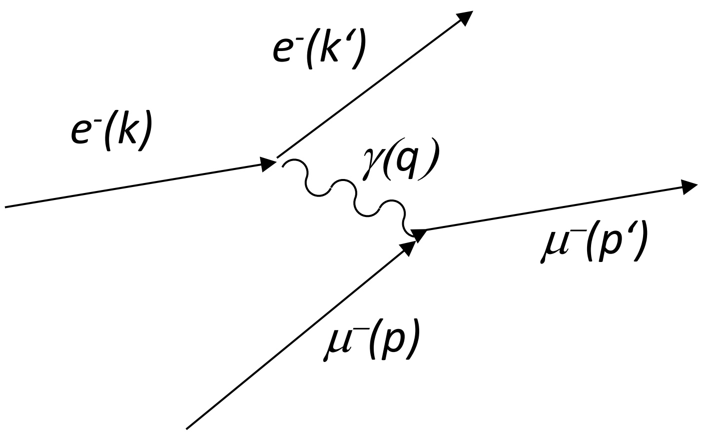
2.7 Electromagnetic interaction of Dirac particles

2.7.1 $e^- \mu^- \to e^- \mu^-$ scattering

Knowing that the Dirac equation describes relativistic particles with spin ½ let us calculate the cross section for the electromagnetic interaction between two such particles, an electron and a muon. The Feynman diagram of such a process is



and the matrix element is $$T_{fi} = -i \int j_\nu^e \left(-\frac{1}{q^2} \right) j_\mu^\nu d^4 x$$

where j_ν^e and j_μ^ν denote the electromagentic current of the electron and muon, respectively:

$$j_\nu^e = -e\overline{u}(k')\gamma_\nu u(k) e^{i(k'-k)x}$$
$$j_\mu^\nu = -e\overline{u}(p')\gamma^\nu u(p) e^{i(p'-p)x}$$

Inserting the currents into the matrix element expression we obtain

$$T_{fi} = -ie^2 \overline{u}(k')\gamma_\nu u(k)\left(-\frac{1}{q^2}\right)\overline{u}(p')\gamma^\nu u(p)\int e^{i(k'+p'-k-p)x} d^4x =$$

$$= -i\left[-e\overline{u}(k')\gamma_\nu u(k)\right]\left(-\frac{1}{q^2}\right)\left[-e\overline{u}(p')\gamma^\nu u(p)\right](2\pi)^4 \delta^4(k'+p'-k-p)$$

where in the last line we used the definition of a 4-dimensional delta function (the latter is just a consequence of energy and momentum conservation in the process). It is again custom to separate the delta function out of the matrix element by defining the amplitude for the process $\mathfrak{M}$ as

$$T_{fi} = -(2\pi)^4 \delta^4(k'+p'-k-p)\mathfrak{M}$$

with

$$-i\mathrm{M} = \left[-e\overline{u}(k')\gamma^\kappa u(k)\right]\left(-\frac{g^{\kappa\nu}}{q^2}\right)\left[-e\overline{u}(p')\gamma^\nu u(p)\right]$$

Before proceeding with the calculation we need to determine what kind of the cross section we wold like to determine. The involved particles carry spin. Quite often the spin orientation (which in principle can be measured, i.e. one can distinguish between positive and negative helicity states) of particles is not measured. In this case one talks about the unpolarized cross section. It is defined as

$$\left|\overline{\mathfrak{M}}\right|^2 = \frac{1}{(2s_a+1)(2s_b+1)} \sum_{\substack{\text{allspin} \\ \text{orientations}}} \left|\mathfrak{M}\right|^2$$

Factors *(2s$_i$+1)* represent possible spin states of the incoming particles (*a* and *b*). In the unpolarized cross section one averages over those possible spin orientations. For a particle of spin ½ this factor equals 2 (two possible spin orientations). The sum in the expression runs over all possible spin orientations of the spinors involved in the amplitude $\mathfrak{M}$. It should be noted that the sum runs over amplitudes squared whch is a consequence of the fact that in principle the spin orientations can be measured. The sum involves currents, for example the electron current $\overline{u}(k')\gamma^\kappa u(k)$ which in the amplitude squared enters twice:

$$\left|\mathfrak{M}\right|^2 \propto \left[\overline{u}(k')\gamma^\kappa u(k)\right]\left[\overline{u}(k')\gamma^\sigma u(k)\right]^+$$

Note that the indices of the gamma matrices are different; the first gamma four-vector is multiplied by the corresponding four-vector in the muon current and analogously the second one. Writing out the current product above

$$\left[\overline{u}(k')\gamma^{\kappa}u(k)\right]\left[\overline{u}(k')\gamma^{\sigma}u(k)\right]^{+}=\left[\overline{u}(k')\gamma^{\kappa}u(k)\right]\left[u^{+}(k)\underbrace{\gamma^{\sigma+}\gamma^{0+}}_{=\gamma^{\sigma+}\gamma^{0}=\gamma^{0}\gamma^{\sigma}}u(k')\right]=$$

$$=\left[\overline{u}(k')\gamma^{\kappa}u(k)\right]\left[u^{+}(k)\gamma^{0}\gamma^{\sigma}u(k')\right]=\left[\overline{u}(k')\gamma^{\kappa}u(k)\right]\left[\overline{u}(k)\gamma^{\sigma}u(k')\right]$$

The sum over spin orientations implies

$$\sum_{s,s'=1,2}\left[\overline{u}^{(s')}(k')\gamma^{\kappa}u^{(s)}(k)\right]\left[\overline{u}^{(s)}(k)\gamma^{\sigma}u^{(s')}(k')\right]=$$

$$=\sum_{s,s'=1,2}\left[\overline{u}_{\alpha}{}^{(s')}(k')\gamma_{\alpha\beta}{}^{\kappa}u_{\beta}{}^{(s)}(k)\right]\left[\overline{u}_{\delta}{}^{(s)}(k)\gamma_{\delta\varepsilon}{}^{\sigma}u_{\varepsilon}{}^{(s')}(k')\right]$$

where in the last line we explicitly wrote out the components of the spinors and gamma matrices to be multiplied (sum over the repeated indicies is implied). Each of the factors $u_i^{(s)}$ and $\gamma_{ij}{}^{\kappa}$ is now a simple scalar and their products are commutative. Hence we can move the last factor $u_{\varepsilon}{}^{(s')}(k')$ to the beginning of the product thus obtaining

$$\sum_{s,s'=1,2}\underbrace{u_{\varepsilon}{}^{(s')}(k')\overline{u}_{\alpha}{}^{(s')}(k')}_{\not{k}'+m_e}\gamma_{\alpha\beta}{}^{\kappa}\underbrace{u_{\beta}{}^{(s)}(k)\overline{u}_{\delta}{}^{(s)}(k)}_{\not{k}+m_e}\gamma_{\delta\varepsilon}{}^{\sigma}$$

The notation below the line denotes what we obtain by applying the completness relation, with m_e denoting the mass of electron. The sum is thus

$$\left(\not{k}'+m_e\right)_{\varepsilon\alpha}\gamma_{\alpha\beta}{}^{\kappa}\left(\not{k}+m_e\right)_{\beta\delta}\gamma_{\delta\varepsilon}{}^{\sigma}$$

Examining the matrix indices we realize that the above product is just the trace of the expression,

$$Tr\left[\left(\not{k}'+m_e\right)\gamma^{\kappa}\left(\not{k}+m_e\right)\gamma^{\sigma}\right]$$

The same can of course be obtained for the other (muon) current in the spin averaged amplitude. The latter reads

$$\left|\overline{\mathfrak{M}}\right|^2=\frac{1}{2\cdot 2}\frac{e^4}{q^4}Tr\left[\left(\not{k}'+m_e\right)\gamma^{\kappa}\left(\not{k}+m_e\right)\gamma^{\sigma}\right]Tr\left[\left(\not{p}'+m_\mu\right)\gamma_{\kappa}\left(\not{p}+m_\mu\right)\gamma_{\sigma}\right]$$

It may be of some comfort to know that once we are aware of this result it is not necessary to repeat the derivation each time when calculating amplitudes for various processes. Already from the form of the amplitude $\mathfrak{M}$ on p. ??? we can directly guess the expression for the spin averaged amplitude above.

In proceeding with the calculation of $\left|\overline{\mathfrak{M}}\right|^2$ we use some known identities in calculation of traces without the need for explicit matrix multiplication. This identities are called the trace theorems.

Specifically for the above example the following trace theorem can be used:

$$Tr\left[(\not{k}'+m_e)\gamma^\kappa(\not{k}+m_e)\gamma^\sigma\right]=4\left[k'^\kappa k^\sigma + k'^\sigma k^\kappa - (k'\cdot k - m_e^2)g^{\kappa\sigma}\right]$$

Upon using the same theorem for the traces of the muon current we obtain

$$\left|\overline{\mathfrak{M}}\right|^2 = \frac{8e^4}{q^4}\left[(k'p')(kp)+(k'p)(kp')-m_e^2 p'p - m_\mu^2 k'k + 2m_e^2 m_\mu^2\right]$$

The most difficult part of the cross section calculation is by this accomplished. We obtained the spin averaged amplitude expressed in terms of four-momenta products (it should be noted that the products of four-vectors are Lorentz invariant).

To obtain the cross section from the spin averaged amplitude we need to add a few further factors. We defined the differential cross section (p. ???) as

$$\frac{d\sigma}{d\Omega} = \frac{dW_{fi}/d\Omega}{\rho_i v_i}, \quad \frac{dW_{fi}}{d\Omega} = \frac{2\pi}{\hbar}\left|T_{fi}\right|^2 \frac{d\rho_f}{d\Omega}$$

Density of final states ρ_f was obtained from

$$d^3N = V\frac{d^3p}{(2\pi\hbar)^3} = \frac{1}{\rho}\frac{d^3p}{(2\pi\hbar)^3}$$

where we used $\rho = 1/V$ to denote the probability density in Schrödinger equation.

The density of final states for relativistic particles must be written using

$$d^3N = \frac{V}{2E}\frac{d^3p}{(2\pi\hbar)^3}$$, taking into account the normalization to *2E* particles in volume *V*.

ρ_f is proportional to d^3p/E. A differential Lorentz transformation (in *x* direction) is

$$\begin{aligned} dp_x' &= \gamma\left(dp_x - \beta dE\right) \\ dp_y' &= dp_y, \quad dp_z' = dp_z \\ dE' &= \gamma\left(dE - \beta dp_x\right) \end{aligned}$$

and the Lorent transformed d^3p/E factor is

$$\frac{d^3p'}{E'} = \frac{\gamma\left(dp_x - \beta dE\right)dp_y dp_z}{\gamma\left(E - \beta p_x\right)} = \frac{dp_x\left(1 - \beta dE / dp_x\right)}{E - \beta p_x} dp_y dp_z =$$

$$= \frac{dp_x\left(1 - \beta E / p_x\right)}{E\left(1 - \beta p_x / E\right)} dp_y dp_z = \frac{d^3p}{E}$$

where we used $E = \sqrt{(p_x^2+p_y^2+p_z^2+m^2)}$ and hence $dE/d^3p_x = p_x/E$. The density of final states proportional to d^3p/E is Lorentz invariant.

The last factor needed for the cross section is the density of incoming particles, $\rho_i v_i$. If the initial particle *a* is moving and particle *b* is resting (in a target) then

$$\rho_i v_i = \underbrace{\frac{2E_a}{V} v_a}_{\substack{current\,density\\ of\ particles\ a}} \quad \underbrace{\frac{2E_b}{V}}_{\substack{density\ of\\ target\ particles}}$$

If both initial state particles are moving, then

$$\rho_i v_i \equiv F = \frac{2E_a}{V}\frac{2E_b}{V}\left|\vec{v}_a - \vec{v}_b\right|$$

Taking into account $\frac{v}{c} = \beta = \frac{\gamma m v}{\gamma m c} = \frac{\gamma m v c}{\gamma m c^2} = \frac{cp}{E}$ the velocity difference can be written as

$$\vec{v}_a - \vec{v}_b = \frac{\vec{p}_a E_b - \vec{p}_b E_a}{E_a E_b} \qquad \text{and} \qquad F \propto \left|\vec{p}_a E_b - \vec{p}_b E_a\right| = \sqrt{\left(p_a p_b\right)^2 - m_a^2 m_b^2}$$

The latter expression can be written in explicitly Lorentz invariant form as shown.

The differential cross section can thus be written as a product of Lorentz invariant factors,

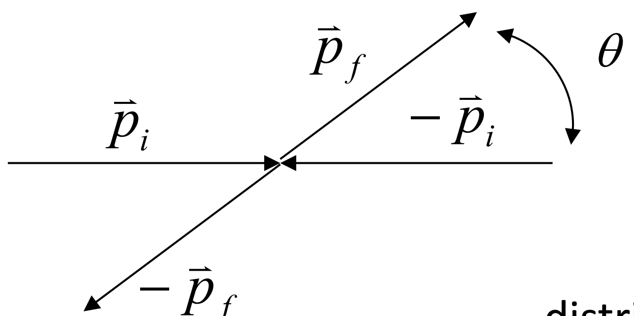
$$d\sigma = \frac{|\mathfrak{M}|^2}{F} dQ$$, with individual factors for the process $a\, b \to c\, d$ written as

$$F = 4\sqrt{(p_a p_b)^2 - m_a^2 m_a^2}$$

$$dQ = (2\pi)^4 \delta^4 (k' + p' - k - p) \frac{d^3 p_c}{(2\pi)^3 2E_c} \frac{d^3 p_d}{(2\pi)^3 2E_d}$$

$$-i\mathfrak{M} = \left[j_{ca}^{\mu}\right]\left[-i\frac{g^{\mu\nu}}{q^2}\right]\left[j_{db}^{\nu}\right]$$

The above ingredients of the differential cross section take specifically compact form if written in the center-of-mass frame (CMS) of the initial and final state particles:



$$\vec{p}_a = \vec{p}_i = -\vec{p}_b$$
$$\vec{p}_c = \vec{p}_f = -\vec{p}_d$$

If we are interested in the angular distribution of final state particles we can write (note that p_i and p_f are not 4-vectors but the magnitudes of the corresponding 3-momenta):

$$d^3 p_c = p_f^2 dp_f d\Omega, \quad d\Omega = 2\pi \sin\theta d\theta$$

and integrate over d^3p_d:

$$\int \frac{d^3 p_d}{2E_d} \underbrace{\delta^4(p_c + p_d - p_a - p_b)}_{=\delta(E_c+E_d-E_a-E_b)\delta^3(\vec{p}_c+\vec{p}_d-\vec{p}_a-\vec{p}_b)} = \frac{1}{2E_d}\delta(E_c + E_d - E_a - E_b)$$

Denoting the CMS collission energy by E $(=E_a+E_b)$ we have

$$dQ = \frac{1}{4\pi^2}\frac{p_f^2 dp_f d\Omega}{4E_c E_d}\delta(E_c + E_d - E)$$

$$E = E_c + E_d = \sqrt{p_f^2 + m_c^2} + \sqrt{p_f^2 + m_d^2}$$

$$\frac{dE}{dp_f} = \frac{p_f}{E_c} + \frac{p_f}{E_d}$$

$$dQ = \frac{1}{4\pi^2}\frac{p_f}{4(E_c + E_d)} d\Omega dE \delta(E_c + E_d - E)$$

