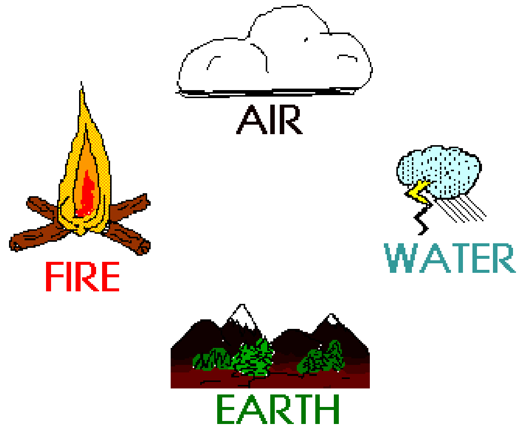
Nuclear and Particle Physics

Introduction

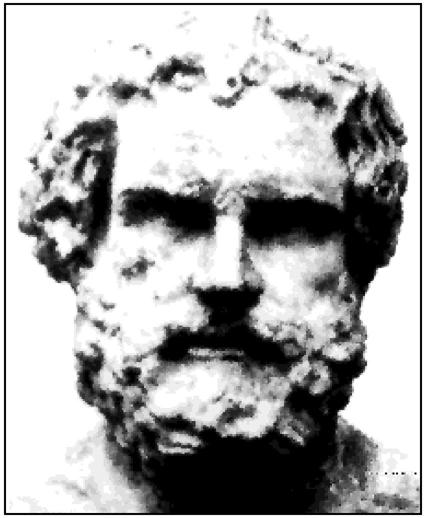
What the elementary particles are: a bit of history

The idea about the elementary particles has changed in the course of history, in accordance with the human's comprehension and later observation of nature.



Ancient Greeks believed that the world is made of four basic elements: air, fire, water and earth.

Demokritos, 4th century B.C.: the world is composed of the smallest indivisable parts – atoms.



A bit of history:

D. Mendeljejev, 1869: periodic system of elements

JJ. Thompson, 1897: discovery of electron (e-)

↓ student

Ernest Rutherford, 1911:
explains the structure of an atom with the atomic nucleus

students

Geiger, Marsden: they do the experimental work

E. Rutherford: “All science is either physics or stamp collecting.”

Ernest Rutherford, 1911: discovers that all nuclei consist of the Hydrogen nucleus, which is considered as the discovery of the proton (p) proton: greek word for “first”, πρῶτον.

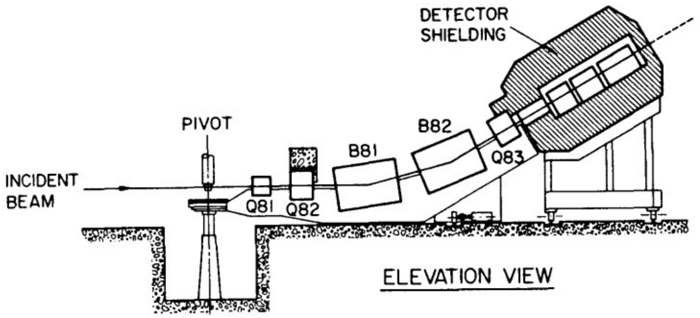
In this laboratory J. Chadwich in 1932 discovered neutrons (n)

A bit of history:

M. Gell-Mann, G. Zweig in 1964 suggest that n and p are composed of quarks
quark: J. Joyce, Finnegans Wake

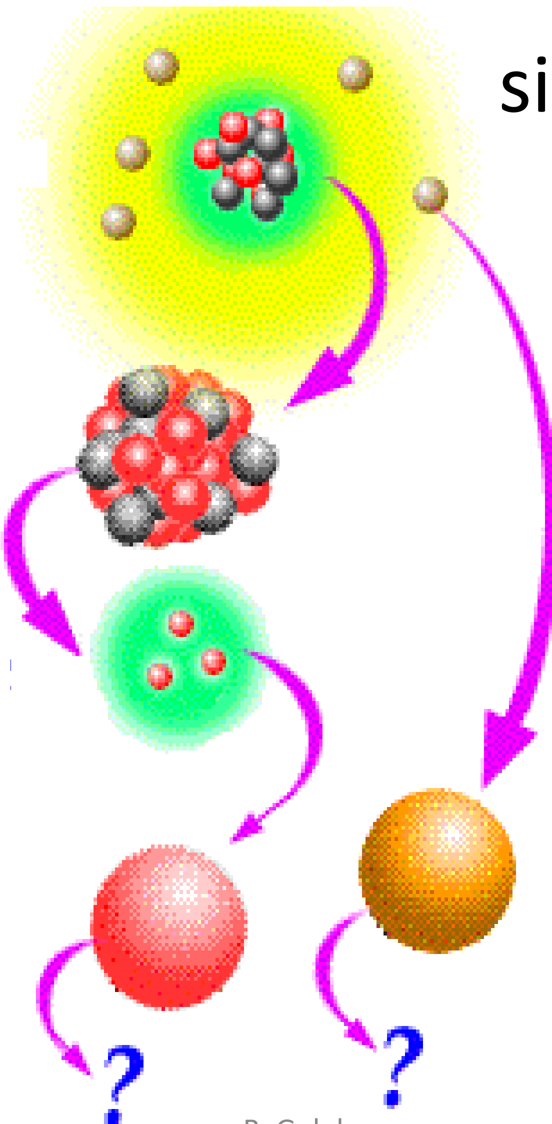
Gell-Mann receives the Nobel prize in physics in 1969, for the classification of the elementary particles; Zweig does not get the Nobel prize

J.I. Friedman, H.W. Kendall in R. Taylor "repeat" the Rutherford experiment (see p. 23) in 1967-73



They experimentally confirm the existence of quarks and live to get the Nobel prize in 1990.

Composition of the world as seen today:



size in m		size in 10^{-18} m
10^{-10}m	atom	100,000,000
10^{-14}m	nucleus	10,000
10^{-15}m	proton	1,000
$\leq 10^{-18}$m	quark / electron	≤ 1

Part 1, Nuclear Physics

1.1 Basic properties of nuclei

1.1.1 Mass

Semi-empirical mass formula (Weizsäcker formula):

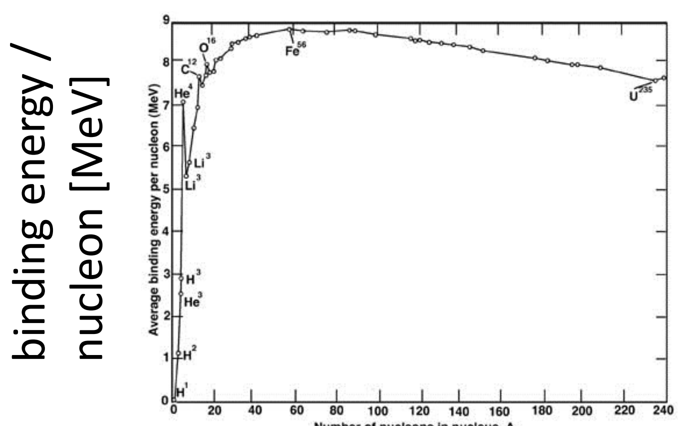
"empirical": written with the intention of describing the experimental data
"semi": there are phenomenological reasons for inclusion of individual terms in the equation

mass of a nuclei with Z protons and (A-Z)=N neutrons (p, n = nucleons):

$$m(A,Z) = Zm_p + (A-Z)m_n + W/c^2$$

W: binding energy of nucleons in the nuclei, *W* < 0;
negative binding energy is a consequence of the strong nuclear force binding the nucleons inside the nuclei; alternatively, the binding energy is a consequence of a attractive potential among the nucleons

1) the average binding energy per nucleon is approximately constant:



Semi-empirical mass formula (Weizsäcker formula):

2) deviations from the average value are seen at low A; this is a consequence of the fact that for low A there are relatively more nucleons at the “surface” of the nuclei and are hence less bound (because of a lower number of neighbouring nucleons); the effect is a positive term in the binding energy, proportional to the surface, i.e. a term proportional to r^2

