??? but now with the inclusion of the EM potential:

$$\vec{\sigma}(\vec{p}+e\vec{A})u_B = (E-m+eA^0)u_A$$

$$\vec{\sigma}(\vec{p}+e\vec{A})u_A = (E+m+eA^0)u_B \quad /\cdot(E+m+eA^0)^{-1}$$

$$\vec{\sigma}(\vec{p}+e\vec{A})u_B = (E-m+eA^0)u_A$$

$$(E+m+eA^0)^{-1}\vec{\sigma}(\vec{p}+e\vec{A})u_A = u_B$$

Inserting the expression for u_B into the first equation we get

$$\vec{\sigma}(\vec{p}+e\vec{A})(E+m+eA^0)^{-1}\vec{\sigma}(\vec{p}+e\vec{A})u_A = (E-m+eA^0)u_A$$

Low energy limit implies

$$\left.\begin{matrix} m >> p \\ m >> eA^0 \end{matrix}\right\} \Rightarrow \begin{cases} E + m + eA^0 \approx 2m \\ E - m + eA^0 = E_{nr} + eA^0 \end{cases}$$

where E_{nr} denotes the non-relativistic energy of the particle. In this limit the equation for u_A becomes

$$\frac{1}{2m}\vec{\sigma}(\vec{p} + e\vec{A})\vec{\sigma}(\vec{p} + e\vec{A})u_A = (E_{nr} + eA^0)u_A \quad .$$

To simplify the notation let us use $\vec{p}' = \vec{p} + e\vec{A}$. For any vector pair $\vec{a}, \vec{b}$

$(\vec{\sigma}\vec{a})(\vec{\sigma}\vec{b}) = \vec{a}\vec{b} + i\vec{\sigma}(\vec{a} \times \vec{b})$, which follows from properties of Pauli matrices. Hence

$(\vec{\sigma}\vec{p}')(\vec{\sigma}\vec{p}') = \vec{p}'\vec{p}' + i\vec{\sigma}(\vec{p}' \times \vec{p}')$. For ordinary 3-dimensional vectors $\vec{p}' \times \vec{p}' = 0$.

However, *p'* should be regarded as an operator (involving derivatives) and hence some care must be taken in deriving the expression.

$$(\vec{p}' \times \vec{p}')u_A = (\vec{p} + e\vec{A}) \times (\vec{p} + e\vec{A})u_A = \underbrace{(\vec{p} \times \vec{p})u_A}_{=0} +$$

$$+ \left[e\vec{p} \times \vec{A} + e\vec{A} \times \vec{p}\right]u_A + e^2 \underbrace{(\vec{A} \times \vec{A})}_{=0} u_A$$

$$\left[e\vec{p}\times\vec{A}+e\vec{A}\times\vec{p}\right]u_A=-ie\left[\underbrace{\vec{\nabla}\times(\vec{A}u_A)}_{=(\vec{\nabla}\times\vec{A})u_A-\vec{A}\times(\vec{\nabla}u_A)}+\vec{A}\times(\vec{\nabla}u_A)\right]=-ie(\vec{\nabla}\times\vec{A})u_A=$$

$$=-ie\vec{B}u_A$$

This reduces the equation for u_A to

$$\frac{1}{2m}\vec{\sigma}(\vec{p}\,')\vec{\sigma}(\vec{p}\,')u_A=(E_{nr}+eA^0)u_A$$

$$\frac{1}{2m}\left[p'^2+e\vec{\sigma}\vec{B}\right]u_A=(E_{nr}+eA^0)u_A$$

$$\underline{\underline{\left[\frac{(\vec{p}+e\vec{A})^2}{2m}+\frac{e}{2m}\vec{\sigma}\vec{B}-eA^0\right]}}u_A=E_{nr}u_A$$

The underlined factor is nothing but the clasical Hamiltonian for an electron in an electromagnetic field (see App. B). The term $-eA^0$ is the electrostatic potential energy, and $\vec{\sigma}\vec{B}$ is the classic interaction of a magnetic dipole moment with the magnetic field. More precisely, the term $\frac{e}{2m}\vec{\sigma}\vec{B}$ represents a classic interaction of a magnetic dipole moment, classically written as $-\vec{\mu}\vec{B}$, with $\vec{\mu}$ representing the dipole moment.