Upon integration over the energy E we get

$$dQ = \frac{1}{4\pi^2}\frac{p_f}{4E}d\Omega$$

Substitution of p_a and p_b expressed in terms of p_i into the expression for F yields

$$F = 4p_i E$$

The differential cross section in CMS is

$$\frac{d\sigma}{d\Omega dE} = \frac{|\mathfrak{M}|^2 p_f}{64\pi^2 p_i (E_c + E_d)^2}\delta(E_c + E_d - E)$$

and the trivial integration over E (because of the delta function) yields

$$\frac{d\sigma}{d\Omega} = \frac{|\mathfrak{M}|^2 p_f}{64\pi^2 p_i E^2}$$

In the ultrarelativistic limit $m_x << p_x$ and $p_i = p_f$,

$$\frac{d\sigma}{d\Omega} = \frac{|\mathfrak{M}|^2}{64\pi^2 E^2}$$

Returning to our example of the $e^-\mu^- \to e^-\mu^-$ scattering, we can now write the differential cross sectiomn in the CMS in the ultrarelativistic limit :

$$k=(E/2,\vec{p}_i),\quad k'=(E/2,\vec{p}_f),\quad p=(E/2,-\vec{p}_i),\quad p'=(E/2,-\vec{p}_f)$$

$$k'p=(E^2/4)+\vec{p}_i\vec{p}_f=(E^2/4)(1+\cos\theta)$$

$$kp=(E^2/4)+\vec{p}_i\vec{p}_i=E^2/2$$

$$q^2=(k'-k)^2=(0,p_f-p_i)^2=-(E^2/2)(1-\cos\theta)$$

$$\frac{d\sigma}{d\Omega}=\frac{e^4}{32\pi^2E^2}\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}$$

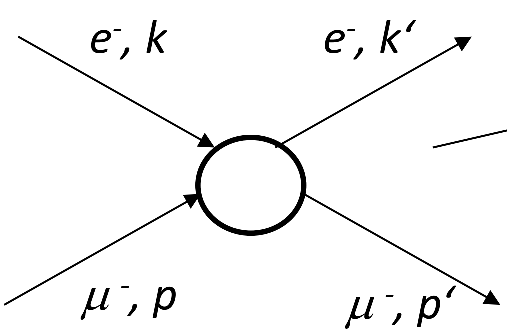
In rewriting the expression to obtain the units m^2 as expected for the cross section one should take into account $\alpha = e^2/4\pi\hbar c$ to obtain

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2(\hbar c)^2}{2E^2}\frac{4+(1+\cos\theta)^2}{(1-\cos\theta)^2}$$

2.7.2 Crossing, $e^- e^+ \to \mu^- \mu^+$

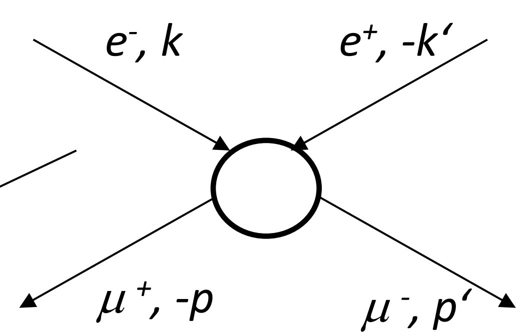
Having calculated the differential cross-section for the $e^- \mu^- \to e^- \mu^-$ scattering, it is easy to obtain cross sections for some related processes involving the anti-particles. The procedure to do so is called the crossing. Let's sketch the original process $e^- \mu^- \to e^- \mu^-$ on the left-hand side, identifying the four-vectors of individual incoming (p_a, p_b) and outgoing (p_c, p_d) particles. Now we can replace one electron (e^-) by its anti-particle positron (e^+). In doing so in accordance with the Feynman- Stückelberg interpretation (see p. ??) the four-vector of the particle reverses its sign (i.e. $k' \to -k'$). This includes the reversal of the three-momentum implying an outgoing particle becoming an ingoing one and vice-versa. We repeat the same procedure for one of the muons. The sketch of the crossed process which we get by this ($e^- e^+ \to \mu^- \mu^+$) is shown on the right-hand side.

$e^- \mu^- \to e^- \mu^-$



incoming		outgoing	
p_a	p_b	p_c	p_d
k	p	k'	p'
k	$-k'$	$-p$	p'

$e^- e^+ \to \mu^- \mu^+$



By comparing the four-vectors of the incoming and outgoing particles for the original and the crossed process we see that the only difference between the two is the replacement $k' \leftrightarrow -p$. Hence we can get the cross-section for the crossed process using the calculated cross-section of the original process and performing the mentioned transformation. The amplitude for $e^- e^+ \to \mu^- \mu^+$) is thus (see p. ???):

$$\left|\overline{\mathfrak{M}}\right|^2 = \frac{8e^4}{q^4}\left[(-pp')(-kk') + (k'p)(kp') + m_e^2 p'k' + m_\mu^2 pk + 2m_e^2 m_\mu^2\right]$$

In the ultrarelativistic limit and in the CMS, by using the four-vectors

$$k = (\frac{E}{2}, \vec{p}_i), \quad k' = (\frac{E}{2}, \vec{p}_f), \quad p = (\frac{E}{2}, -\vec{p}_i), \quad p' = (\frac{E}{2}, -\vec{p}_f)$$

(the same as for the original process), and $$q^2 = (k'-k)^2 \underbrace{\rightarrow}_{\text{after crossing}} (p+k)^2 = E^2$$

we get $$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 (\hbar c)^2}{4E^2}(1+\cos^2\theta)$$

Integration of the differential cross-section yields

$$\sigma = \frac{4\pi\alpha^2 (\hbar c)^2}{3E^2}$$

For example at *E*=10 GeV the total cross-section is

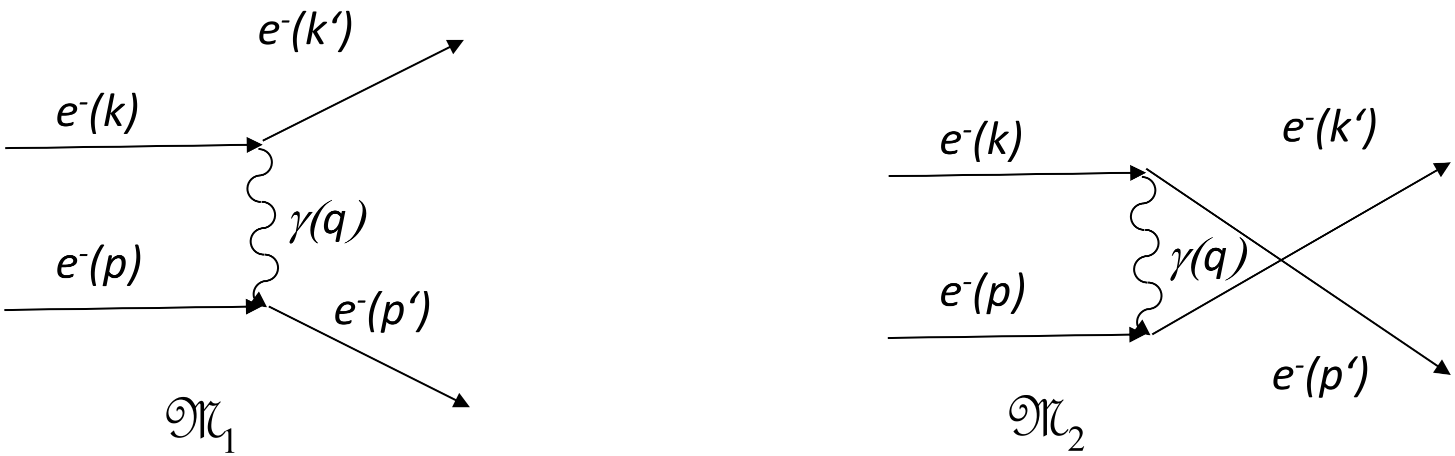
$$\sigma \approx \frac{4\pi (0.2\ GeV\ \ fm)^2}{(137)^2 \cdot 3 \cdot 100\ GeV^2} \approx 9 \cdot 10^{-38}\ m^2 = 0.9\ nb$$

where barn (b) is an appropriate unit for measuring the cross-sections (1 b = 10^{-28} m^2).

2.7.3 $e^- e^- \to e^- e^-$, $e^- e^+ \to e^- e^+$

In the ultrarelativistic limit there is no difference between the e^- and μ^- (the only difference between the two particles is their mass which is neglected in the ultrarelativistic limit). Hence one expects in this limit the cross-section for the process $e^- e^- \to e^- e^-$ to be the same as the one for the $e^- \mu^- \to e^- \mu^-$

However, in the former process one deals with two indistiguishable particles in the final state. Hence we can not distinguish between the two Feynman diagrams shown below:



In words, one doesn't know whether the final state electron with the four-momentum k' arises from the vertex with the initial state electron of four-momentum k or p. Due to this one has to make the amplitude symmetric with respect to the interchange $k' \leftrightarrow p'$. Mathematically this corresponds to

$$\mathfrak{M}_1 \propto \left[\overline{u}(k')\gamma^\kappa u(k)\right]\left[\overline{u}(p')\gamma_\kappa u(p)\right]$$

$$\mathfrak{M}_2 \propto \left[\overline{u}(p')\gamma^\kappa u(k)\right]\left[\overline{u}(k')\gamma_\kappa u(p)\right]$$

$$\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2$$

We don‘t need to repeat the whole calculation but rather symmetrize the result for the $e^- \mu^- \to e^- \mu^-$ cross-section:

$$\left|\overline{\mathfrak{M}}\right|^2_{e^-\mu^-} = \frac{8e^4}{q^4_{e^-\mu^-}}\left[(k'p')(kp) + (k'p)(kp') - m_e^2 p'p - m_\mu^2 k'k + 2m_e^2 m_\mu^2\right]$$

$$q_{e^-\mu^-} = (k' - k)$$

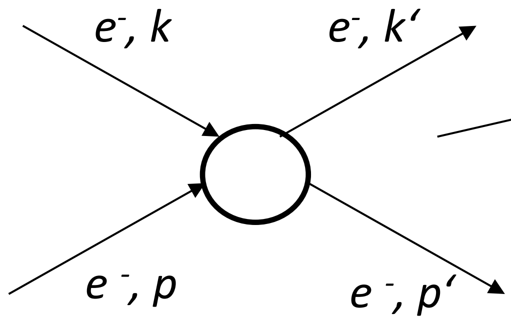
$$\left|\overline{\mathfrak{M}}\right|^2_{e^-e^-} = \frac{8e^4}{q^4_{e^-e^-}}\left[(k'p')(kp) + (k'p)(kp') - m_e^2 p'p - m_e^2 k'k + 2m_e^4\right] +$$

$$\frac{8e^4}{q^4_{e^-e^-}}\left[(k'p')(kp) + (p'p)(kk') - m_e^2 k'p - m_e^2 p'k + 2m_e^4\right]$$

$$q_{e^-e^-} = (k' - k) + (p' - k)$$

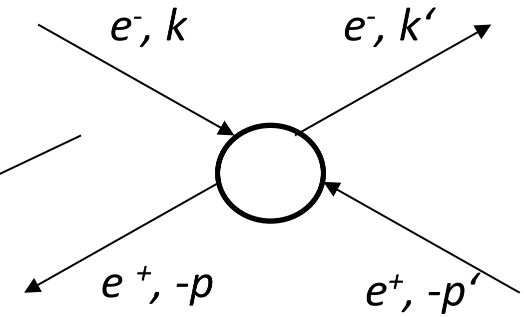
Having at hands the amplitude for the $e^-e^- \rightarrow e^-e^-$ it is now straightforward to apply the crossing method to obtain the amplitude for the $e^-e^+ \rightarrow e^-e^+$:

$e^-e^- \rightarrow e^-e^-$



incoming		outgoing	
p_a	p_b	p_c	p_d
k	p	k'	p'
k	$-p'$	k'	$-p$

$e^-e^+ \rightarrow e^-e^+$



The necessary transformation is $p \leftrightarrow p'$, and hence

$$\left|\overline{\mathfrak{M}}\right|^2_{e^-e^+} = \frac{8e^4}{q^4_{e^-e^+}}\left[(k'p)(kp') + (k'p')(kp) - m_e^2 p'p - m_e^2 k'k + 2m_e^4\right] +$$

$$\frac{8e^4}{q^4_{e^-e^+}}\left[(k'p)(kp') + (p'p)(kk') - m_e^2 k'p' - m_e^2 pk + 2m_e^4\right]$$

$$q_{e^-e^+} = (k'-k) + (p-k)$$

Upon the inspection of the amplitude for the $e^-e^+ \to e^-e^+$ we realize there are still two terms in the xpression corresponding to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ from the $e^-e^- \to e^-e^-$. In the latter process two amplitudes were assigned to two indistiguishable Feynman diagrams (see p. ???). Indeed also for the $e^-e^+ \to e^-e^+$ we have two possible indistinguishable diagrams:

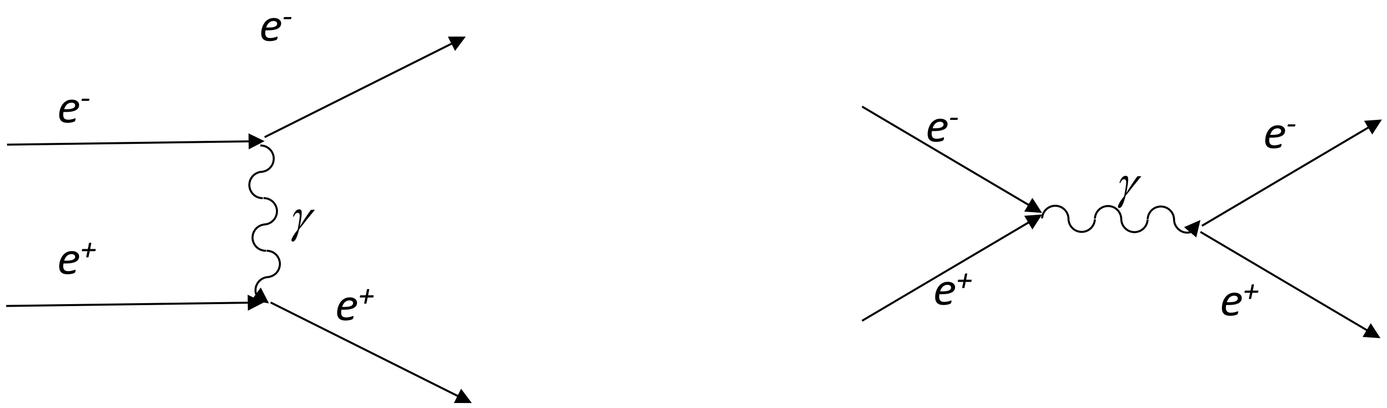


Diagram on the left is sometimes called the scattering process and the one on the right the annihilation process.

The quantum electro-dynamics (QED) processes discussed above are among the experimentally most accurately measured processes, confirming the calculations in the framework of QED to high precission. These calculations involve not only the leading order calculations as shown here but also higher order processes (i.e. processes involving more vertices).

Experimentally, the processes are accurately measured using the electron – positron colliders.