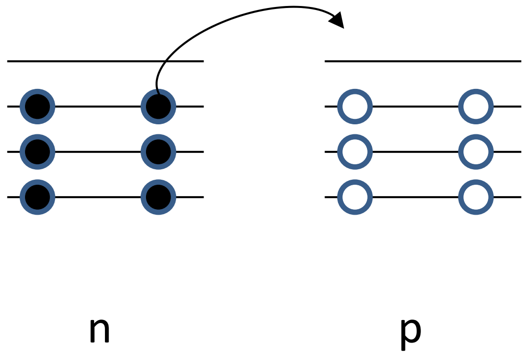
3) deviations are seen also at high Z values; this is a consequence of the Coulomb repulsion among positively charged protons; the effect is a positive term in the binding energy, proportional to the electrostatic potential energy of Z protons, i.e. proportional to Z^2/r

$$E = \frac{Ze}{4\pi\varepsilon_0 r^2}; U(\infty) - U(r) = -\int_r^\infty E dr = \frac{Ze}{4\pi\varepsilon_0 r} = 0 - U(r)$$

$$W_{elec} = ZeU(r) = \frac{Z^2 e^2}{4\pi\varepsilon_0 r}$$

Semi-empirical mass formula (Weizsäcker formula):

4) it is energetically favourable for nuclei to have the same number of n and p; if we consider energy levels of nucleons in the nuclei (similarly at the energy levels of electrons in the atom):



p and n are fermions (spin 1/2 particles); in accordance with the Pauli exclusion principle each energy level is occupied with two identical fermions differing in the 3rd component of the spin

if in the above picture one n is replaced by a p, it must occupy a higher energy level which means it is less bound; the effect is a positive term per nucleon in the binding energy proportional to $(Z/A - 1/2)^2$ (a quadratic effect since it is less favourable for a nuclei to have a larger number of p or n); for the total binding energy this is $A\,(Z/A - 1/2)^2 \propto (2Z - A)^2/A$

5) it is experimentally observed that nuclei with an even number of p and n (even-even nuclei) are stronger bound than the nuclei with an odd number of either p or n and even stronger bound than the nuclei with an odd number of both p and n; this is a consequence of the Pauli exclusion principle, similarly as mentioned under 4);

Semi-empirical mass formula (Weizsäcker formula):

the effect is a term in the binding energy proportional to

$$\delta(A,Z)=\begin{cases}-1 & even-even\\ 0 & even-odd\\ +1 & odd-odd\end{cases}$$

There are terms in the binding energy depending on the "radius" of a nuclei; experimentally (as described on p. 24) we can see that the "radius" of nuclei is given approximately by $r=r_0\,A^{1/3}$, with $r_0 \approx 1.2$ fm.

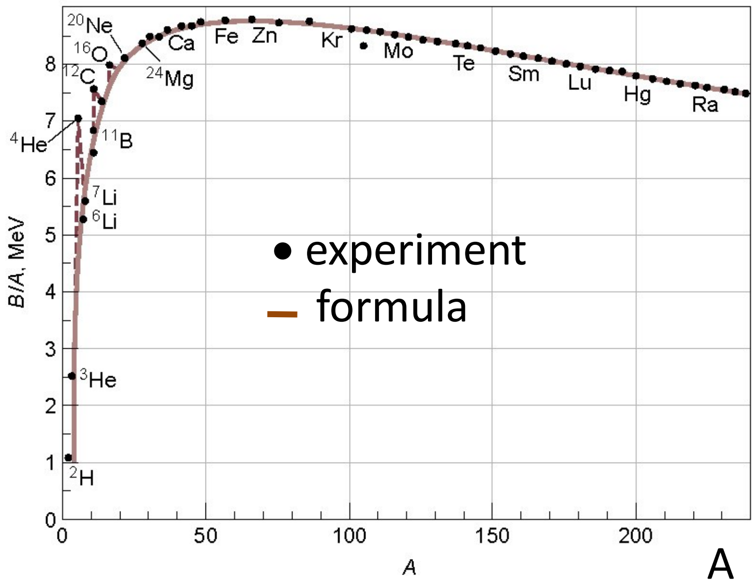
Summing all the mentioned terms the binding energy is

$$W(A,Z)=-w_0A+w_1A^{2/3}+w_2\frac{Z^2}{A^{1/3}}+w_3\frac{(2Z-A)^2}{A}+w_4A^{-3/4}\delta(A,Z)$$

Constants $w_{0\text{-}4}$ are determined to describe data, typical values found are
w_0=15.6 MeV, w_1=17.2 MeV, w_2=0.7 MeV, w_3=23.2 MeV, w_4=12 MeV

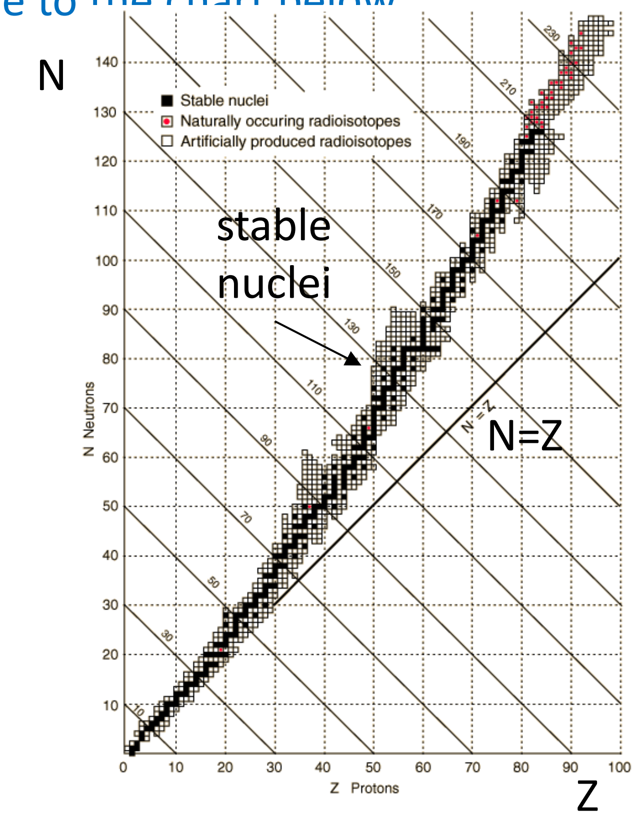
Such a formula describes well experimentally observed values:

Semi-empirical mass formula (Weizsäcker formula):



Homework 1: calculate masses for some of the nuclei in the chart

Homework 2: estimate which nucleus of a given mass number is the most stable; compare to the chart below

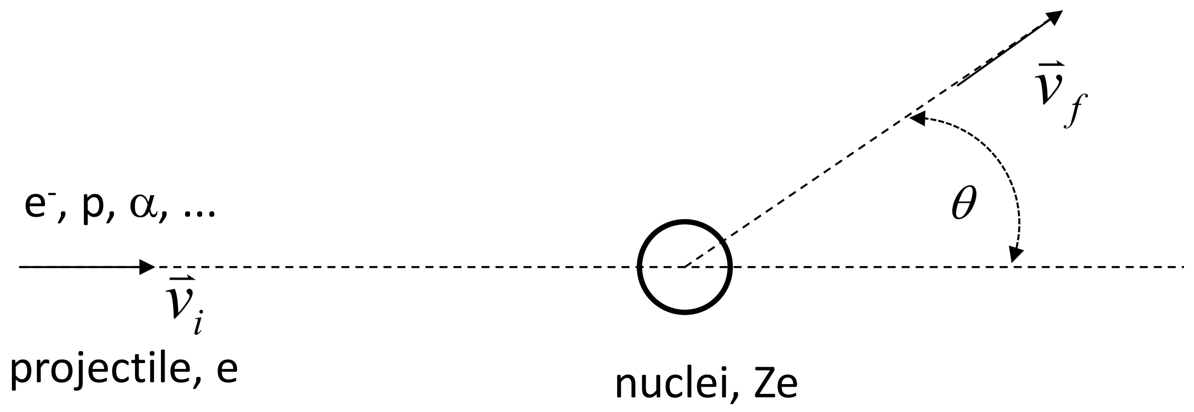


1.1 Basic properties of nuclei

1.1.2 Charge distribution

EM scattering

Let’s consider an experiment where a charged projectile
is scattered off the nuclei due to the Coulomb force (electromagnetic
potential; an experiment similar to the one which Rutherford did to discover the atomic
nuclei)



we write the quantum mechanical (but classical - as opposed to relativistic) expression for
probability of scattering the projectile through a scattering angle θ
(interval of solid angle $d\Omega = 2\pi \sin\theta \, d\theta$)