The dipole moment of a particle described by the Dirac equation is thus

$$\frac{e}{2m}\vec{\sigma}\vec{B} = -\vec{\mu}\vec{B}$$

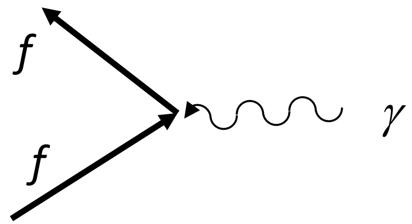
$$\vec{\mu} = -\frac{e}{2m}\vec{\sigma} = -\frac{e}{2m} g_s \frac{\vec{\sigma}}{2}$$

where we intorduced the spin gyromagnetic ratio g_s =2 (see p. ???). From the commutator between the Hamiltonian and the angular momentum (p. ???) we know that $\vec{\sigma}/2$ is the operator of spin, and hence

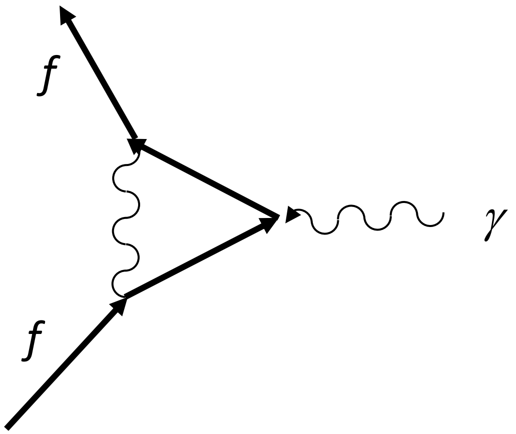
$$\vec{\mu} = -\frac{e}{2m} g_s \vec{s}$$

with g_s =2 which represents a big success of the Dirac equation.

It should be noted, however, that $g_s=2$ is the result of the lowest order in perturbation theory. In other words, the result following from the Dirac equation represents the electromagnetic interaction of a fermion with a single vertex:



There are higher order corrections to this interaction, represented by Feynman digrams involving more vertices (also called the loop diagrams), for example:



This and similar processes cause the g_s of a given fermion to slightly deviate from 2. This deviations represent on of the most thorough tests of calculations within the quantum electrodynamics (involving calculations of various loop diagrams as the one shown in the figure) and are compared to one of the most precise measurements in particle physics – those of the so called anomalous magnetic moment of the muon – the observable by which one means $(g_s(\mu) - 2)/2$. The principle of this measurement is described in the following.

Schematics of the muon anomalous magnetic moment measurement:

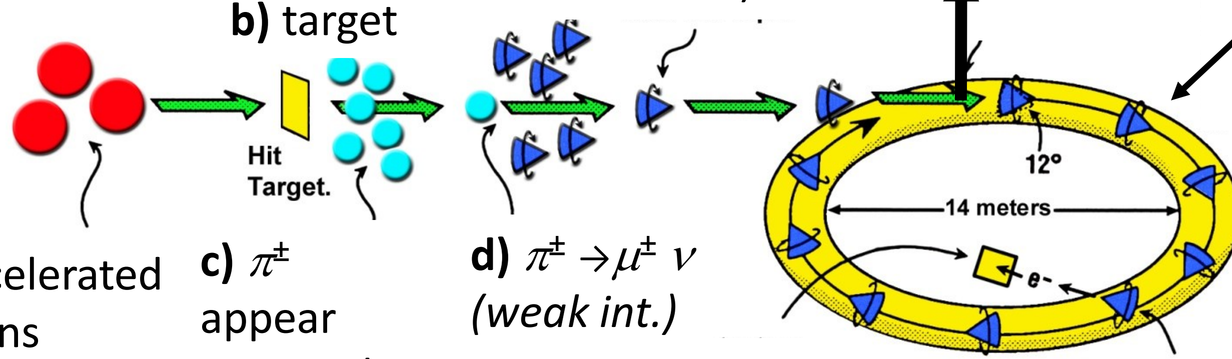
g) μ^- circulate in a magnetic field the direction of which is $\perp$ to their spin (magn. dipole moment)

1)

e) μ^- spin points in the directiction of its momentum (this is a property of weak int.)