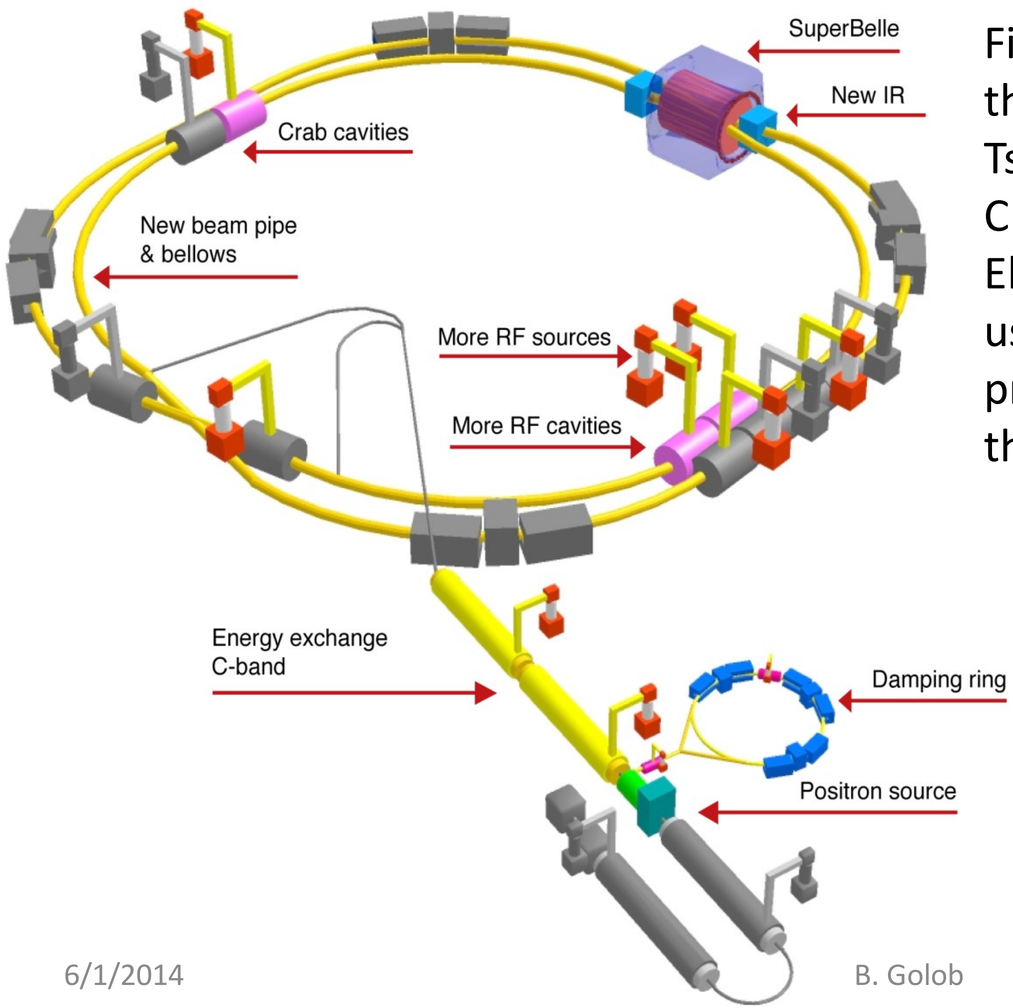
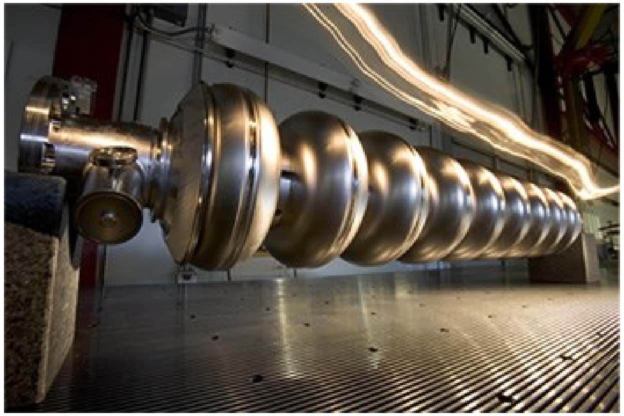
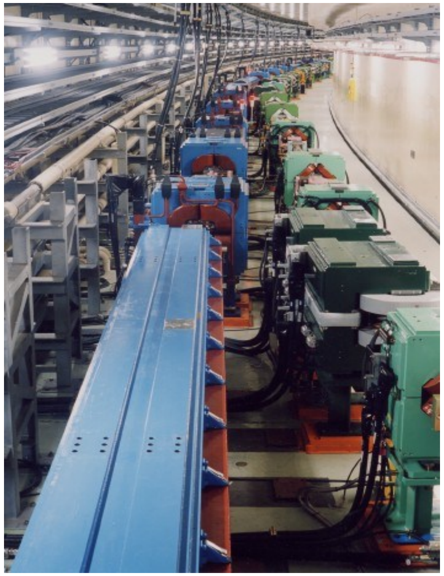


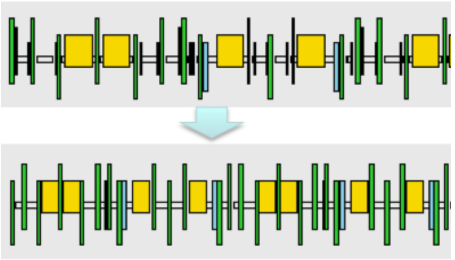
Figure on the left shows an example of the e^+e^- collider (Super KEK-B) built in Tsukuba, Japan, to study the collissions at the CMS energies around 10 GeV.
Electrons and positrons are accelerated using the standing electromagnetic waves produced in the radio-frequency cavities as the one shown on the figure below.



In order to make the trajectory of the accelerated particles (approximately) circular bending dipole magnets are used. Charged particles traveling through a magnetic field perpendicular to their velocity experience the Lorentz force which keeps them in a circular orbit.



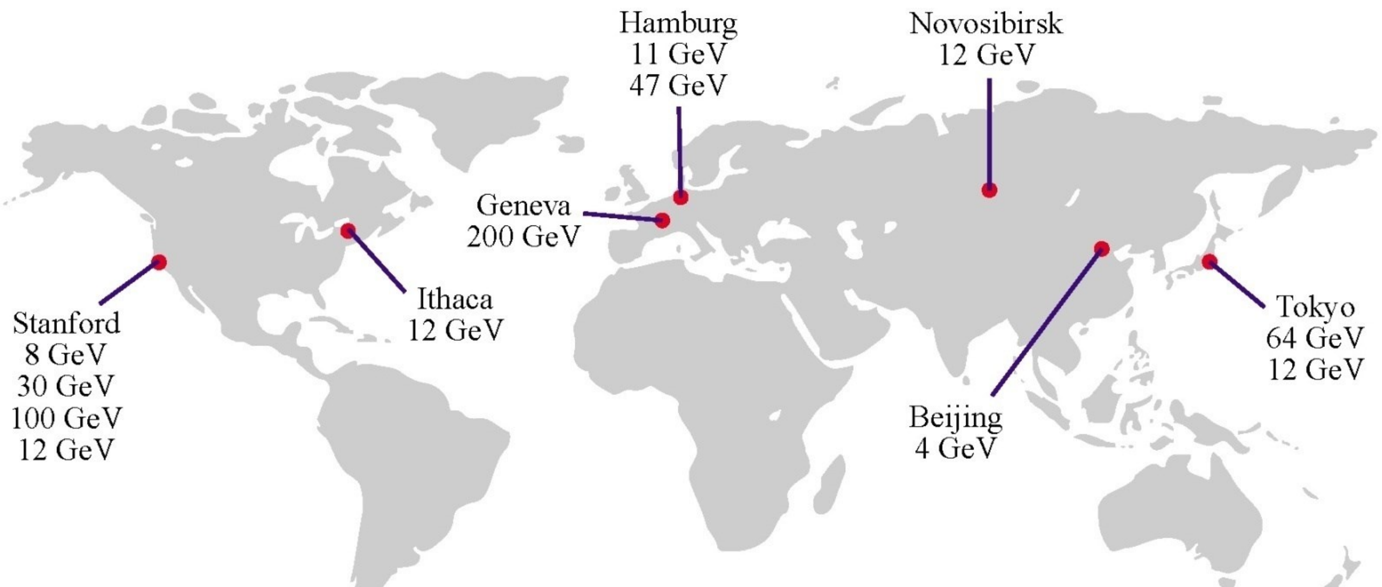
Apart from the dipole magnets other magnets are used in the accelerators, for example quadrupole magnets used to focus the beams of accelerated particles infront of the point where one wants the interactions to take place. Consequently, the accelerator is a complicated lattice of various magnets and accelerating cavities.



Part of the lattice for the KEK-B accelerator. Each yellow box represents a specific magnet used.

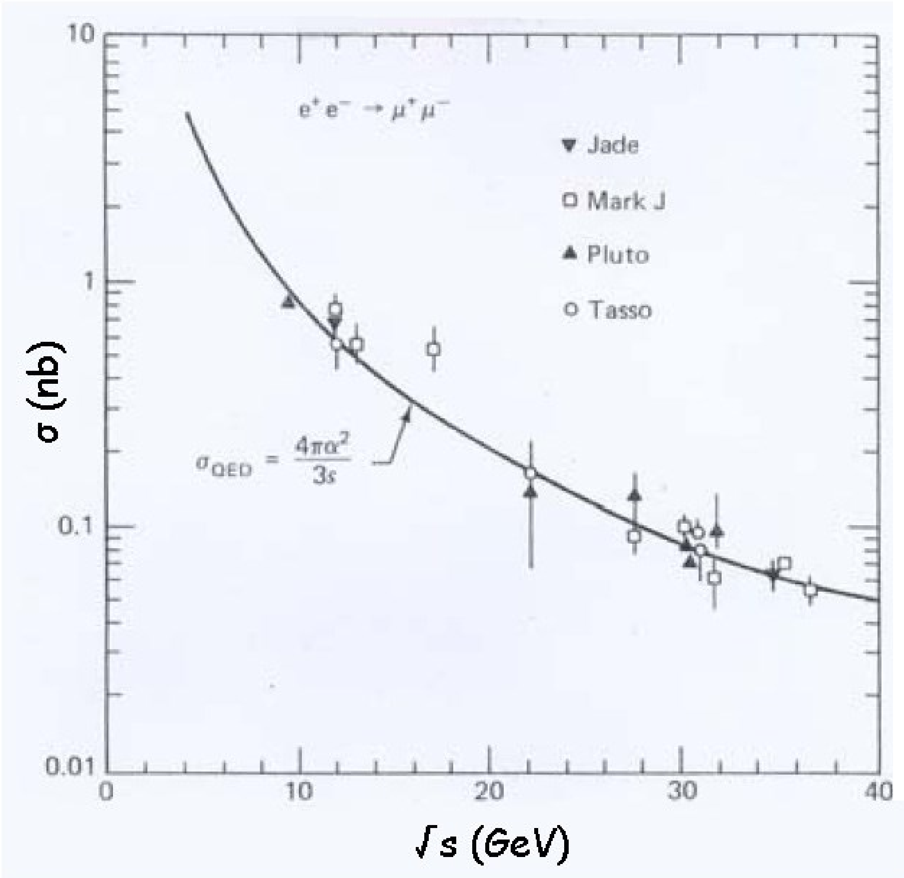
Long bending magnets (in blue) used at the KEK-B accelerator.

Particle accelerators are expensive infrastructure. Consequently there are only few infrustructure centers around the world at which particle physicist from all over the world perfrom various measurements. Some of the past and existing e^+e^- accelerators are shown in the map below:



A very rough estimate for the costs of an e^+e^- accelerator can be obtained using the formula cost $\sim a\,R + b\,E^4 / R$. The first term scales with the length of the accelerator (proportional to radius R) and roughly accounts for the price of the civil engineering work needed, number of magnets, etc. The second term accounts for the synchrotron radiation causing the accelerated (light) particles to loose their energy and hence takes into account the price of accelerating units, cooling equipment, etc. The parameters a and b can be estimated to approximately $a \sim 1{,}2 \cdot 10^5$ \$/m and $b \sim 1{,}3 \cdot 10^3$ \$m/GeV^4 from approximate costs of the SPEAR (Stanford, USA, E=8 GeV, R=40 m, cost $\sim 5{\cdot}10^6$ US\$;) and LEP (CERN, Geneva, E=200 GeV, R=4,3 km, cost $\sim 10^9$ US\$) accelerators.

The figure below on the left shows some of the earlier measurements of the total $e^-e^+ \to \mu^- \mu^+$ cross-section at various CMS energies. The solid line is the leading order prediction as calculated on p. ???.



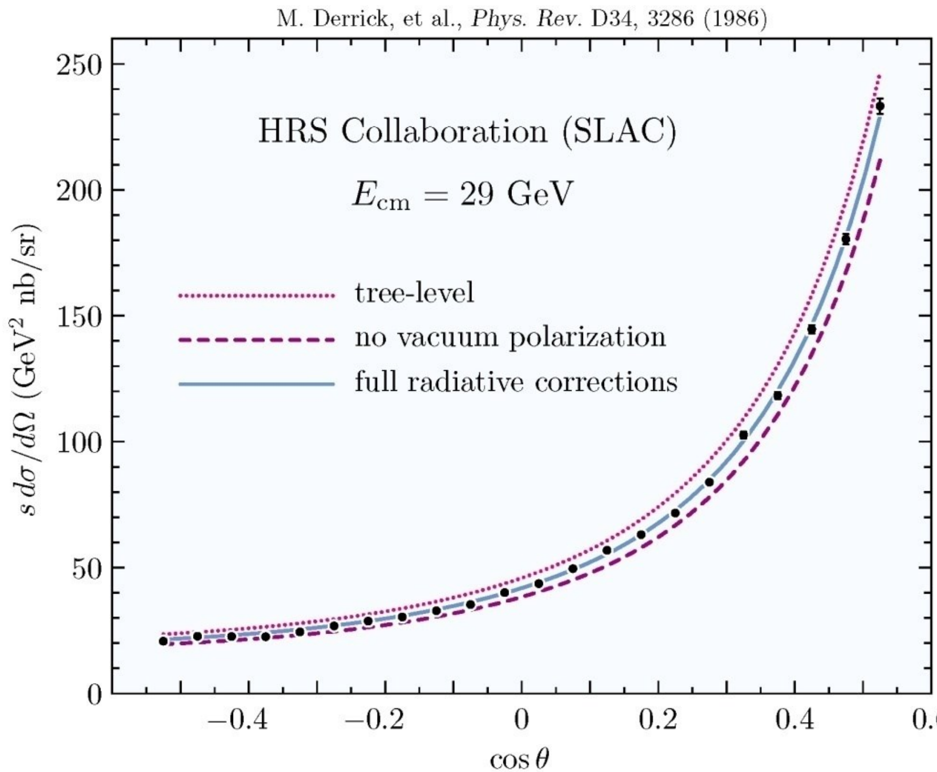
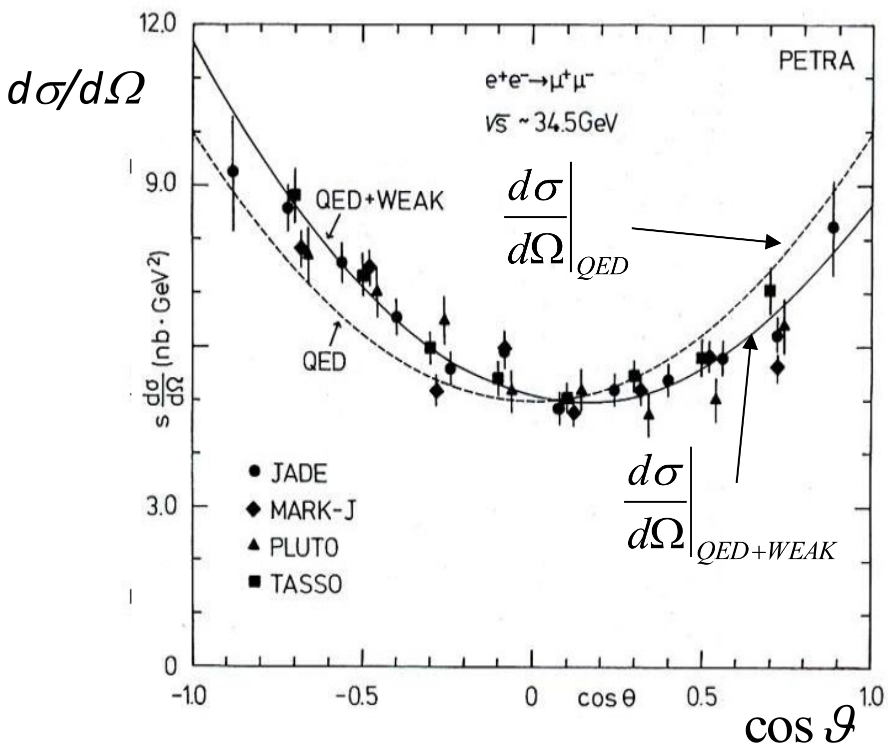


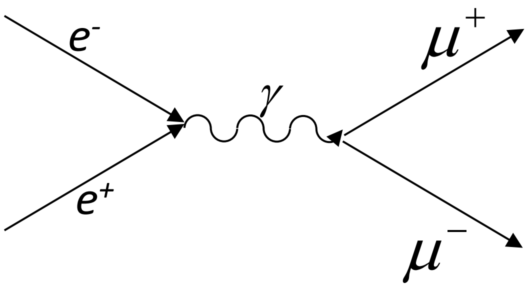
Figure on the right illustrates experimental tests of hihger order corrections to the $e^-e^+ \to e^-e^+$ differential cross-section at E=29 GeV. The leading order calculation is shown by the solid red line. The measurements are accurate enough to exhibit the need for the corrections.