EM scattering

Fermi golden rule:

$$W_{fi} = \frac{2\pi}{\hbar} \left| V_{fi} \right|^2 \rho_f(E_i)$$

W_{fi}: probability for a system (in our case projectile) transition from an initial state *i* to a final state *f* per interval of time (units s^{-1})
V_{fi}: matrix element for the $i \rightarrow f$ transition
$\rho_f(E_i)$: density of final states at the energy of the initial state E_i

initial state *i*: projectile with the velocity , described by the wave function $\psi_i(\vec{r})$
final state *f*: projectile with the velocity , described by the wave function $\psi_f(\vec{r})$
wave function: $|\psi|^2$ is the probability density, integrated over a volume gives probability of finding the state in this volume;

$$\int_{all\,space} \left| \psi_{i,f} \right|^2 d^3 r = 1$$

the simples approximation is to describe the projectile (far away from the scattering centre) as a plane wave:

$$\psi_{i,f} = \frac{1}{\sqrt{V_n}} e^{i\vec{k}_{i,f}\vec{r}}$$

EM scattering

V_n is the normalization volume, the wave function is normalized as 1 particle per V_n;
V_n is arbitrary and has to cancel in any final expression describing any observable
$\vec{k}$ is the wave vector:
λ: de Broglie wavelength

$$k = \frac{2\pi}{\lambda}; \lambda = \frac{h}{p}; k = \frac{p}{\hbar}$$

number of final states in a volume element of space and momentum:

$$d^6 N_f = \frac{d^3 r d^3 p}{h^3}; d^3 N_f = V_n \frac{d^3 p}{h^3}$$

ρ_f: density of final states

$$d\rho_f = \frac{d^3 N_f}{dE_f} = V_n \frac{p_f^2 dp_f d\Omega}{h^3 dE_f}$$

$$\frac{d\rho_f}{d\Omega} = V_n \frac{p_f^2 dp_f}{h^3 dE_f}; E_f^2 = p_f^2 / 2m; dE_f = p_f dp_f / m$$

$$\frac{d\rho_f}{d\Omega} = V_n \frac{m p_f}{(2\pi\hbar)^3}; \frac{d\rho_f}{d\Omega}(E_i) = V_n \frac{m^2 v_i}{(2\pi\hbar)^3}$$

EM scattering

V_{fi} : matrix element; expectation value of the potential causing the $i \rightarrow f$ transition

$$V_{fi} = \int \psi_f^*(\vec{r}) V(\vec{r}) \psi_i(\vec{r}) d^3 r$$

From the Fermi golden rule we construct a new observable, more directly related to the measurement - the differential cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{dW_{fi} / d\Omega}{\rho_i v_i}$$

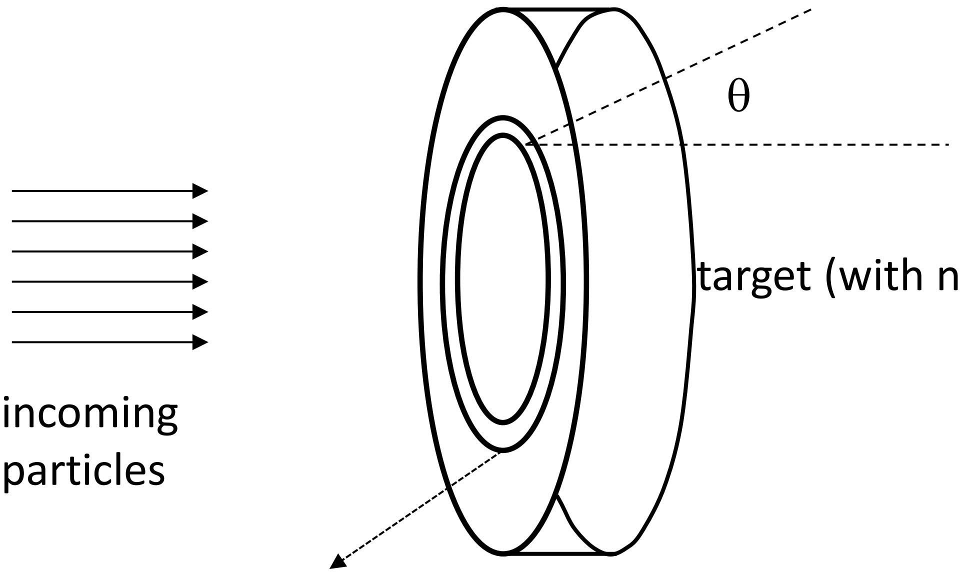
it represents the probability for the transition per unit of time and interval of the solid angle Ω, normalized to the incoming flux of projectiles $\rho_i v_i$; since we introduced the normalization volume V_n, the density of incoming particles is just $1/V_n$

units for $d\sigma/d\Omega$:

$$\frac{1/s}{1/m^3 \; m/s} = m^2$$

EM scattering

the differential cross-section can be measured in the following way:



number of scattered projectiles in the interval of the solid angle dΩ:

$$\frac{dN}{d\Omega} = \frac{d\sigma}{d\Omega}\frac{dN_t}{dS}; \quad \frac{dN_t}{dV} = \rho_t \frac{dN_t}{dm} = \rho_t \frac{Z}{M}$$

$$\frac{dN_t}{dV} = \frac{1}{l}\frac{dN_t}{dS}$$

$$\frac{dN}{d\Omega} = l\rho_t \frac{Z}{M}\frac{d\sigma}{d\Omega}$$

by measuring the number of scattered projectiles into the solid angle interval dΩ and knowing the properties of the target ρ_t, *Z, M* (mass of nucleus), *l*, one can determine $d\sigma/d\Omega$

EM scattering

to calculate the matrix element

$$V_{fi} = \int \psi_f^*(\vec{r}) V(\vec{r}) \psi_i(\vec{r}) d^3 r$$

we first insert the plane wave approximation for the wave functions:

$$V_{fi} = \frac{1}{V_n} \int e^{i(\vec{k}_i - \vec{k}_f)\vec{r}} V(\vec{r}) d^3 r = \frac{1}{V_n} \int e^{i\vec{q}\vec{r}} V(\vec{r}) d^3 r$$

with $\vec{q}=\vec{k}_i-\vec{k}_f$

Then we make use of the Green's formula:

$$\int_V \left[u\nabla^2 v - v\nabla^2 u \right] d^3 r = \oint_S \left[u\vec{\nabla} v - v\vec{\nabla} u \right] d\vec{S}$$

where we take $u = e^{i\vec{q}\vec{r}}, v = V(\vec{r})$

The potential energy of the projectile and the nucleus is $V(r) = \frac{Ze^2}{4\pi\varepsilon_0 r} = eU(r)$

and the potential *U(r)* is related to the electric charge distribution in the nucleon by

$$\nabla^2 U(r) = -\frac{\rho(r)}{\varepsilon_0}$$

$\rho(r)$ is normalized so that $\int \rho(r) d^3 r = Ze$

EM scattering

far away from the nucleus we can assume $$V(r), \vec{\nabla} V(r) \xrightarrow[r\to\infty]{} 0$$

and hence the right hand side of the Green's formula equals 0.
The left hand side, taking into account

$$\nabla^2 V(r) = -e\frac{\rho(r)}{\varepsilon_0}, \nabla^2 e^{i\vec{q}\vec{r}} = -q^2 e^{i\vec{q}\vec{r}}$$

yields $$-\frac{e}{\varepsilon_0}\int e^{i\vec{q}\vec{r}}\rho(r)d^3r + q^2\int e^{i\vec{q}\vec{r}}V(r)d^3r = 0$$

$$V_{fi} = \frac{e}{\varepsilon_0 V_n q^2}\int e^{i\vec{q}\vec{r}}\rho(r)d^3r = \frac{e}{\varepsilon_0 V_n q^2}F(\vec{q})$$

The matrix element V_{fi} is thus related to the charge distribution in the nucleus, more precisely to its Fourier transform, which we denote by $F(\vec{q})$ and call the form factor.
Now we insert all the ingredients to the expression for the differential cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{dW_{fi}/d\Omega}{\rho_i v_i} = \frac{2\pi}{\hbar}\left[\frac{e}{\varepsilon_0 V_n q^2}\right]^2 \left|F(\vec{q})\right|^2 \frac{V_n^2}{v_i}\frac{mp_f}{(2\pi\hbar)^3}$$