B

b) target



a) accelerated protons

c) $\pi^\pm$ appear among the products

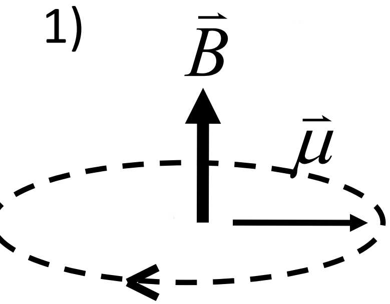
d) $\pi^\pm \rightarrow \mu^\pm \, \nu$ *(weak int.)*

h) detectors of $e^\pm$ from the $\mu^\pm \rightarrow e^\pm \, \nu$ decays

2)

f) accumulation ring; because of the magnetic filed μ circulate within the ring

1)



μ magn. moment (spin) precesses around the direction of the external magnetic field;

$\omega_p \propto g_s\text{-}2$

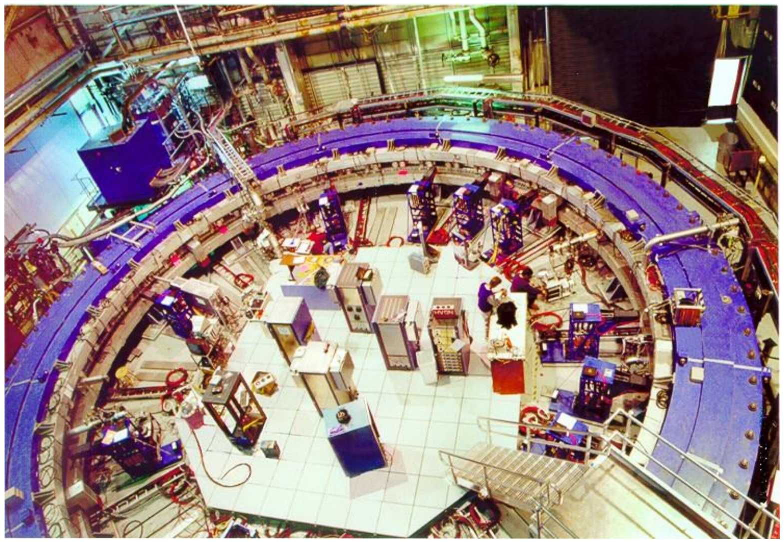
2)

in $\mu^\pm \rightarrow e^\pm \, \nu$ decay $e^\pm$ fly preferentially in the direction of the μ spin (another property of weak int.)

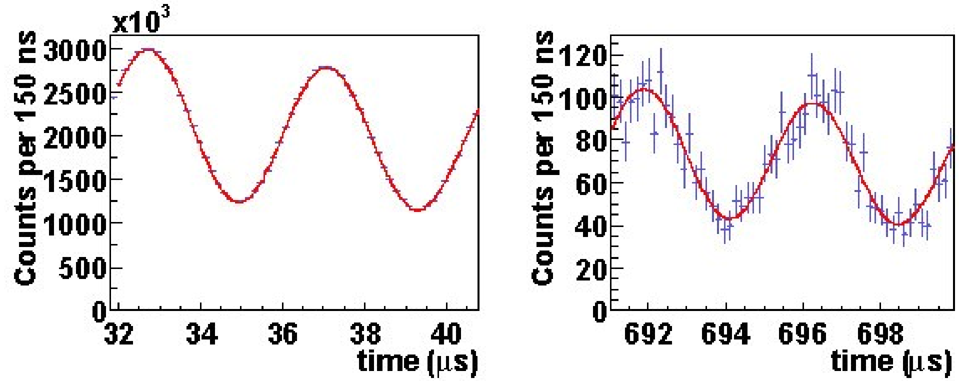
number of detected $e^\pm$ along the ring is proportional to $\mu^\pm$ average direction and by this to ω_p