The differential cross-section for $e^- e^+ \to \mu^- \mu^+$ is an even function of cos θ (see p. ???). An example of measurements is shown below. The measurement cleraly exhibits an asymmetry in the angular distribution. This is a consequence of the weak interaction (which contributes to the process beside the pure electromagnetic interaction). The QED prediction is shown by the line denoted $d\sigma/d\Omega|_{QED}$ and the prediction taking into account also the weak interaction by the dasehd line ($d\sigma/d\Omega|_{QED+WEAK}$).

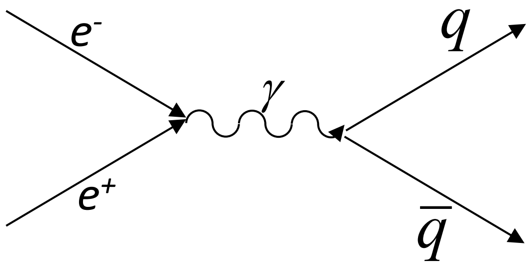


2.7.4 $e^-e^+ \to q\bar{q}$

In the electron positron annihilation also pairs of quarks can be produced. The Feynman diagram is similar to the one for the production of the muon pair:



$$-i\mathfrak{M} = \left[-e_e \bar{u}(k')\gamma^\kappa u(k)\right]\left(-\frac{g^{\kappa\nu}}{q^2}\right)\left[-e_\mu \bar{u}(p')\gamma^\nu u(p)\right]$$



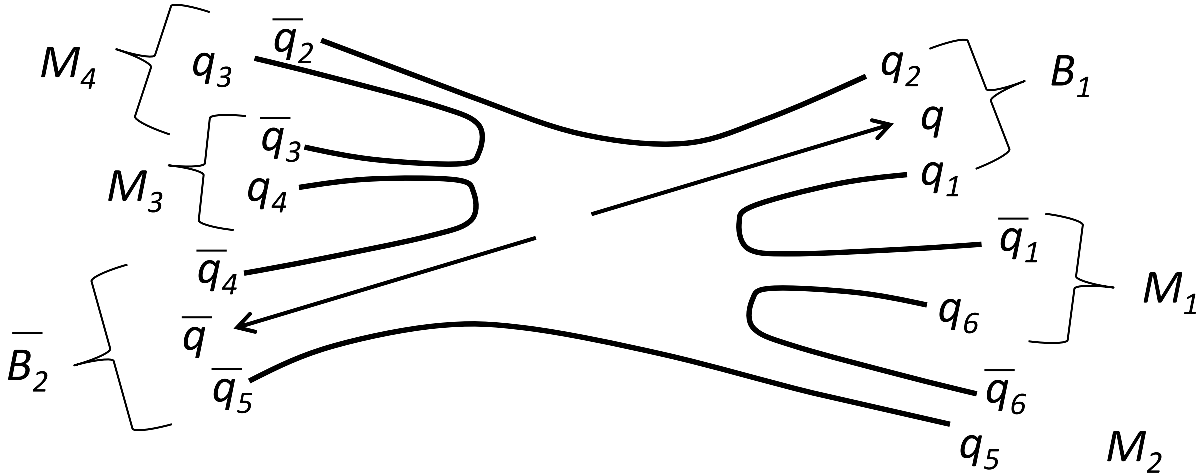
$$-i\mathfrak{M} = \left[-e_e \bar{u}(k')\gamma^\kappa u(k)\right]\left(-\frac{g^{\kappa\nu}}{q^2}\right)\left[-e_q \bar{u}(p')\gamma^\nu u(p)\right]$$

Since quarks are fermions, like muons, the amplitude for the process $e^-e^+ \to q\bar{q}$ follows from the amplitude for the $e^-e^+ \to \mu^+\mu^-$ process. The only difference is that in the latter one encounters the charge of the muon, which in the former should obviously be replaced by the corresponding charge of the quark. The cross-section $\sigma(e^-e^+ \to \mu^+\mu^-) \propto |\mathfrak{M}|^2 \propto e_e^2 e_\mu^2$ and hence $\sigma(e^-e^+ \to q\bar{q}) \propto e_e^2 e_q^2$. The only other difference between the two cross sections arises from the quntum number assigned to quarks but not to muons, the color. Since quarks arise in three possible colors (see p. ??) the cross section must be multiplied by 3.

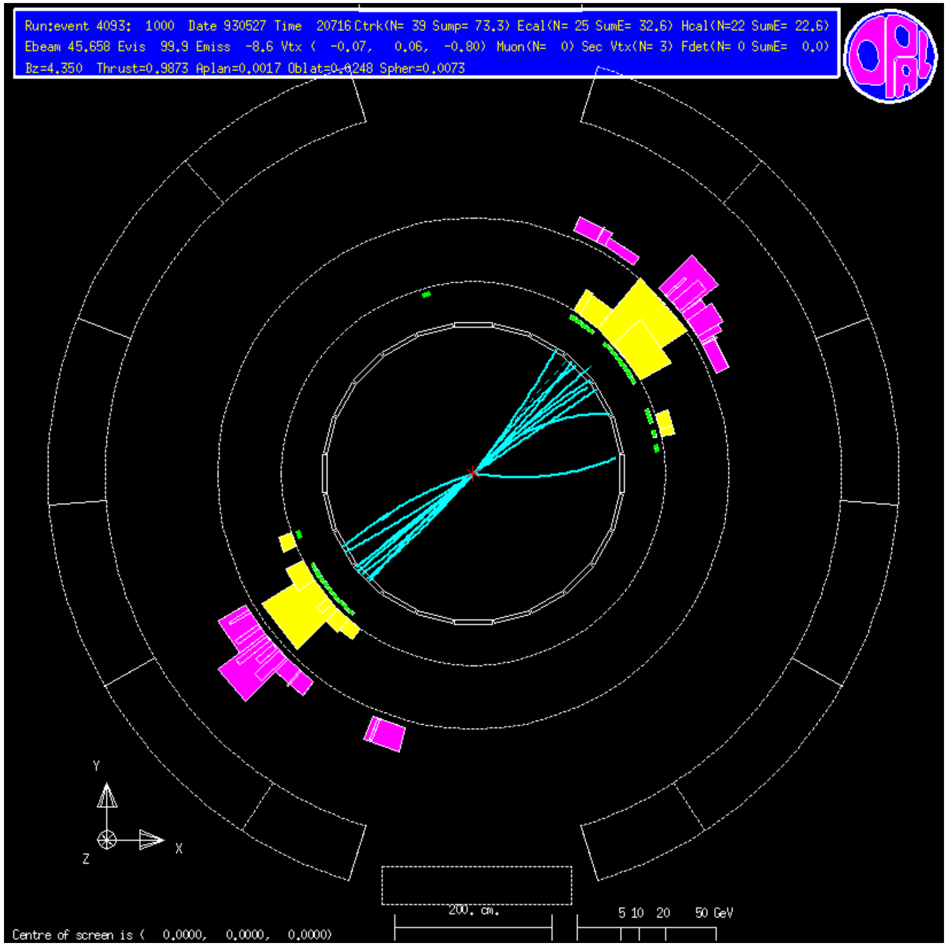
Hence the ratio of the two cross sections is

$$\frac{\sigma(e^+e^- \to q\overline{q})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \frac{3e_e^2 e_q^2}{e_e^2 e_\mu^2} = 3Q_q^2$$

where Q_q denotes the charge of the quark in units of the elementary charge e_0.
As discussed on p. ??? produced quarks immediately „dress“ with other quarks in the process of so called hadronization, for example



In the above illustration q and $\overline{q}$ denote the original quark pair, while q_i are (anti)quarks produced from vaccuum. Quarks form hadrons, mesons (M_i) or baryons (B_i). The final result of the process are two jets of hadrons that can be detected in a particle detector, as shown in the next figure.



Computer reconstruction of an $e^-e^+ \rightarrow q\bar{q}$ annihilation detected by the Opal detector at the Large Electron Positron collider (operating at Cern in the period 1989 - 2000; in the same tunnell nowadays the Large Hadron Collider is located) resulting in two back-to-back jets of hadrons. Blue lines represent detected charged hadrons in the detector.

The production of quarks always results in various hadrons in the final state. Summing over all possible quark flavors one obtains the total cross section for the production of hadrons in electron positron annihilations:

$$\sigma(e^+e^- \rightarrow hadrons) = \sum_{q=u,d,s,c...} \sigma(e^+e^- \rightarrow q\bar{q})$$

Of course the centre-of-mass energy of the collission must be high enough for the production of a pair of specific quark flavor (more precisely it must be high enough for the production of at least two lightest hadrons composed of these two quarks). Over which quark flavors the sum runs over thus depends on the collission energy.

Ratio of the cross section for the hadron and the muon pair production *R* is

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3 \sum_{q=u,d,s,c...} Q_q^2$$

At the energies sufficient to produce pions only (composed of u and d quarks) the ratio is

$$R = 3\left((2/3)^2 + (1/3)^2\right) = 5/3$$

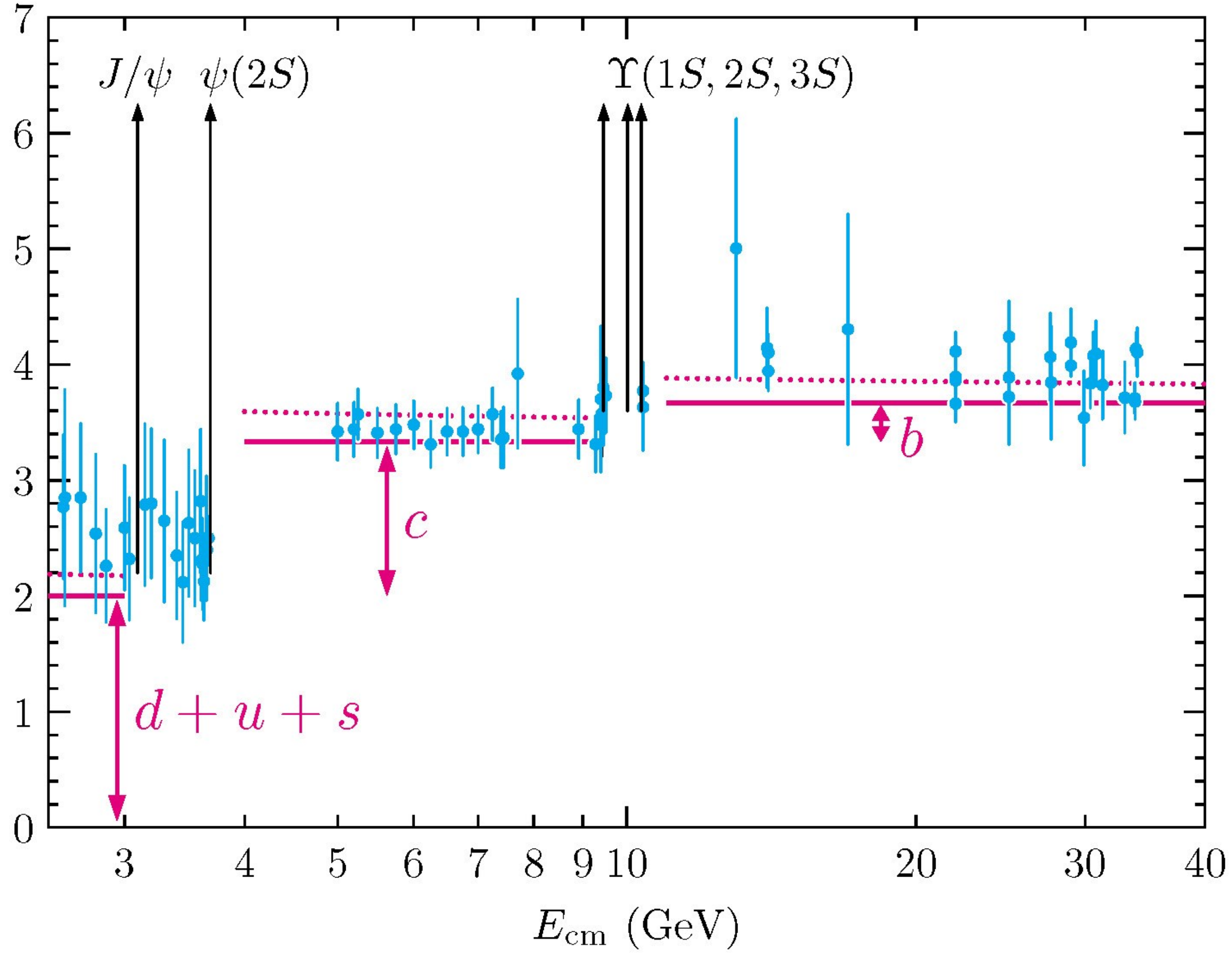
Once the energy becomes high enough for producing *s* quark pairs, the ratio becomes

$$R = 3\left((2/3)^2 + (1/3)^2 + (1/3)^2\right) = 6/3$$

and at even higher energies

$$R = 3\left(\underbrace{(2/3)^2 + (1/3)^2}_{u,d} + \underbrace{(1/3)^2}_{s} + \underbrace{(2/3)^2}_{c} + \underbrace{(1/3)^2}_{b}\right) = 11/3$$

An example of measured ratio is shown in the figure below. Note that no e^+e^- accelerator has so far achieved energies to produce *t* quark pairs.