EM scattering

after rearrangement we can write a more compact form

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar}\left[\frac{me}{\varepsilon_0 q^2}\right]^2 \left|F(\vec{q})\right|^2 \frac{1}{(2\pi\hbar)^3}$$

in case of elastic scattering (E_i=E_f), q^2 equals

$$q^2 = (k_i - k_f)^2 = k_i^2 + k_f^2 + 2k_i k_f \cos\vartheta$$

$$k_i^2 = k_f^2 = \frac{p^2}{\hbar^2}; q^2 = 4\frac{p^2}{\hbar^2}\sin^2\frac{\vartheta}{2}$$

and so

$$\frac{d\sigma}{d\Omega} = \left[\frac{me}{8\pi\varepsilon_0 p^2}\right]^2 \frac{1}{\sin^4\frac{\vartheta}{2}}\left|F(\vec{q})\right|^2$$

EM scattering

in case of a point-like nucleon, an assumption valid if $\lambda >> R;\ p << h / R$
where *R* is the size of the nucleus,
charge distribution is just $\rho(r) = Ze\delta(r - r_0)$

and the form factor is

$$F(\vec{q}) = Ze\int e^{i\vec{q}\vec{r}}\delta(r - r_0)d^3r = Ze\,e^{i\vec{q}\vec{r}_0}$$

$$\left|F(\vec{q})\right|^2 = Z^2e^2$$

the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \left[\frac{Ze^2m}{8\pi\varepsilon_0 p}\right]^2 \frac{1}{\sin^4\frac{\vartheta}{2}}$$

and describes what is called the Rutherford scattering (most of projectiles at small scattering angles, some also at large).

EM scattering

We can use a Taylor expansion in the form-factor expression:

$$F(\vec{q}) = \int e^{i\vec{q}\vec{r}} \rho(r) d^3r = \int_0^\infty \rho(r) dr \int_0^\pi e^{iqr\cos\alpha} 2\pi r^2 \sin\alpha d\alpha =$$

$$= 4\pi \int_0^\infty \rho(r) \frac{\sin(qr)}{qr} r^2 dr \approx \int_0^\infty \rho(r) \left(1 - \frac{(qr)^2}{3!} + \frac{(qr)^4}{5!} + \dots\right) 4\pi r^2 dr =$$

$$= Ze\left(1 - \frac{q^2}{6}\langle r^2 \rangle + \dots\right)$$

the integration goes to $r\to\infty$; however, if $\rho(r)$ is null for some $r >> R$ values, and $qR<<1$, we can do the Taylor expansion

By measuring $d\sigma/d\Omega$ we can determine the average square radius of the charge distribution within the nucleus. This is how one can experimentally verify the previously given relation $r = r_0 A^{1/3}$ (p. 12).

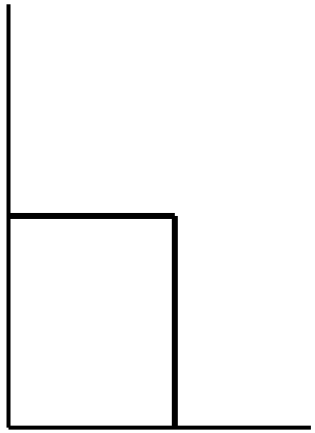
One can of course do more, and try to determine the shape of $\rho(r)$ more precisely.

EM scattering

for an assumed $\rho(r)$ in the form of a step function

homework 3: calculate this $|F(q)|^2$

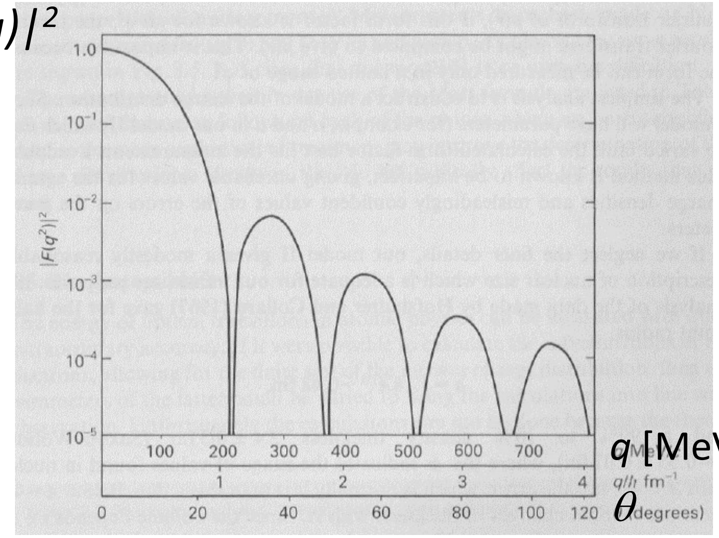
$\rho(r)$



4 r [fm]

the calculated $|F(q)|^2$ is:

$|F(q)|^2$

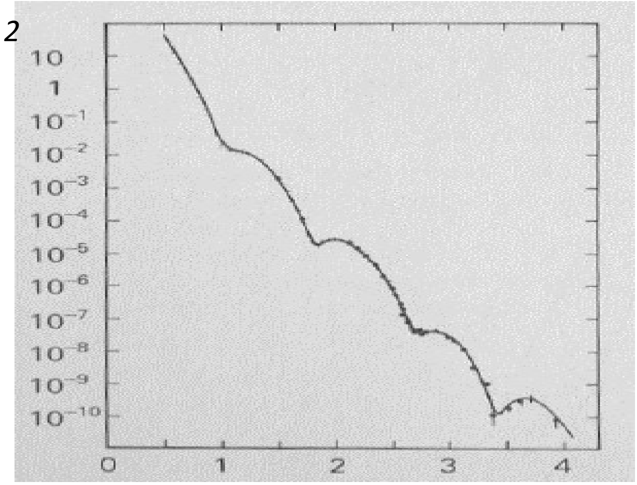


q [MeV/c]

θ

The measurement of 450 MeV e^- scattering on $^{58}_{28}Ni$ nuclei gives

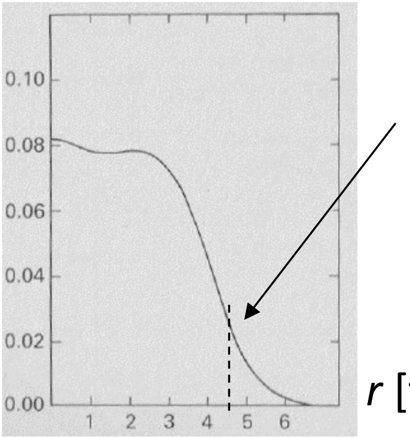
$|F(q)|^2$



q [MeV/c]

one can try various $\rho(r)$ ansatzs and the one which fits the measured data best is

$\rho(r)$



r [fm]

r calculated from $r = r_0 A^{1/3}$ for $A=58$

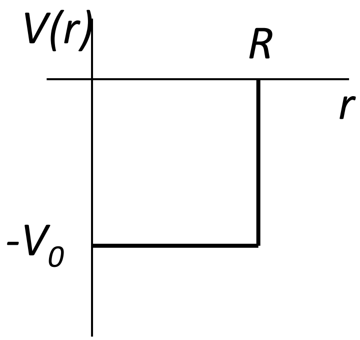
1.1 Basic properties of nuclei

1.1.3 Spin

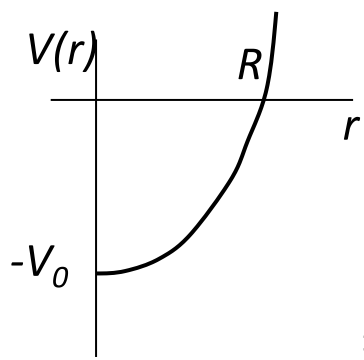
Principle of Schrödinger equation solving – shell model of nuclei

1) assume an average potential felt by nucleons
2) solve the Schrödinger equation
3) nucleons fill the calculated energy levels in accordance with the Pauli exclusion principle
4) check the magic numbers (Z and N of nuclei exhibiting larger than the average binding energy
5) in case of discrepancy with the experimental data correct the potential $\rightarrow$1)

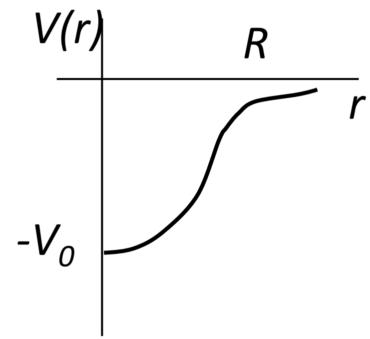
Some possible potential shapes:



$\Rightarrow$ improvement



$\Rightarrow$ improvement



potential well

harm. oscillator

$$V(r) = -V_0 + \frac{1}{2}\mu\omega^2 r^2$$

Saxon-Woods potential

$$V(r) = \frac{-V_0}{1+e^{(r-R)/a}}$$

Principle of Schrödinger equation solving

Schrödinger equation

$$\hat{H}\psi = E\psi$$

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\psi = E\psi$$

μ: reduced mass of a nucleon and the rest of the nucleus, $\mu = m_n m_N/(m_n+m_N) \sim m_n$

solutions of the equation are E's - single nucleon energy levels

solutions are typically searched for by the ansatz

$$\psi = R(r)Y_{lm}(\theta,\varphi)$$

$$R(r) = u(r)/r$$

with Y_{lm} denoting the spherical harmonics

$$-\frac{\hbar^2}{2\mu}\nabla^2 = -\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{\ell}^2}{2\mu r^2}$$

with $\hat{\ell}^2$ denoting the operator of the square of the angular momentum

$$\hat{\ell}^2 Y_{lm}(\theta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\theta,\varphi)$$

The Schrödinger equation simplifies to

$$-\frac{\hbar^2}{2m_n}\frac{\partial^2 u(r)}{\partial r^2} + \left[V(r) + \frac{\hbar^2 l(l+1)}{2m_n r^2}\right]u(r) = Eu(r)$$

The angular momentum term acts as an additional potential

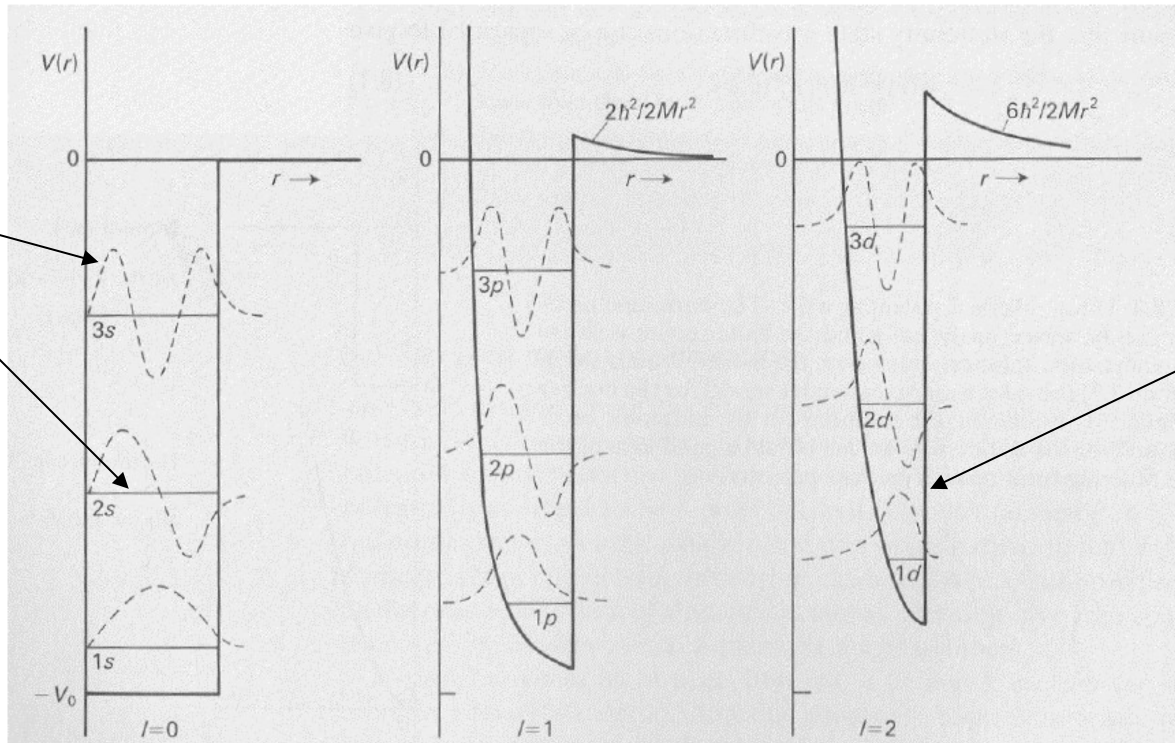
Homework 4: verify that with $\psi = \frac{u(r)}{r}Y_{lm}(\theta,\varphi)$ eq. $\hat{H}\psi = E\psi$ reduces to this form!

Principle of Schrödinger equation solving

solutions for the potential well: solutions for E and ψ depend on the main quantum number n (due to the boundary conditions) and orbital angular momentum quantum number l

wave functions

energy levels



potential

$$-V_0 + \frac{\hbar^2 l(l+1)}{2m_n r^2}$$

individual solutions are marked by *nl*, with *l=0* marked as *s*, *l=1* as *p*, *l=2* as *d*, ...
for higher *l* the nucleons are “pushed” to higher radii

Principle of Schrödinger equation solving

solutions for the harmonic oscillator ($-V_0+(1/2)m_N\omega^2r^2$): $E_{n,\ell}=(2n+\ell-1/2)\hbar\omega$

for the harmonic oscillator the first three magic numbers (see p. 33, left column) are in accordance with the measurements (2,8,20; separately for n and p), the higher are not.

Using a finite potential changes the energy levels to some extent although the situation with the magic numbers remains the same (the magic numbers are those,
where there are larger energy gaps between groups of energy levels; see p. 33, middle column).

To better explain the energy levels and hence the magic numbers of nuclei one needs to consider the spin-orbit interaction. This is an additional term in the potential arising from the interaction between the total spin of two nucleons and their relative orbital angular momentum. It takes the form $E_{ls}=-2\eta\vec{\ell}\cdot\vec{s}$

where $\vec{\ell}$ denotes the orbital angular momentum and $\vec{s}$ the spin.

Principle of Schrödinger equation solving

Inclusion of the spin-orbit term in the potential changes the quantum numbers by which the individual solutions can be labelled; instead of good quantum numbers *n, l,* l_z, *s* and s_z (good quantum numbers determine the expectation values of the corresponding operators, in the above case of the operators of energy, magnitude and 3rd component of the orbital angular momentum, $\hat{\ell}^2, \hat{\ell}_z$, and the magnitude and the 3rd component of the spin, $\hat{s}^2, \hat{s}_z$; these operators commute with the Hamiltonian operator, and hence the expectations values of those operators are conserved) one has good quantum numbers *n, j,* j_z, *l*, and *s* (due to the fact that the Hamiltonian now includes the $\hat{\vec{\ell}} \cdot \hat{\vec{s}}$ term, it does not commute any more with the operators $\hat{\ell}_z, \hat{s}_z$; it does, however commute, with the magnitude of the total angular momentum magnitude and its 3rd component $\hat{j}^2, \hat{j}_z$).

In order to understand the solutions of the Schrödinger eq. with the inclusion of the spin-orbit term in the potential, first one needs the relation between the total angular momentum $\hat{j}^2$ and $\hat{\vec{\ell}} \cdot \hat{\vec{s}}$:

$$\hat{\vec{j}} = \hat{\vec{\ell}} + \hat{\vec{s}}$$

$$\hat{j}^2 = (\hat{\vec{\ell}} + \hat{\vec{s}})^2 = \hat{\ell}^2 + \hat{s}^2 + 2\hat{\vec{\ell}} \cdot \hat{\vec{s}}$$

$$\left\langle \hat{\vec{\ell}} \cdot \hat{\vec{s}} \right\rangle = \frac{1}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]\hbar^2$$