Results of the measurement:

accumulation ring
(Brookhaven National Laboratory)



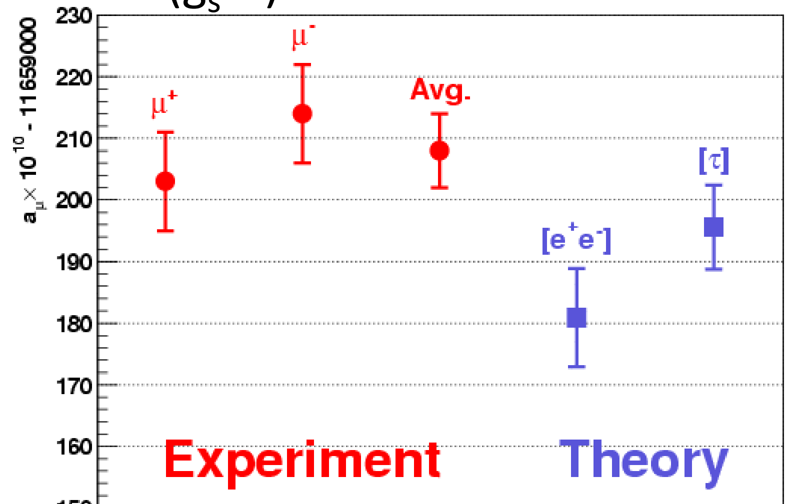
number of detected $e^{\pm}$ after time t following the $\mu^{\pm}$ injection



(g_s-2)=11,659214 x 10^{-4} $(1\pm 7\text{x}10^{-7})$

some discrepancy between theory and measurement is still present

$(g_s\text{-}2)\text{x } 10^{10}\text{-}11659000$



2.6.5 Spinor normalization and completness relations

In discussion about the probability density following from the Klein-Gordon equation (p. ???) we argued it is correct to normalize a wave function describing a relativistic particle in such a way as to have *2E* particles in an arbitrary normalization volume *V*. Since the Dirac equation also describes relativistic particles one must follow the same prescritpion also here,

$$\rho = \overline{\psi}\gamma^0\psi = \psi^+\psi$$

$$\int\limits_{norm.\,volume\,V} \psi^+\psi dV = u^+u = 2E$$

$$u^{(s)} = N\begin{bmatrix} \chi^{(s)} \\ \dfrac{\vec{\sigma}\vec{p}}{E+m}\chi^{(s)} \end{bmatrix}$$

$$u^+u = |N|^2 \begin{bmatrix} \chi^{(s)T} & \left[\dfrac{\vec{\sigma}\vec{p}}{E+m}\chi^{(s)}\right]^T \end{bmatrix} \begin{bmatrix} \chi^{(s)} \\ \dfrac{\vec{\sigma}\vec{p}}{E+m}\chi^{(s)} \end{bmatrix} =$$

$$= |N|^2 \left[1 + \frac{p^2}{(E+m)^2}\right] = |N|^2 \frac{2E}{E+m} = 2E \Rightarrow$$

$$\Rightarrow |N| = \sqrt{E+m}$$

We wrote the Dirac equation in a form

$$\left[\gamma^{\mu} p_{\mu} - m\right]\psi = 0$$

It is custom to use another shorthand notation for any four-vector multiplied by the foour-vector of gamma matrices:

$$\gamma^{\mu} a_{\mu} \equiv \not{a}$$

and the Dirac equation is sometimes written in this form

$$\left[\not{p} - m\right]\psi = 0$$

Using the properties of the bispinors we can check (see App. C) that the following identity is satisfied

$$\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p} + m$$

The equation is called the completness relation.