2.8 Weak interaction

2.8.1 Introduction

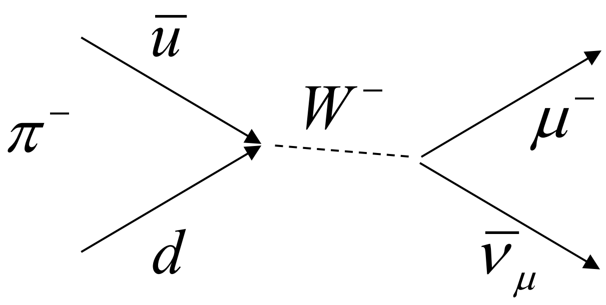
An obvious hint that beside the strong and the electromagnetic there must exist yet another interaction are lifetimes of charged and neutral pions:

$\tau(\pi^-) = 2.6\ 10^{-8}$ s

$\tau(\pi^0) = 8.4\ 10^{-17}$ s

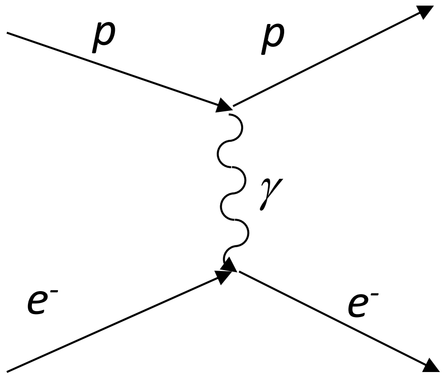
Why the two mesons, both composed of *u* and *d* quarks, have lifetimes differing by 9 orders of magnitude?

Pions are the lightest hadrons and hence can not deacy through the strong interaction into lighter hadrons. The neutral pion can, however, decay through an electromagnetic process, $\pi^0 \to \gamma\gamma$. On the other hand electromagnetic decays with photons in the final state are not possible for the charged pion. The decay $\pi^- \to \mu^- \gamma$, for example, is forbidden by the lepton number conservation. By far the most abundant decay mode of charged pions is $\pi^- \to \mu^- \nu_\mu$, proceeding through a (charged) weak interaction. The Feynman diagram of the decay is

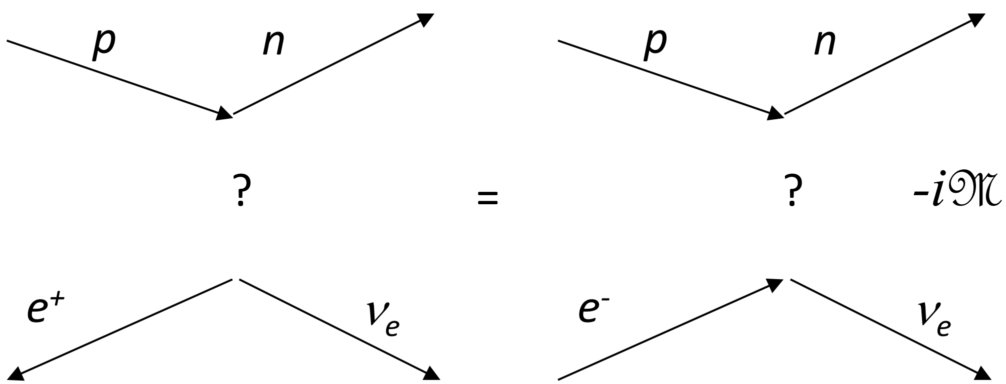


The charged weak interaction propagated by charged weak bosons $W^\pm$ is the only one that changes the flavor of quarks (or in other words couples the quarks of different flavors as seen in the pion vertex in the figure).

As mentioned already on p. ??? the weak interaction causes β decays of nuclei, for example $^{10}C \rightarrow {}^{10}B\, e^+\, \nu_e$. In this particular case a proton inside the initial nuclei decays into a neutron, positron and a neutrino. In 1932 Fermi wrote the matrix element for such a process in analogy with the electromagnetic interaction:



$$-i\mathfrak{M} = e^2 \left[\bar{u}_p \gamma^\kappa u_p\right] \left(-\frac{g^{\kappa\nu}}{q^2}\right) \left[\bar{u}_{e^-} \gamma^\nu u_{e^-}\right]$$



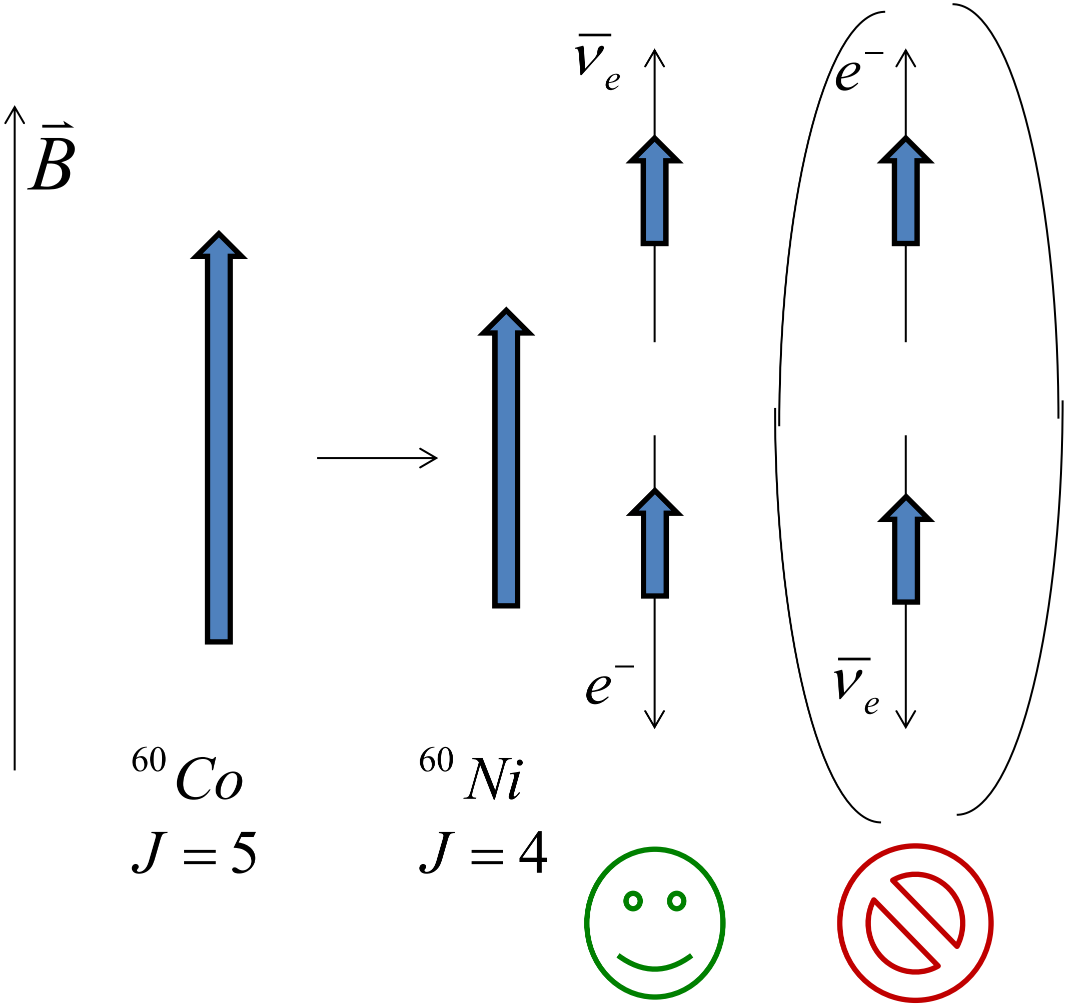
$$-i\mathfrak{M} = \frac{G_F}{\sqrt{2}} \left[\bar{u}_p \gamma^\kappa u_p\right] \left(-\frac{g^{\kappa\nu}}{q^2}\right) \left[\bar{u}_{e^-} \gamma^\nu u_{e^-}\right]$$

Because he didn't know what kind of particle propagates the interaction he skipped the $1/q^2$ term and changed the coupling constant (e^2 or α for the EM interaction). The constant G_F is nowadays known as the Fermi constant. Surprisingly enough the proposed description was successful in description of β decays. And indeed it only needs slight modification to account for some of the properties of weak interaction, most importantly the parity violation.

2.8.2 Parity violation

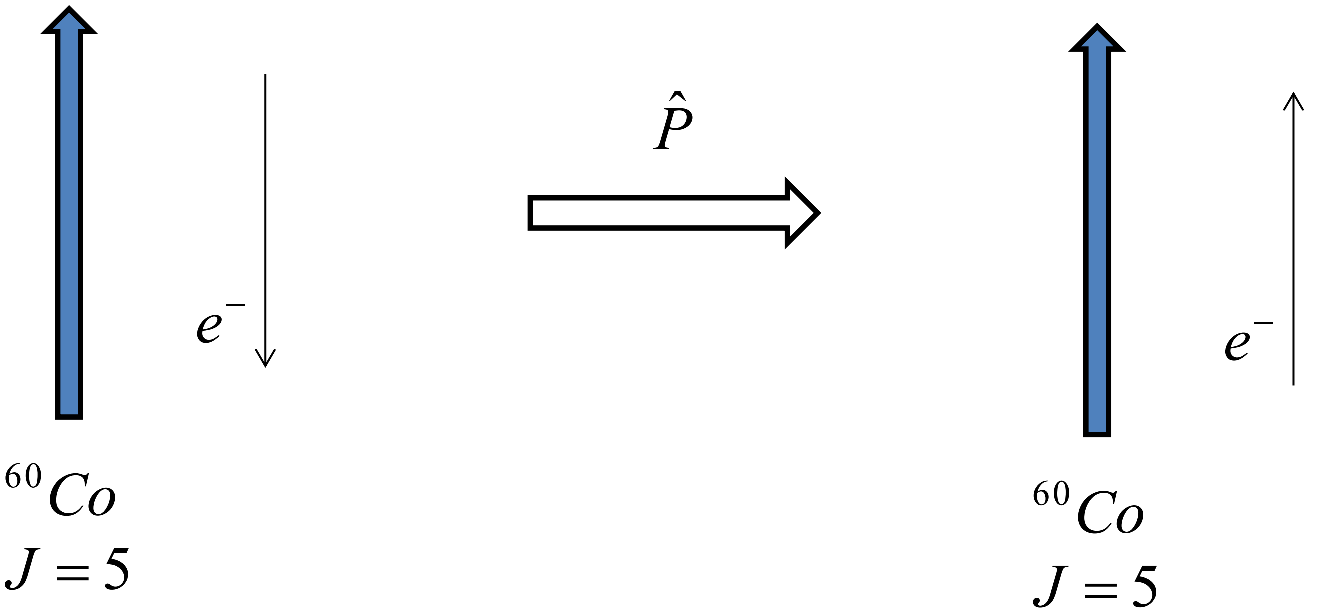
In 1950's the so called $\theta-\tau$ puzzle was one of important unanswered questions of particle physics. It consisted of two different decays of what was at that time believed to be two different particles, θ^+ and τ^+ (note that τ^+ has nothing to do with the contemporary τ lepton): $\theta^+ \to \pi^+\pi^0$, $\tau^+ \to \pi^+\pi^+\pi^0$. Considering the properties under the parity operator P (reflection of spatial coordinates, see p. ???) pions (composed of a quark and an anti-quark) have a negative parity value. The parity is a multiplicative quantum number and hence the two pion final state has a P value of +1, while the three pion final state has a P value of -1. What was puzzling was increasing experimental evidence that the two particles, θ^+ and τ^+, are the same (in terms of their mass and other properties). An obvious question was how could the same particle decay into final states with different parity? The electromagentic and strong interaction, experimentally already well known, conserved the parity, i.e. the parity of inital and final states were equal in all known processes proceeding through these two interactions.
In 1956 Tsung Duo Lee and Chen-Ning Yang examined the available experimental data and suggested that they can be interpreted by the weak interaction causing the above and similar

decays to violate parity (i.e. the parity value of the initial and final states in the processes proceeding through the weak interaction are not necessary the same). They proposed an experiment carried out by Chien SHiung Wu, called the Cobalt-60 experiment.



Nuclei of ^{60}Co were put into external magnetic field $\vec{B}$ at low temperature. ^{60}Co nucleus has a spin $J=5$ (spins are denoted by  in the figure). At low tempreature almost all nuclei oriented with spin parallel to the external mag. field. The Cobalt nucleus undergoes a β^- decay into ^{60}Ni nucleus with $J=4$. Knowing that the spin of electron and neutrino is ½ and the fact that fermions according to the Dirac equation have positive or negative helicity (projection of the spin to momentum direction, see p. ???) one is left with two extreme configurations of electron and neutrino spins and momenta as sketched in the figure. The result of the experiment, in which electrons were detected, showed

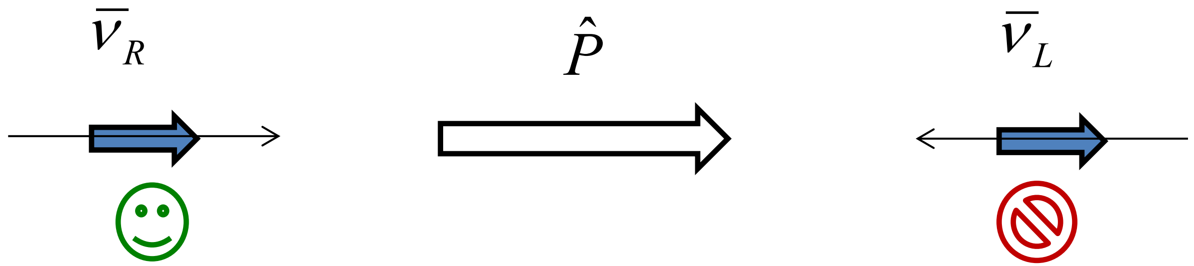
the large majority of electrons were flying in the direction oposite to the magnetic field and no electrons were found to fly in the direction of the magnetic field. This proved that the parity is indeed violated in β decays – a tipycal process proceeding through the weak interaction.
Why is this violation of the *P* symmetry? We are facing the configuration shown on the left:



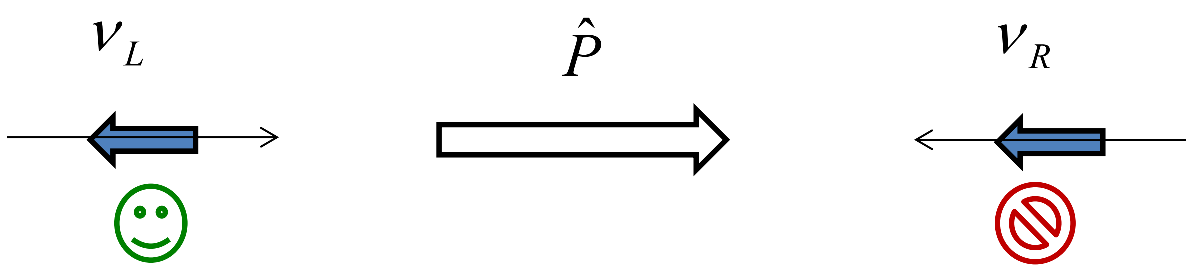
Electrons fly in the direction oposite to the external magnetic field, i.e. in the direction oposite to the *Co* nucleus spin. Under the parity transformation the electron momentum revrses its sign. On the other hand the nucleus spin, being an axial vector, does not change sign. Hence the *P* transformed configuration is represented by electrons flying in the direction of the nucleus spin. This is experimentally not observed. This is an obvious asymmetry between the two configurations related by the parity operation. Hence the interaction responsible for such a system does not obey symmetry under the parity transformation.