note: $\hat{\vec{\ell}}, \hat{\vec{s}}$: operators
l, s : quantum numbers

Principle of Schrödinger equation solving

the last line in the equation was derived using the fact that

$$\hat{\ell}^2\psi = l(l+1)\hbar^2\psi, \hat{s}^2\psi = s(s+1)\hbar^2\psi$$

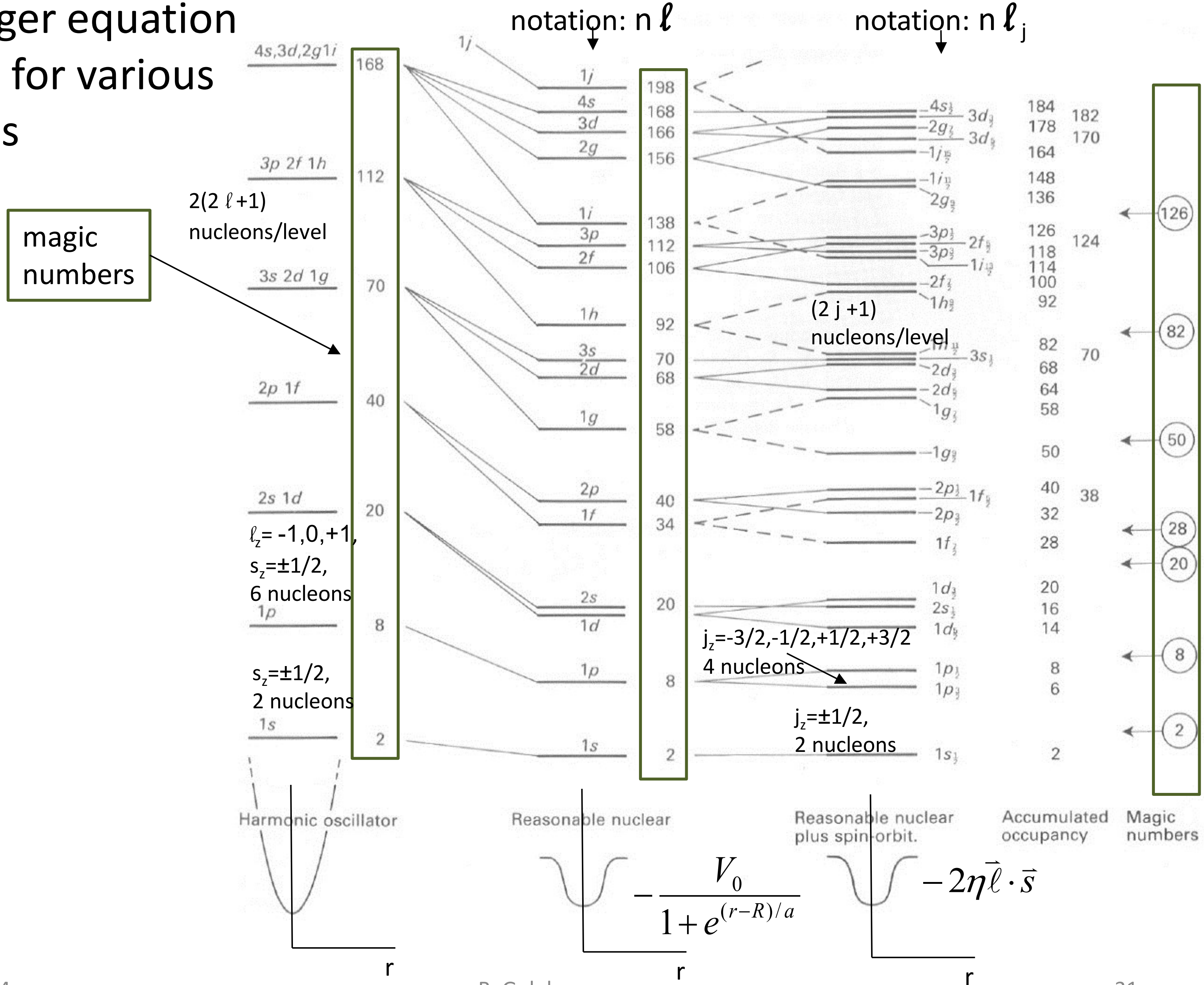
if one deals with fermions (like nucleons), s=1/2, and hence the total angular momentum of a nucleon can be only j = l ± ½:

$$\left\langle \hat{\vec{\ell}} \cdot \hat{\vec{s}} \right\rangle = \left\{ \begin{array}{cc} 1/2 l\hbar^2 & j = l + 1/2 \\ -1/2(l+1)\hbar^2 & j = l - 1/2 \end{array} \right.$$

Every energy level with a given *l* is now splits into two levels, one with *j=l+1/2* and one with *j=l-1/2* (apart from the level with *l=0*).

With this inclusion of the spin-orbit interaction the calculated magic numbers using the finite potential agree with the measured ones (see p. 33, right column).

Schrödinger equation solutions for various potentials



notation:

ℓ=0 s

ℓ=1 p

ℓ=2 d

ℓ=3 f

ℓ=4 g

:

:

Spin of nuclei within the shell model

The total angular momentum of the nucleus (also called the nucleus spin) is a vector sum of the angular momenta of individual nucleons.

Individual orbits (energy level determined by a given set of values *n, l, j*) fully filled by nucleons have the total angular momentum equal to zero. This is easy to see since at such energy level all the sublevels (degenerated, i.e. all having the same energy) corresponding to the 3rd component $j_z=-j, -j+1, \ldots, j-1, j$ are equally populated and hence the vector sum of those nucleons is zero.

Moreover, if the nucleus has a single nucleon more than needed to fully populate the lower energy orbits, spin of the nucleus is determined by the total angular momentum of this additional nucleon. Similar is true for nuclei which have one nucleon less than the number required to fully populate energy orbits.

Example: let's take the $^{17}_{8}O$ nucleus (one of oxygen isotopes). It has 8 protons and 9 neutrons. Looking at the energy levels shown at the right column of the scheme on p. 33, we see that 8 protons fully populate the $1s_{1/2}$, $1p_{3/2}$ and $1p_{1/2}$ levels. The same is true for 8 neutrons. The additional neutron populates the $1d_{5/2}$ level. Since the first three energy levels are fully populated, the nucleus spin is 5/2, as a consequence of the additional neutron on $j=5/2$ level.
Homework 5: check the energy level population for some other nuclei, e.g. ^{13}C, ^{39}K, ^{101}Sn, and try to determine their spin.

Interesting facts related to the nuclear shell model

Maria Goeppert-Mayer, Hans Jensen: Nobel prize in physics in 1963 for the nuclear shell model
(shared with **Eugene Wigner**)

Hans Jensen: participated in the development of centrifuges for Uranium separation in Germany during the WWII
Eugene Wigner: one of the initiators of the Manhattan project in the U.S.

Wikipedia: huge amount of information, e.g. on the Epsilon operation; a U.S. operation to seize the leading German physicists, including **Werner Heisenberg ,** with the aim of gathering information on the German nuclear program. Heisenberg later cleared Jensen of accusations of cooperation with the Nazis.

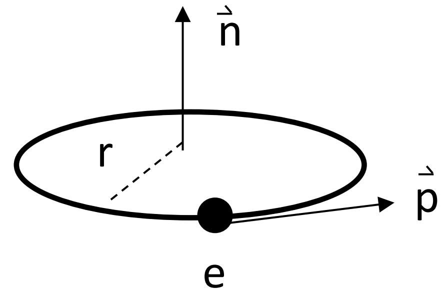
Nobel prize **H. Jensen**, 1963:
"for his contributions to the theory of the atomic nucleus and the elementary particles, particularly through the discovery and application of fundamental symmetry principles"

Nobel prize **M. Kobayashi, T. Maskawa**, 2008*:*
"for the discovery of the origin of the broken symmetry which predicts the existence of at least three families of quarks in nature"

1.1 Basic properties of nuclei

1.1.4 Dipole magnetic moment

Dipole magnetic moment



classic dipole magnetic moment of circulating charge:

$$\vec{\mu} = IS\vec{n};\ I = \frac{dq}{dt} = \frac{e}{t_0} = \frac{ev}{2\pi r} = \frac{ep}{2\pi rm}$$

$$\vec{\mu} = \frac{ep}{2\pi rm}\pi r^2 \vec{n} = \frac{e}{2m}\vec{r}\times\vec{p} = \frac{e}{2m}\vec{\ell}$$

The correspondence principle in quantum mechanics tells us that the quantum mechanical description of an observable can be obtained by exchanging classical quantities by corresponding operators; for the above case this means that

$$\hat{\vec{\mu}} = \frac{e}{2m}\hat{\vec{\ell}}$$

$$\vec{\mu}^2\psi = \left(\frac{e}{2m}\right)^2 \hat{\vec{\ell}}^2\psi = \left(\frac{e\hbar}{2m}\right)^2 l(l+1)\psi$$

μ_B: Bohr's magneton

$$\mu = \left(\frac{e\hbar}{2m}\right)\sqrt{l(l+1)} = \mu_B\sqrt{l(l+1)}$$

Dipole magnetic moment

What if the charged particle carries also spin beside the orbital angular momentum? According to the correspondence principle one would assume

$$\hat{\vec{\mu}} = \frac{e}{2m}\left(\hat{\vec{\ell}} + \hat{\vec{s}}\right)$$

However, this is not the case. The reason for this is that spin (intrinsic angular momentum of a particle) does not have a classical equivalent, and hence the correspondence principle can not be applied.