Piece of paper with what is supposingly one of discussions between T.D. Lee and C.N. Yang about the parity violation. They shared the Nobel prize in physics in 1957 for the discovery of parity violation. T.D. Lee was at the age of 30 the third youngest Nobel prize laureate (after W.L. Bragg, 25 in 1915, and W. Heisenberg, 30 in 1932).

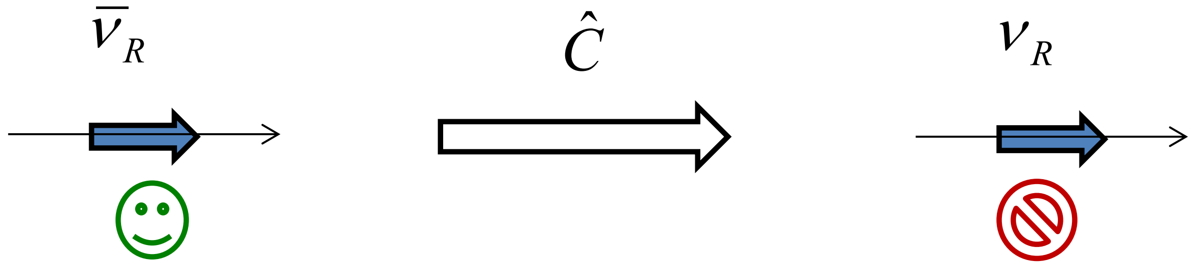
One can also have a slightly different look at the parity violation: in the Cobalt-60 experiment only positive helicity anti-neutrinos ($\bar{\nu}_R$) were observed (see illustration on p. ???).



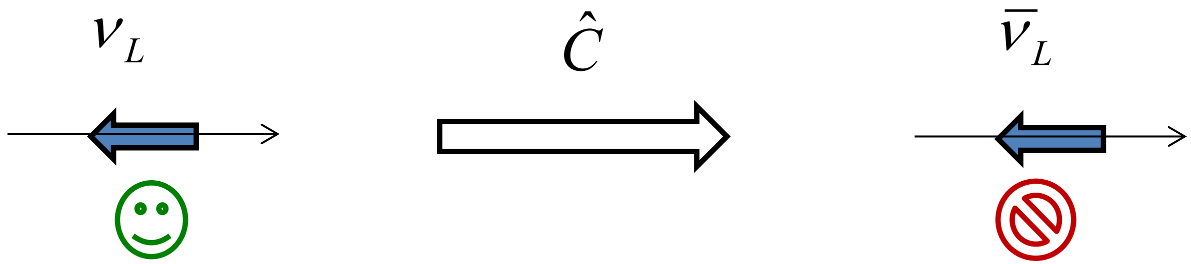
Under parity transformation the aniti-neutrino with positive helicity transforms into an anti-neutrino with negative helicity ($\bar{\nu}_L$, because momentum changes sign and spin does not). However, the latter was not observed nor in the Cobalt-60 or any other experiment so far. Similar is true for neutrinos: while neutrinos with negative helicity exist, neutrinos with positive helicity are not observed.



This implies that the weak interaction violates another symmetry: the symmetry under the charge conjugation C (which transfroms particles into anti-particles and vice versa). Namely, if one starts with a positive helicity anti-neutrino and performs the C transformation

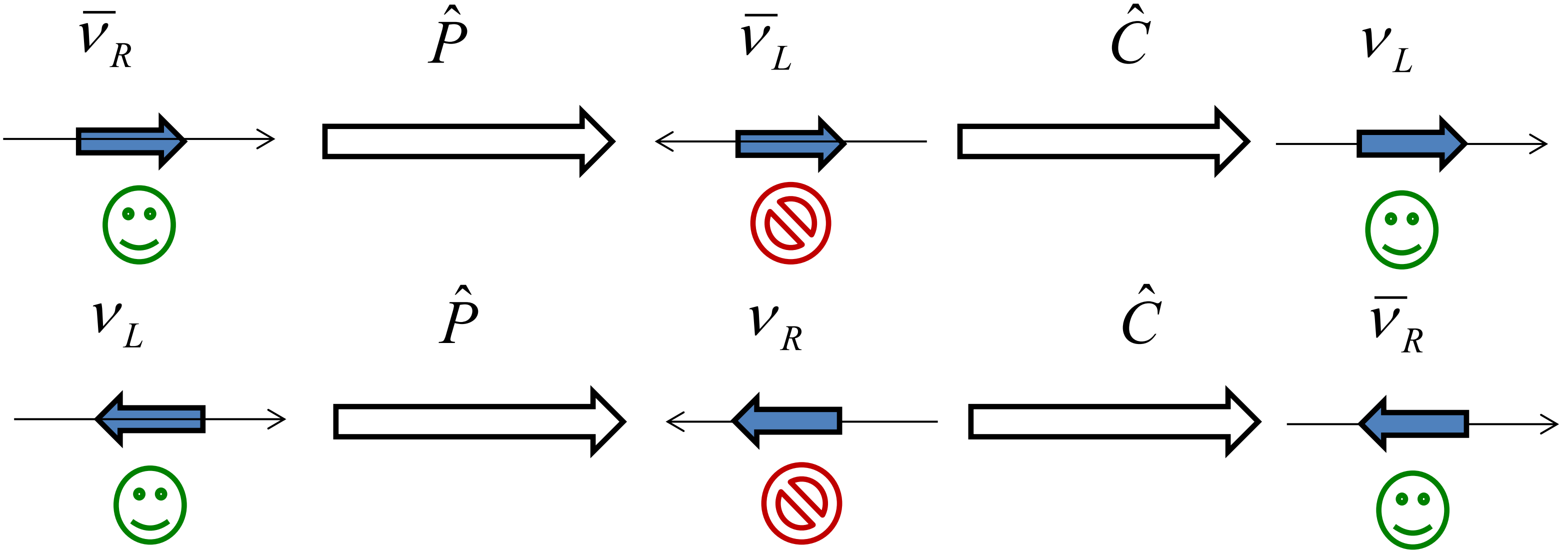


the result is a non-existing positive helicity neutrino. Analogously, starting with a negative helicity neutrino one arrives to a non-existing negative helicity anti-neutrino*.



The weak interaction thus violates both, the P as well as the C parity simmetry.
In 1957 Lev Landau proposed that the true symmetry which is preserved (also) by the weak interaction is the symmetry under a combined CP transformation:

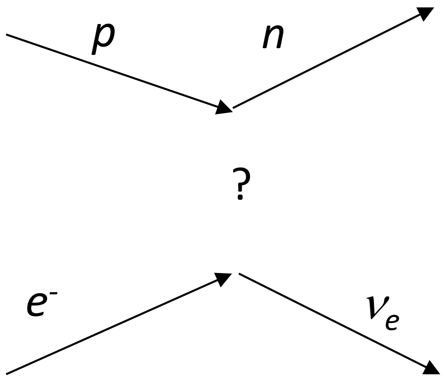
* Actually, the terminology here is not copmpletely correct. What one observes is that in the charged weak interaction only negative helicity neutrinos and positive helicity anti-neutrinos are involved. Since the neutrinos interact only through the weak interaction one can do a slightly sloppy generalization about the existence of the two mentioned states and non-existence of the other two.



The idea remained valid until 1964 when it was experimentally verified that also the combined CP symmetry is violated in charged weak inetraction (see p. ???).

2.8.3 Theory of weak interaction

E. Fermi in 1930's didn't know about the parity violation when writing down the amplitude for a process proceeding through the weak interaction (p. ???). It turns out that the correction needed in writing down the amplitude in order to account for this property of the weak interaction is reralively small:



$$-i\mathfrak{M} = \frac{G_F}{\sqrt{2}} \left[\bar{u}_p \gamma^\kappa (1-\gamma^5) u_p\right] \left(-\frac{g^{\kappa\nu}}{q^2}\right) \left[\bar{u}_{e^-} \gamma^\nu (1-\gamma^5) u_{e^-}\right]$$

All what is needed is an inclusion of the factor (1-γ^5) as written above, where γ^5 is a product of all four γ matrices:

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\gamma^{5^+} = \gamma^5; \left(\gamma^5\right)^2 = 1; \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 \underbrace{=}_{\mu=0,1,2,3} 0$$

What does the inclusion of this factor mean?

γ^5 is called the handedness operator. Factor (1-γ^5) projects the so called left- and righ-handed component of a bispinor:

$$u_L = \frac{1}{2}(1-\gamma^5)u, \quad u_R = \frac{1}{2}(1+\gamma^5)u, \quad u = u_L + u_R$$

$$\gamma^5 u_{L,R} = \gamma^5 \frac{1}{2}(1\pm\gamma^5)u = \frac{1}{2}(\gamma^5 \pm \left(\gamma^5\right)^2)u = -\frac{1}{2}(1\pm\gamma^5)u = \mp u_{L,R}$$

What is important is the helicity of the left- and righ-handed components in the ultrarelativistic limit:

$$u_L = \frac{1}{2}(1-\gamma^5)u = \ldots = \frac{1}{2}\left[\chi - \frac{\vec{\sigma}\,\vec{p}}{E+m}\chi\right] \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$u_L \underbrace{\approx}_{E>>m} \frac{1}{2}\left[\chi - \vec{\sigma}\,\hat{\vec{p}}\chi\right] \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where $\hat{\vec{p}}$ is a unit vector in the direction of fermion‘s momentum. The helicity of u_L is negative:

$$\vec{\sigma}\,\hat{\vec{p}}u_L \underbrace{\approx}_{E>>m} \frac{1}{2}\left[\vec{\sigma}\,\hat{\vec{p}}\chi - \left(\vec{\sigma}\,\hat{\vec{p}}\right)^2\chi\right] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{2}\left[\chi - \vec{\sigma}\,\hat{\vec{p}}\chi\right] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -u_L$$

Similarly one obtains $\vec{\sigma}\,\hat{\vec{p}}u_R \underbrace{\approx}_{E>>m} u_R$

In other words, the handedness (eigenvalue of the γ^5 operator) coincides with the helicity (eigenvalue of the $\vec{\sigma}\,\hat{\vec{p}}$ operator) in the ultrarelativistic limit.
One should also note that

$$\bar{u}_{f,L} = u_{f,L}^{+}\gamma^0 = u_f^{+}\frac{1}{2}\left(1-\gamma^5\right)^{+}\gamma^0 = u_f^{+}\frac{1}{2}\left(1-\gamma^5\right)\gamma^0 = u_f^{+}\gamma^0\frac{1}{2}\left(1+\gamma^5\right) =$$

$$= \bar{u}_f\frac{1}{2}\left(1+\gamma^5\right)$$

and hence
$$\bar{u}_f \gamma^\mu u_i = \bar{u}_{f,L} \gamma^\mu u_{i,L} + \bar{u}_{f,R} \gamma^\mu u_{i,R}$$
because
$$\bar{u}_{f,L} \gamma^\mu u_{i,R} = \bar{u}_f \frac{1}{2}\left(1+\gamma^5\right)\gamma^\mu \frac{1}{2}\left(1+\gamma^5\right)u_i = \frac{1}{4}\bar{u}_f \left(1+\gamma^5\right)\left(1-\gamma^5\right)\gamma^\mu u_i =$$
$$= \frac{1}{4}\bar{u}_f \left(1-\left(\gamma^5\right)^2\right)\gamma^\mu u_i = 0$$

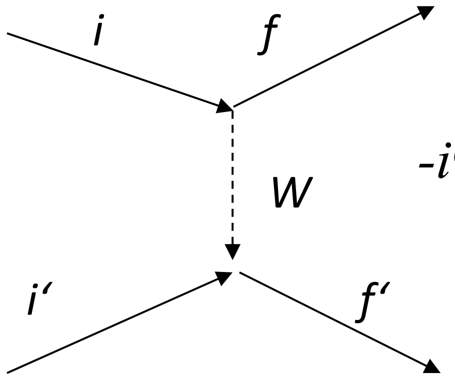
(and similarly $\bar{u}_{f,R} \gamma^\mu u_{i,L} = 0$).

The effect of the (1-γ^5) factor in the amplitude is thus

$$\bar{u}_f \gamma^\mu (1-\gamma^5) u_i = \bar{u}_f \gamma^\mu 2 u_{i,L} = 2\bar{u}_{f,L} \gamma^\mu u_{i,L}$$

where $u_{f,i}$ are bispinors of any final and initial state fermion involved in the process. The factor (1-γ^5) projects out only left-handed component of the bispinors in the amplitude, and (reminding that the handedness coincides with the helicity in the ultrarelativistic limit, which is always fulfilled for neutrinos) only negative helicity neutrinos (and positive helicity anti-neutrinos) take part in the weak interaction. By this the parity violation property of the weak interaction is properly accounted for.

A full amplitude for the process proceeding through the charged weak interaction is thus



$$-i\mathfrak{M} = \left[\frac{g_W}{\sqrt{2}}\bar{u}_f\gamma^\kappa(1-\gamma^5)u_i\right]\left(-\frac{g^{\kappa\nu}}{M_W^2-q^2}\right)\left[\frac{g_W}{\sqrt{2}}\bar{u}_{f'}\gamma^\nu(1-\gamma^5)u_{i'}\right]$$

where instead of the Fermi constant we wrote out the „true" weak interaction coupling constant g_W as well as the factor exposing the interaction carriers, weak charged bosons $W^\pm$, with the mass M_W. In the limit $M_W^2 >> q^2$ we see that the g_W and M_W yields the Fermi constant,

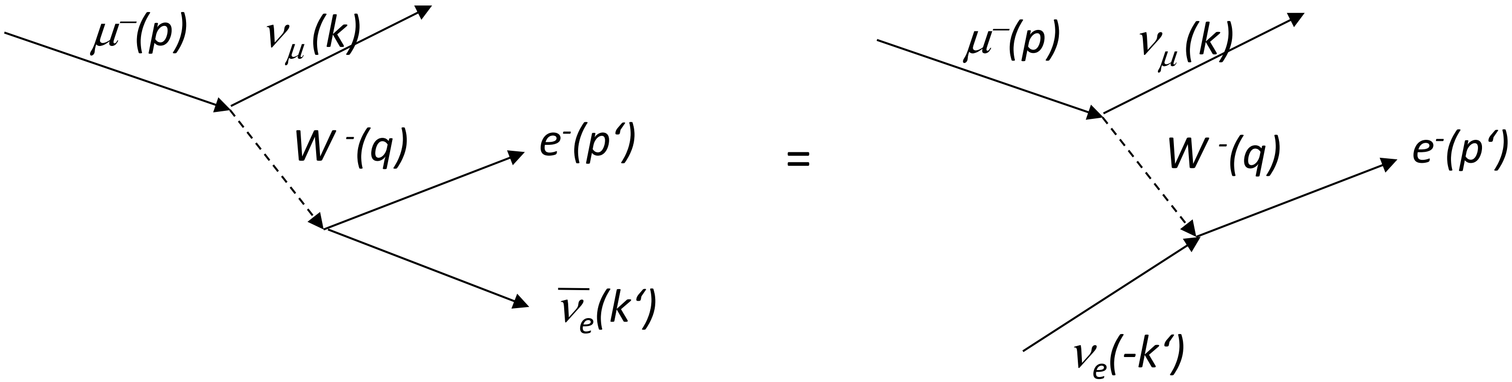
$$G_F = \frac{g_W^2}{\sqrt{2}M_W^2}$$

In 1960's A. Salam, S. Glashow and S. Weinberg published a series of articles in which they derived the properties of charged weak interaction and the Lagrangian for the description of the weak, electromagentic and strong interaction among elementary particles. In doing so they exposed relations pointing to the fact that the electromagnetic and weak interaction are actually just a low energy manifestations of a unified electroweak interaction. Furthermore they predicted the existence ow neutral weak interaction. Their work is nowadays regarded as the basis of the Standard Model of the weak, electromagnetic and strong interaction, one of the experimentally best verified physics theories. For their work they shared the Nobel prize for physics in 1979.