The evaluation of the dipole magnetic moment of a spin ½ particles (fermions) is actually a big success of the equation named after Paul Dirac – the Dirac equation. The equation (see p. ???) is a relativistic equivalent of the Schrödinger equation (which is non-relativistic) for description of fermions. It can be shown (see p. ???) that it predicts the dipole magnetic moment for such particles to be

$$\hat{\vec{\mu}}_s = g_s \frac{e}{2m}\hat{\vec{s}}$$

where g_s is called the spin gyromagnetic ratio and $g_s=2$ for fermions (and not 1 as would be implied by the correspondence principle).

Dipole magnetic moment

A consequence of is that for a charged fermion having a non-zero orbital angular momentum the dipole magnetic moment does not point in the direction of the total angular momentum:

$$\hat{\vec{\mu}} = \frac{e}{2m}\left(g_l \hat{\vec{\ell}} + g_s \hat{\vec{s}}\right) \underbrace{=}_{charged\ fermion} \frac{e}{2m}\left(\hat{\vec{\ell}} + 2\hat{\vec{s}}\right) = \frac{e}{2m}\left(\hat{\vec{j}} + \hat{\vec{s}}\right)$$

In the above equation we introduced g_l, an equivalent of the g_s for the orbital angular momentum, which equals 1 for charged and 0 for neutral particles.

Nucleons (p and n) are fermions and hence one would expect :

$$\hat{\vec{\mu}}_{s,p} = g_{s,p}\frac{e}{2m}\hat{\vec{s}};\ g_{s,p} = 2$$

$$\hat{\vec{\mu}}_{s,n} = g_{s,n}\frac{e}{2m}\hat{\vec{s}} = 0;\ (n\ is\ neutral)$$

Surprisingly, the measured dipole magnetic moments of p and n give

$$g_{s,p} = 5.6$$

$$g_{s,n} = -3.8$$

The unexpected dipole moments of p and n are due to their constituents; p and n are not elementary fermions but particles composed of quarks and can not be described by the Dirac equation.

Dipole magnetic moment

Hence the spin dipole magnetic moment eigenvalues of nucleons are

$$\mu_{s,p} = g_{s,p}\mu_N\sqrt{s(s+1)}$$
$$\mu_{s,p} = g_{s,n}\mu_N\sqrt{s(s+1)}$$

with $\mu_N = \frac{e\hbar}{2m_N}$ called the nuclear magneton.

As the dipole magnetic moment of a nucleus one usually quotes the 3rd component of $\vec{\mu}$ in the direction of an external magnetic field, at the maximal spin projection. In terms of an expression for the expectation value this means

$$\mu = \langle JJ \mid \hat{\mu}_z \mid JJ \rangle$$

where we used the short-hand notation $|JJ_z\rangle$ for the nucleus state (wave function) of total angular momentum J and its 3rd component J_z. $\langle JJ_z|$ represents the conjugate wave function. The above expression is hence a short-hand notation for the expectation value

$$\mu = \langle \hat{\mu}_z \rangle = \int \psi^*(\vec{r}; J, J_z = J)\hat{\mu}_z \psi(\vec{r}; J, J_z = J) d^3 r$$

Dipole magnetic moment

Due to $g_l \neq g_s$ the dipole magnetic moment of a nucleus does not point in the same direction as the total angular momentum (spin) of the nucleus. One can nevertheless define an effective gyromagnetic ratio g so that $\vec{\mu}$ and $\vec{J}$ are parallel:

$$\hat{\vec{\mu}} \equiv g \frac{e}{2m_N} \hat{\vec{J}}$$

$$\mu = \langle JJ \mid \hat{\mu}_z \mid JJ \rangle = g \frac{e}{2m_N} \langle JJ \mid \hat{J}_z \mid JJ \rangle = g \frac{e}{2m_N} \hbar J = g \mu_N J$$

If the nucleus has a single nucleon out of the otherwise fully populated orbits the nucleus spin is determined by the total angular momentum of this unpaired nucleon (see p. xx). We can write either

$$\mu = \langle jj \mid \hat{\mu}_z \mid jj \rangle = g \mu_N j$$

or

$$\mu = \frac{e}{2m_N} \langle jj \mid g_l \hat{\ell}_z + g_s \hat{s}_z \mid jj \rangle$$

where j denotes the total angular momentum of the unpaired nucleon.

Dipole magnetic moment

The effective gyromagnetic ratio *g* can be determined by evaluating the expression

$$\left\langle jm \,|\, \hat{\vec{\mu}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle$$

On one hand, using the definition of *g*, this equals to

$$\left\langle jm \,|\, \hat{\vec{\mu}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle = \left\langle jm \,\middle|\, g \frac{e}{2m_N} \hat{j}^2 \,\middle|\, jm \right\rangle = g\mu_N \hbar j(j+1)$$

On the other hand, using g_l and g_s, this is

$$\left\langle jm \,|\, \hat{\vec{\mu}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle = \frac{e}{2m_N} \left\langle jm \,|\, g_l \hat{\vec{\ell}} \cdot \hat{\vec{j}} + g_s \hat{\vec{s}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle$$

To evaluate the later expression we need to determine

$\left\langle jm \,|\, \hat{\vec{\ell}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle$ and $\left\langle jm \,|\, \hat{\vec{s}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle$

Dipole magnetic moment

Using a shortened notation we can get the following expressions:

$$\left\langle jm \,|\, \hat{\vec{\ell}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle \equiv \left\langle \hat{\vec{\ell}} \cdot \hat{\vec{j}} \right\rangle = \left\langle \hat{\vec{\ell}} \cdot (\hat{\vec{\ell}} + \hat{\vec{s}}) \right\rangle = \left\langle \hat{\ell}^2 + \hat{\vec{\ell}} \cdot \hat{\vec{s}} \right\rangle =$$

$$= \hbar^2 l(l+1) + \left\langle \hat{\vec{\ell}} \cdot \hat{\vec{s}} \right\rangle$$

$$\left\langle jm \,|\, \hat{\vec{s}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle \equiv \left\langle \hat{\vec{s}} \cdot \hat{\vec{j}} \right\rangle = \ldots = \hbar^2 s(s+1) + \left\langle \hat{\vec{\ell}} \cdot \hat{\vec{s}} \right\rangle$$

The $\left\langle \hat{\vec{\ell}} \cdot \hat{\vec{s}} \right\rangle$ we have evaluated already before (see p. xx). Using this we get

$$\left\langle \hat{\vec{\ell}} \cdot \hat{\vec{j}} \right\rangle = \frac{1}{2}\hbar^2 \left[j(j+1) + l(l+1) - s(s+1) \right]$$

$$\left\langle \hat{\vec{s}} \cdot \hat{\vec{j}} \right\rangle = \frac{1}{2}\hbar^2 \left[j(j+1) - l(l+1) + s(s+1) \right]$$

Dipole magnetic moment

Finally, we arrive at

$$\left\langle jm \,|\, \hat{\vec{\mu}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle = \frac{e}{2m_N} \left\langle jm \,|\, g_l \hat{\vec{\ell}} \cdot \hat{\vec{j}} + g_s \hat{\vec{s}} \cdot \hat{\vec{j}} \,|\, jm \right\rangle =$$

$$= \mu_N \hbar \left[g_l \frac{j(j+1)+l(l+1)-s(s+1)}{2} + g_s \frac{j(j+1)-l(l+1)+s(s+1)}{2} \right]$$

Equating this expression and the one obtained using g (p. xx), we get to the effective gyromagnetic ratio

$$g = g_l \frac{j(j+1)+l(l+1)-s(s+1)}{2j(j+1)} + g_s \frac{j(j+1)-l(l+1)+s(s+1)}{2j(j+1)}$$

with

$$g_l = \begin{cases} 1 & p \\ 0 & n \end{cases} \quad g_s = \begin{cases} 5.6 & p \\ -3.8 & n \end{cases}$$