S. Glashow A. Salam S. Weinberg

2.8.3 Muon decay

As a specific example of a process proceeding through the charged weak interaction let us examine the decay of a muon, $\mu^- \to e^- \ \overline{\nu_e} \ \nu_\mu$. The Feynman diagram is



where in parenthesis we denoted the four-momenta of particles. The diagram on the right is an analogous diagram where the anti-particle ($\overline{\nu_e}$) is replaced by the particle (ν_e) with a reversed sign of the four-momentum.

The observable related to a particle decay is its total decay width, $\Gamma = 1/\tau$, where τ is particle's lifetime (or, written in non-natural units, $\Gamma = \hbar c/c\tau$). In calculating the decay width, the expression for the cross section (p. ???) is slightly modified:

$$d\Gamma = \frac{|\mathfrak{M}|^2}{2E} dQ$$

where the factor *F* appropriate for the scattering process is replaced by *2E*, the density of initial state particles (a single particle), and *E* is the enrgy of the initial state particle.
The phase space *dQ* for the specific decay is written as

$$dQ = \frac{d^3 p'}{(2\pi)^3 2E'} \frac{d^3 k}{(2\pi)^3 2\omega} \frac{d^3 k'}{(2\pi)^3 2\omega'} (2\pi)^4 \delta^4\left(p - p' - k - k'\right)$$

where ω and ω' denote energies of the muon and electron neutrino, respectively. Considering the fact that neutrinos are difficult to detect, in the muon decay one is primarily interested in the electron energy spectrum ($d\Gamma/dE'$). Hence one can integrate *dQ* over d^3k, taking into account the following identity:

$$\int \frac{d^3 k}{2\omega} = \int d^4 k \theta(\omega) \delta(k^2)$$

$\theta(\omega)$ in the equation above is the Heaviside function (=1 if $\omega>0$ and =0 if $\omega<0$). Hence

$$dQ = \frac{1}{(2\pi)^5} \frac{d^3 p'}{2E'} \frac{d^3 k'}{2\omega'} \int d^4 k \theta(\omega) \delta(k^2) \delta^4\left(p - p' - k - k'\right)$$

which is trivial because of the $\delta^4(\ldots)$ function:

$$dQ = \frac{1}{(2\pi)^5}\frac{d^3p'}{2E'}\frac{d^3k'}{2\omega'}\theta(E-E'-\omega')\delta\left((p-p'-k')^2\right)$$

The matrix element is

$$\mathfrak{M} = \frac{G_F}{\sqrt{2}}\left[\overline{u}(k)\gamma^\mu(1-\gamma^5)u(p)\right]\left[\overline{u}(p')\gamma_\mu(1-\gamma^5)u(-k')\right]=$$
$$=\frac{G_F}{\sqrt{2}}\left[\overline{u}(k)\gamma^\mu(1-\gamma^5)u(p)\right]\left[\overline{u}(p')\gamma_\mu(1-\gamma^5)v(k')\right]$$

where in the last line we introduced a bispinor of anti-particle (i.e. of a fermion with a negative energy), *v(k')*, for easier notation. Remembering that the negative energy solutions of the Dirac equation are interpreted as anti-partilce solutions, i.e. we denote the solutions as

$$u^{(1,2)}e^{-ipx};\quad E>0$$
$$u^{(3,4)}e^{-(-ipx)} \equiv v^{(2,1)}e^{ipx};\quad E>0$$

While the compact form of the Dirac equation for bispinors *u* is

$$(\not{p}-m)u=0$$

the form for the bispinors *v* is

$$\left(\not{p} + m\right)v = 0$$

The other formal difference between bispinors *u* and *v* is in the form of the completness relation (see p. ???):

$$\sum_{s=1,2} u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p} + m$$
$$\sum_{s=1,2} v^{(s)}(p)\overline{v}^{(s)}(p) = \not{p} - m$$

The average matrix element for decays of unpolarized muons is

$$\left|\overline{\mathfrak{M}}\right|^2 = \frac{1}{2}\frac{G_F^2}{2}\sum\left[\overline{u}(k)\gamma^{\mu}(1-\gamma^5)u(p)\right]\left[\overline{u}(p')\gamma_{\mu}(1-\gamma^5)v(k')\right]$$
$$\left[\overline{u}(k)\gamma^{\sigma}(1-\gamma^5)u(p)\right]^{+}\left[\overline{u}(p')\gamma_{\sigma}(1-\gamma^5)v(k')\right]^{+}$$

The leading factor ½ arises from two possible spin orientations of the initial muon (*1/(2*s_μ*+1)*). Inspecting the third *[...]* factor we see

$$\left[\overline{u}(k)\gamma^{\sigma}(1-\gamma^5)u(p)\right]^{+} = u^{+}(p)(1-\gamma^5)^{+}\gamma^{\sigma+}\overline{u}(k)^{+} =$$

$$= u^{+}(p)(1-\gamma^{5})\gamma^{\sigma +}\gamma^{0}u(k) = -u^{+}(p)\gamma^{0}(1+\gamma^{5})\gamma^{\sigma +}u(k) =$$
$$= -\overline{u}(p)\gamma^{\sigma +}(1-\gamma^{5})u(k) = \overline{u}(p)\gamma^{\sigma}(1-\gamma^{5})u(k)$$

Similarly for the other *[...]⁺* term one obtains

$$\left[\overline{u}(p')\gamma_{\sigma}(1-\gamma^{5})v(k')\right]^{+} = \overline{v}(k')\gamma_{\sigma}(1-\gamma^{5})u(p')$$

The average square of the matrix element is thus

$$\left|\overline{\mathfrak{M}}\right|^{2} = \frac{G_F^2}{4}\sum_{\substack{\mu,\nu_\mu \\ spins}}\left[\overline{u}(k)\gamma^{\mu}(1-\gamma^{5})u(p)\right]\left[\overline{u}(p)\gamma^{\sigma}(1-\gamma^{5})u(k)\right]$$
$$\sum_{\substack{e,\nu_e \\ spins}}\left[\overline{u}(p')\gamma_{\mu}(1-\gamma^{5})v(k')\right]\left[\overline{v}(k')\gamma_{\sigma}(1-\gamma^{5})u(p')\right]$$

The sum over spin configurations leads in the same manner as in the case of the elctromagnetic interaction using the completness relations (see p. ???) to traces of matrices:

$$\left|\overline{\mathfrak{M}}\right|^2 = \frac{G_F^2}{4} Tr\left[(\not{k} + m_{\nu_\mu})\gamma^\mu(1-\gamma^5)(\not{p} + m_\mu)\gamma^\sigma(1-\gamma^5)\right]$$

$$Tr\left[(\not{p}' + m_e)\gamma_\mu(1-\gamma^5)(\underbrace{\not{k}' - m_{\nu_e}}_{\substack{-\text{ sign because of different}\\ \text{completness relation for } \nu}})\gamma_\sigma(1-\gamma^5)\right]$$

For clarity masses of all particles were explicitly written in the above expression. In the ultrarelativistic limit we neglect masses of neutrinos and of the electron. On the other hand, mass of the muon can not be neglected (in the muon rest frame the total energy is just m_μ). However, one of the most usefule trace theorems states that the trace of the product of an odd number of γ matrices always equal 0. By inspection one can see that all the terms with m_μ appears in products of an odd number of γ matrices in the above expression (note that γ^5 should be counted not as one but as four γ matrices, because $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$). Hence also the terms with m_μ yields 0:

$$\left|\overline{\mathfrak{M}}\right|^2 = \frac{G_F^2}{4} Tr\left[\not{k}\gamma^\mu(1-\gamma^5)\not{p}\gamma^\sigma(1-\gamma^5)\right] Tr\left[\not{p}'\gamma_\mu(1-\gamma^5)\not{k}'\gamma_\sigma(1-\gamma^5)\right]$$

One of the trace theorems (or an explicit calculation of the traces) yields:

$$\left|\overline{\mathfrak{M}}\right|^2 = \frac{G_F^2}{4} 256(kp')(pk')$$

In the muon rest frame $p = (m_\mu, 0)$ and

$$(p-k')^2 = (p'+k)^2 = \underbrace{p'^2}_{m_e^2 \sim 0} + \underbrace{k^2}_{m_{\nu\mu}^2 \sim 0} + 2p'k$$

$$p'k \approx \frac{1}{2}(p-k')^2$$

$$\left|\overline{\mathfrak{M}}\right|^2 = 32 G_F^2 (p-k')^2 (pk') = 32 G_F^2 (m_\mu - \omega', -\vec{k}')^2 m_\mu \omega' =$$
$$= 32 G_F^2 m_\mu^2 \omega' (m_\mu - 2\omega')$$

Inserting the matrix element into the expression for $d\Gamma$ we obtain

$$d\Gamma = 16 G_F^2 m_\mu \omega' (m_\mu - 2\omega') \frac{1}{(2\pi)^5} \frac{d^3 p'}{2E'} \frac{d^3 k'}{2\omega'} \delta\left((p - p' - k')^2\right)$$

(we left out $\theta(\omega)$ in the above expression since always ω>0). The remaining differentials in $d\Gamma$ can be written as

$$d^3 p' = 4\pi E'^2\, dE'$$
$$d^3 k = 2\pi\omega'^2\, d\omega'\, d(\cos\vartheta)$$

where θ denotes the agle between the electron and electron neutrino 3-momenta.
The δ function can be written as

$$\delta\left((p-p'-k')^2\right) = \ldots = \delta\left(m_\mu^2 - 2m_\mu E' - 2m_\mu\omega' + 2E'\omega'(1-\cos\vartheta)\right) =$$
$$= \delta\left(\ldots + 2E'\omega'\cos\vartheta\right) = \frac{1}{2E'\omega'}\delta\left(\ldots + \cos\vartheta\right)$$

enabling a trivial integration over $\cos\theta$:

$$d\Gamma = \frac{G_F^2}{2\pi^3} m_\mu \omega' (m_\mu - 2\omega') dE'\, d\omega'$$

with an additional requirement following from the δ function:

$$\cos\vartheta = \frac{m_\mu^2 - 2m_\mu E' - 2m_\mu\omega'}{2E'\omega'} + 1$$

which yields the integration boundaries for the final integration over ω'.

$$-1 \le \cos\vartheta \le 1$$

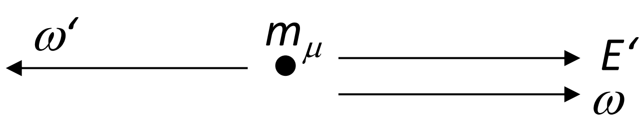
$$-2 \le \frac{m_\mu^2 - 2m_\mu E' - 2m_\mu \omega'}{2E'\omega'} \le 0$$

$$\omega'(2E' - m_\mu) \ge \frac{m_\mu}{2}(2E' - m_\mu)$$

Before continuing one has to clarify whether $(2E'-m_\mu) > 0$ or <0.
If in the decay there would be only two particles in the final state, the electron would have an energy $E' = m_\mu/2$ (in the ultrarelativistic limit). However, since three particles are produced, the electron energy is $E' \le m_\mu/2$. Hence

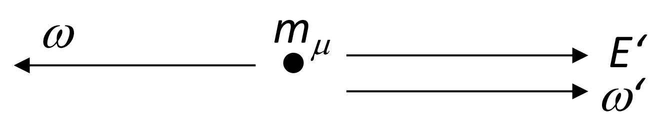
$$\omega' \le \frac{m_\mu}{2}$$

This corresponds to the decay ($\theta=\pi$)



$$\omega' \ge \frac{m_\mu}{2} - E'$$

This corresponds to the decay ($\theta=0$)

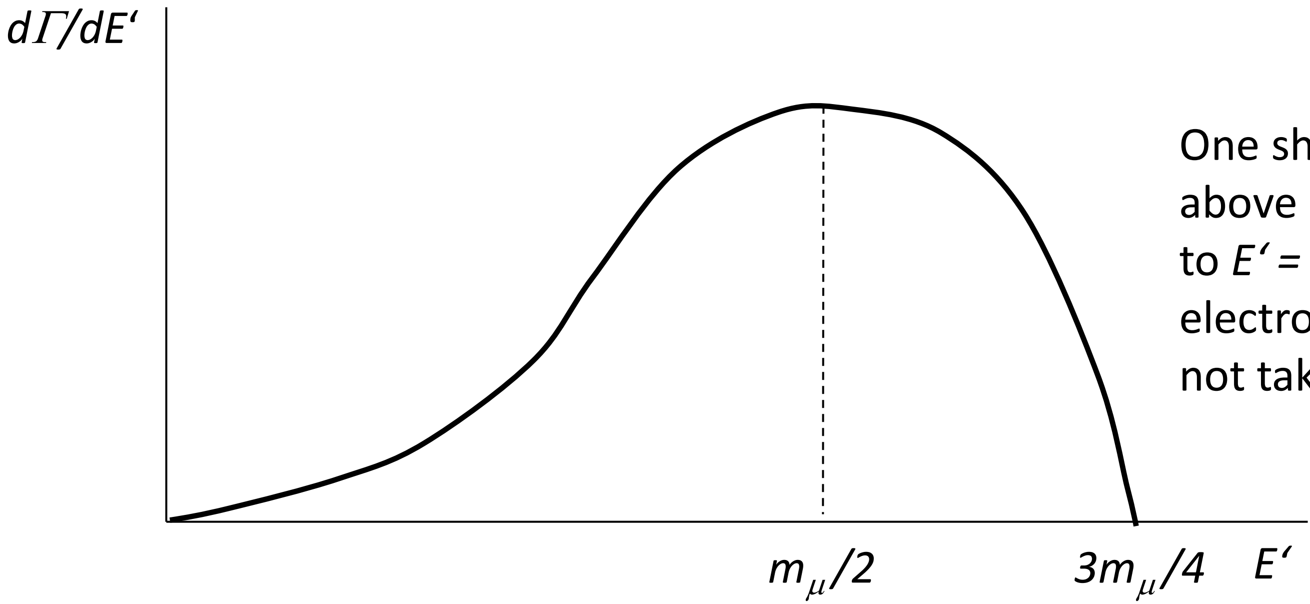


Finally, we arrive at

$$d\Gamma = \frac{G_F^2}{2\pi^3} m_\mu \int_{\frac{m_\mu}{2}-E'}^{\frac{m_\mu}{2}} \omega'(m_\mu - 2\omega')\,d\omega'$$

$$\frac{d\Gamma}{dE'} = \frac{G_F^2}{12\pi^3} m_\mu^2 E'^2 \left(3 - \frac{4E'}{m_\mu}\right)$$

The energy spectrum of electrons from muon decay looks like

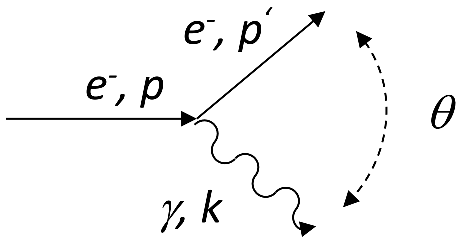


One should note that the above expression is only valid to $E' = m_\mu/2$, because the electron in the decay can not take larger energy.

The total decay width is

$$\Gamma = \frac{G_F^2}{12\pi^3} \int_0^{\frac{m_\mu}{2}} m_\mu^2 E'^2 \left(3 - \frac{4E'}{m_\mu}\right) dE' = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

2.2 Homeworks Solutions



Homework 1:

the simplest way may be to consider the invariant mass of the initial electron;

$p=(mc^2,0)$ 4-momentumof initial e^- in its rest frame

$$p'=(\sqrt{m^2c^4+c^2p'^2},c\vec{p})$$
$$k=(ck,c\vec{k})$$

4-momenta of final e^- and γ in laboratory frame

The magnitude of 4-vectors is invariant to Lorentz transformation. Hence the square of *p* (written in one frame) must be the same as the square of p in the laboratory frame, and this in turn must equal to the square of (*p'+k*) (written in laboratory frame).

$$(mc^2,0)^2=(\sqrt{m^2c^4+c^2p'^2}+k,c\vec{p}+c\vec{k})^2$$
$$m^2c^4=m^2c^4+c^2p'^2+k^2+2k\sqrt{m^2c^4+c^2p'^2}-p^2-k^2-c^2pk\cos\theta$$

this mass is called the „invariant mass" of the initial particle since it's calculated from energies and momenta of final state particles in another frame

With some rearrangements of the above equation we arrive to

$$\cos\theta=\sqrt{1+\frac{m^2c^2}{p'^2}}>1$$ which is clearly impossible.

Homework 2:

operator of infinitezimal rotation around the *z*-axis for an angle ε is written as

$$\hat{R}(\varepsilon)\psi(x,y,z)=\psi(x+\varepsilon y,y-\varepsilon x,z)\underbrace{\approx}_{Taylor\ series}\psi(x,y,z)+\varepsilon(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y})\psi(x,y,z)=$$

$$(1-\frac{i}{\hbar}\varepsilon\hat{\ell}_z)\psi(x,y,z) \qquad ,$$

where $\hat{\ell}_z$ is the z-component angular momentum operator, $\hat{\vec{\ell}}=\hat{\vec{r}}\times\hat{\vec{p}}=-i\hbar\hat{\vec{r}}\times\vec{\nabla}$.

The above equation is jujst the first order in the Taylor expansion, the operator of rotation for a finite angle can be written as

$$\hat{R}(\varepsilon)\psi(x,y,z)=(1-\frac{i}{\hbar}\varepsilon\hat{\ell}_z+\ldots\hat{\ell}_z^2+\ldots)\psi(x,y,z)=e^{-i\varepsilon\hat{\ell}_z/\hbar}\psi(x,y,z)$$

Homework 3:

operator of infinitezimal rotation around the *z*-axis for an angle ε is written as

$$\pi^+ \to \mu^+ \quad \nu_\mu$$

L:	0	-1	+1
B:	0	0	0

The process conserves lepton and baryon number. It conserves charge and is also energetically allowed since $m_\pi c^2$ = 139.6 MeV, $m_\mu c^2$ = 105.7 MeV and $m_\nu \sim 0$.
The above charged pion decay is indeed almost the only pion decay, proceeding through the weak interaction (99.99% of pions decay through this process, see p. ??).

Homework 4:

$\pi^0 \to e^+ e^-$	conserves *B, L,* L_i, charge, allowed
$p \to n\, e^+ \nu_e$	conserves *B, L,* L_i, charge; since $m_p < m_n$ it is only possible for *p*'s bound inside nuclei (β^+ decay)
$K^+ n \to \Sigma^+ \pi^0$	conserves *B, L,* L_i, charge; it would be allowed, however, it turns out that strange quarks carry an additional quantum number – strangeness (see p. ??) which should also be conserved in processes proceeding through the strong interaction; hence this process is forbidden
$K^- p \to \Sigma^0 \pi^0$	conserves *B, L,* L_i, charge; since it also conserves the above mentioned strangeness this process is also allowed.

Homework 5:

Σ^- : Since all baryons have B=1 the hypercharge value determines the strangeness and thus the s quark content. For Σ^- Y=0 $\Rightarrow S$= -1 $\Rightarrow$ one s quark. There should be additional two d quarks in order to match the electric charge, which is also in agreement with I_3 = -1.

Ξ^- : Y=-1 $\Rightarrow S$= -2 $\Rightarrow$ two s quarks, 1 d quark, in agreement with I_3 = -1/2.

Δ^-: Y=1 $\Rightarrow S$= 0 $\Rightarrow$ no s quarks, 3 d quark, in agreement with I_3 = -3/2.

Ω^-: Y=-2 $\Rightarrow S$= -3 $\Rightarrow$ 3 s quarks, no d quark, in agreement with I_3 = 0.

Homework 6:

Neutron wave function is similar to the proton one with the exception of the flavor composition which is of course *d, d, u*.

$$\psi_n = \frac{1}{\sqrt{18}}\left[2\left|d\uparrow d\uparrow u\downarrow\right\rangle - \left|d\uparrow d\downarrow u\uparrow\right\rangle - \left|d\downarrow d\uparrow u\uparrow\right\rangle + \ldots\right]$$

In the same manner as for the proton (see p. ??) one can determine

$$\mu_n = -\frac{2}{3}\frac{e_0}{2m_q}$$

Homework 7:

The flavor parts of the wave function for the mesons are

$$|\omega\rangle = \frac{1}{\sqrt{2}}\left[|u\bar{u}\rangle + |d\bar{d}\rangle\right]$$

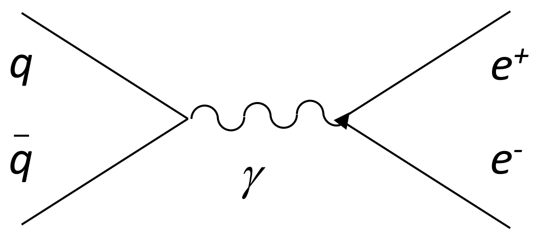
$$|\rho^0\rangle = \frac{1}{\sqrt{2}}\left[|d\bar{d}\rangle - |u\bar{u}\rangle\right]$$

$$|\phi\rangle = |s\bar{s}\rangle$$

$$|J/\psi\rangle = |c\bar{c}\rangle$$

(they are all vector mesons (*J*=1) and hence the flavor part is symmetric; furthermore ω, ϕ are linear combinations of ϕ_0 and ϕ_8 , but the mixing angle is such that ϕ is entirely $s\bar{s}$ and ω entirely $u\bar{u}$, $d\bar{d}$; similar is true for the J/ψ)

Feynman diagram of the process:



Each vertex in the diagram is proportional to the charge of the fermions (electromagnetic interaction, see p. ??). Hence the amplitude is proportional to

$$\left\langle M \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle$$

, where *M* is the corresponding meson, and e_q and e_e are the operators of the quark and electron charges.

For the listed mesons we get

$$\left\langle \omega \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle = \frac{1}{\sqrt{2}} \left[\left\langle u\bar{u} \right| + \left\langle d\bar{d} \right| \right] \hat{e}_q \hat{e}_e \left| e^+ e^- \right\rangle \propto$$

$$\frac{1}{\sqrt{2}} \left[\frac{2}{3} + \left(-\frac{1}{3} \right) \right] = \frac{1}{\sqrt{2}} \frac{1}{3}$$

$$\left\langle \rho \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle = \frac{1}{\sqrt{2}} \left[\left\langle d\bar{d} \right| - \left\langle u\bar{u} \right| \right] \hat{e}_q \hat{e}_e \left| e^+ e^- \right\rangle \propto$$

$$\frac{1}{\sqrt{2}} \left[\left(-\frac{1}{3} \right) - \frac{2}{3} \right] = \frac{1}{\sqrt{2}} (-1)$$

$$\left\langle \phi \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle = \left\langle s\bar{s} \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle \propto -\frac{1}{3}$$

$$\left\langle J/\psi \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle = \left\langle c\bar{c} \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle \propto \frac{2}{3}$$

Ratios of decay rates are

$$\left|\left\langle \omega \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle\right|^2 : \left|\left\langle \rho \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle\right|^2 : \left|\left\langle \phi \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle\right|^2 : \left|\left\langle J/\psi \middle| \hat{e}_q \hat{e}_e \middle| e^+ e^- \right\rangle\right|^2 =$$

$$\frac{1}{2\cdot 9} : \frac{1}{2} : \frac{1}{9} : \frac{4}{9} = 1 : 9 : 2 : 8$$

to be compared to the experimentally determined ratios of 1 : 11.8 : 2.1 : 9.3. The deviations point to defficiencies of the simplest quark model.

Homework 8:

$$\partial^\nu \partial_\nu A_\mu = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \left(\frac{1}{q^2} e\overline{u}(p')\gamma_\mu u(p) e^{iqx} \right) =$$

$$= \frac{1}{q^2} e\overline{u}(p')\gamma_\mu u(p) \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) e^{iqx} =$$

$$= \frac{1}{q^2} e\overline{u}(p')\gamma_\mu u(p) \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) e^{iq_0 t - i\vec{q}\vec{r}} =$$

$$= \frac{1}{q^2} e\overline{u}(p')\gamma_\mu u(p) \left(-q_0^2 + |\vec{q}|^2 \right) e^{iqx} = \frac{1}{q^2} e\overline{u}(p')\gamma_\mu u(p) \left(-q^2 \right) e^{iqx} =$$

$$= -e\overline{u}(p')\gamma_\mu u(p) e^{iqx} = j_\mu$$

Appendix A: Covariant form of Maxwell equations

Classical form of Maxwell equations is

$$\nabla\vec{E} = \rho, \quad \nabla\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0, \quad \vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = \vec{j}$$

If we introduce the scalar and vector potentials A^0 and $\vec{A}$:

$$A^\mu = (A^0, \vec{A}), \quad \vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}A_0, \quad \vec{B} = \vec{\nabla}\times\vec{A} \;,$$

then two of the Maxwell eqs. are automatically satisfied, since

$$\nabla\vec{B} = \nabla(\vec{\nabla}\times\vec{A}) = 0$$
$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}A^0\right) + \frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right) =$$
$$= -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A} - \vec{\nabla}\times\vec{A}\right) - \vec{\nabla}\times\left(\vec{\nabla}A^0\right) = 0$$

Furthermore we can show that the other two eqs. can be written as

$\partial^\nu \partial_\nu A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = j^\mu$, with the current four-vector defined as $j^\mu = (\rho, \vec{j})$.

To see this we write out the above covariant form of the equations:

$$\partial^\nu \partial_\nu A^\mu - \partial^\mu \left(\partial_\nu A^\nu\right) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(A^0, \vec{A}\right) - \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\left(\frac{\partial A^0}{\partial t} + \vec{\nabla}\vec{A}\right) = \left(\rho, \vec{j}\right)$$

The time component of this equation is

$$\left(\frac{\partial^2 A^0}{\partial t^2} - \nabla^2 A^0\right) - \left(\frac{\partial^2 A^0}{\partial t^2} + \frac{\partial}{\partial t}\vec{\nabla}\vec{A}\right) = \rho$$

$$-\nabla^2 A^0 - \frac{\partial}{\partial t}\vec{\nabla}\vec{A} = \rho$$

Inserting $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A_0$ into equation $\nabla \vec{E} = \rho$ we obtain the same equation.

An analogous test can be performed for the space component proving that the covariant form reproduces the equation $\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}$ (in the derivation one can use the relation $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\nabla^2 \vec{A} + \vec{\nabla}\left(\vec{\nabla}\vec{A}\right)$)

On p. ??? we used the covariant form $\partial^{\nu}\partial_{\nu}A^{\mu} - \partial^{\mu}\left(\partial_{\nu}A^{\nu}\right) = j^{\mu}$ without the second term.

The reason is that observable fields *E* and *B* are invariant to gauge transfromations of the type $A^{\mu} \to A^{\mu} + \partial^{\mu}\chi$, where χ is any scalar function.

This can be proven by explicit calculation of *E'* and *B'* fields, with

$$A^{\mu'} = (A^{0}{}', \vec{A}') = \left(A0 + \frac{\partial\chi}{\partial t}, \vec{A} - \vec{\nabla}\chi \right),$$

$$\vec{E}' = -\frac{\partial\vec{A}'}{\partial t} - \vec{\nabla}A_{0}{}', \quad \vec{B}' = \vec{\nabla} \times \vec{A}'$$

and showing that *E'=E* and *B'=B*.

Beacuse of this invariance we can always choose χ such that

$$\partial_{\mu}A^{\mu} \to \partial_{\mu}A^{\mu} + \partial_{\mu}\partial^{\mu}\chi = 0$$

Appendix B: Classic Hamiltonian of a particle in electromagnetic field

Force acting on a point charge *q* in electric field $\vec{E}$ and magnetic field $\vec{B}$ is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

One can see that tthe Lagrangian leading to this force is

$$L = \frac{1}{2} m v^2 - q A_0 + q \vec{v} \vec{A},$$ where A_0 and $\vec{A}$ are the scalar and vector potential

introduced on p. ???. We can prove this using the Euler-Lagrange equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_i}\right] - \frac{\partial L}{\partial x_i} = 0$$

$$\frac{\partial L}{\partial \dot{x}_i} = m v_i + q A_i, \quad \frac{\partial L}{\partial x_i} = -q \frac{\partial A_0}{\partial x_i} + q v_j \frac{\partial A_j}{\partial x_i} \Rightarrow$$

$$\frac{d}{dt}[m v_i + q A_i] + q \frac{\partial A_0}{\partial x_i} - q v_j \frac{\partial A_j}{\partial x_i} = 0$$

$$m a_i + q \frac{\partial A_i}{\partial t} + q \frac{\partial A_0}{\partial x_i} - q v_j \frac{\partial A_j}{\partial x_i} = 0$$

$$F_i = -q \frac{\partial A_i}{\partial t} - q \frac{\partial A_0}{\partial x_i} + q v_j \frac{\partial A_j}{\partial x_i}$$

$$\vec{F} = -q \frac{\partial \vec{A}}{\partial t} - q \vec{\nabla} A_0 + q \vec{v} \times (\vec{\nabla} \times \vec{A})$$

The last term in the above equation follows from

$$\vec{v}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{v}\vec{A})-\vec{A}(\underbrace{\vec{\nabla}\vec{v}}_{=0})=\vec{v}\vec{\nabla}\vec{A}$$

Taking into account

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}A_0\,,\quad \vec{B}=\vec{\nabla}\times\vec{A}$$ one indeed arrives at $\vec{F}=q(\vec{E}+\vec{v}\times\vec{B})$

Hamiltonian is

$$H=\sum_i p_i\dot{x}_i-L(x_i,\dot{x}_i)$$

with $p_i=\frac{\partial L}{\partial\dot{x}_i}\Rightarrow p_i=mv_i+qA_i$

$$H=\sum_i(m_i v_i+qA_i)v_i-\frac{1}{2}mv^2+qA_0-q\vec{v}\vec{A}=$$

$$=\frac{1}{2}mv^2+\underbrace{qA_0}_{potential\ energy}$$

$$\vec{p}=m\vec{v}+q\vec{A}\Rightarrow\vec{v}=\frac{\vec{p}-q\vec{A}}{m}\Rightarrow H=\frac{1}{2}\frac{(\vec{p}-q\vec{A})^2}{m}+qA_0$$

Table of contents