Dipole magnetic moment

For nucleus with a single unpaired nucleon *j=l ± ½,* and

$$\frac{\mu}{\mu_N} = \begin{cases} g_l(j-1/2) + g_s/2 & j = l+1/2 \\ \frac{j}{j+1}\left[g_l(j+3/2) - g_s/2\right] & j = l-1/2 \end{cases}$$

So for nuclei with odd Z (one unpaired p) the dipole magnetic moment is

$$\frac{\mu}{\mu_N} = \begin{cases} j+2.3 & j = l+1/2 \\ \frac{j(j+3/2)}{j+1} - 2.8\frac{j}{j+1} & j = l-1/2 \end{cases}$$

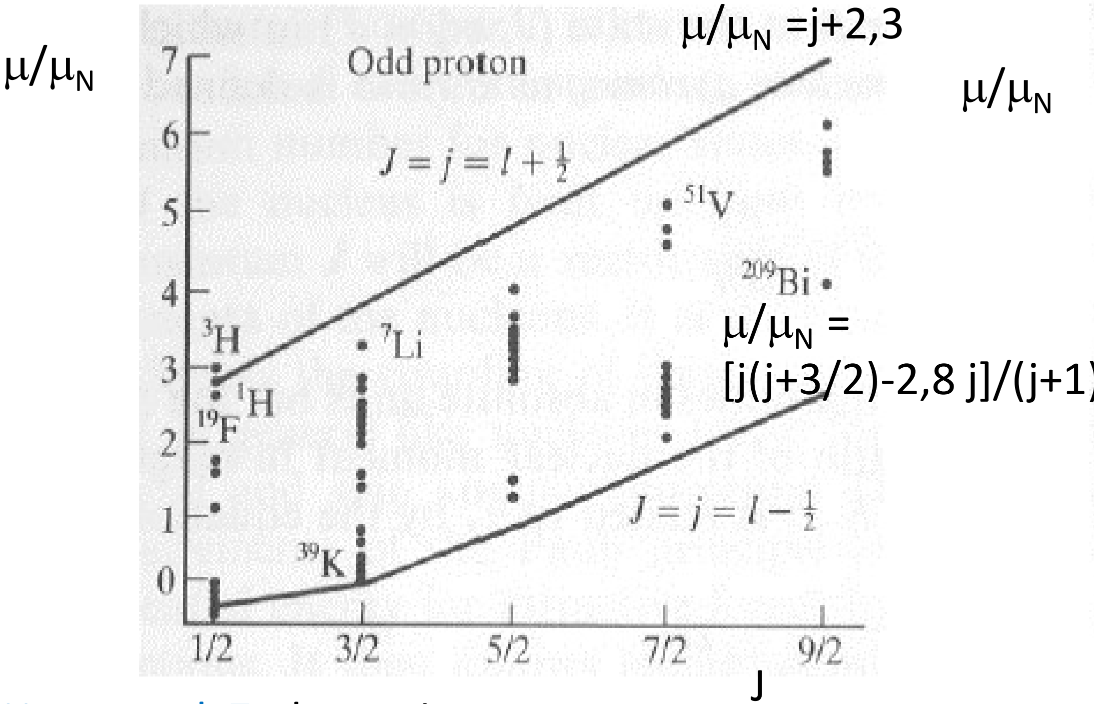
while for the nuclei with odd N (one unpaired n) the dipole magnetic moment is

$$\frac{\mu}{\mu_N} = \begin{cases} -1.9 & j = l+1/2 \\ 1.9\frac{j}{j+1} & j = l-1/2 \end{cases}$$

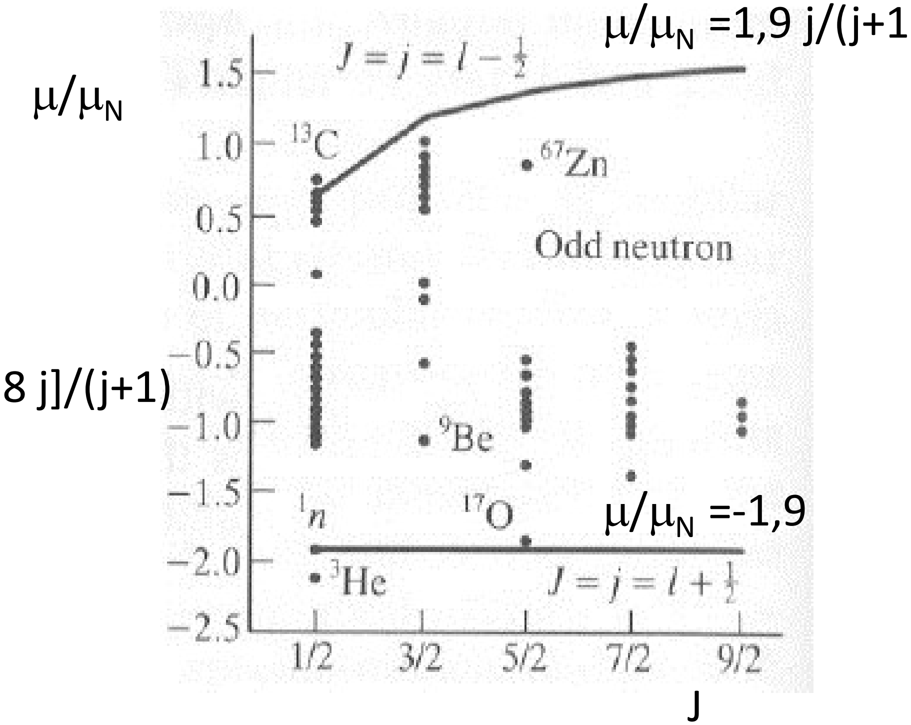
Dipole magnetic moment

The expressions for the dipole magnetic moment of nuclei as a function of the nuclear spin J are known as the Schmidt lines. The measured dipole magnetic moments for various nuclei lie between those lines (since not all of them have only a single unpaired nucleon):

odd Z



odd N



Homework 7: determine the dipole mag. moment of ^{13}C and ^{39}K.

lines: expected μ for nuclei with $J = l \pm ½$ (Schmidt lines)

Interesting facts:

in basic science it happens often that the discoveries made there find their way to various applications, although such an outcome can not be foreseen before the purely scientific research is carried out. Such applications are called **“spin-off”**.

Example 1: www (**World Wide Web**) was developed at Cern due to communication needs of international scientific collaborations

Tim Berners-Lee

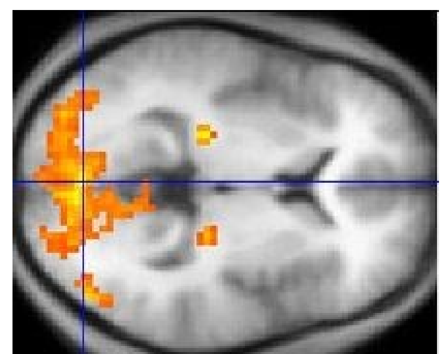
Robert Cailiau

first server

part of today’s computing center at Cern

Example 2: knowing what the dipole magnetic moments of nuclei are, their measurements through the **Nuclear Magnetic Resonance** (NMR) technique represent a nowadays essential diagnostic technique in the medicine.

NMR diagnostic apparatus (B ~1,5 T)



NMR pictures of various tissues

1.2 Nuclear decays

1.2.1 α decays

α decays

In α decays the initial nucleus reduces its energy by emitting a He nucleus:
$(Z,A) \rightarrow (Z-2,A-4) + {}^4_2He$
(He nucleus is also known as the α particle; it's the same particle as used by Rutherford , Geiger and Marsden in the experiment exposing the atomic nuclei, see p. xx yy)

The energy conservation in this process is:
$m(Z,A)\,c^2 = m(Z-2, A-4)\,c^2 + m_\alpha c^2 + T_\alpha + T_{Z-2}$,
where $T_{\alpha,\,Z-2}$ denotes the kinetic energies of the α particle and final state nucleus.
Conservation of momentum requires
$p_{Z-2} = p_\alpha$ and hence $T_\alpha = p_\alpha^2/2m_\alpha^2 = T_{Z-2}\, m(Z-2,A-4)/m_\alpha$.
Since usually $m_\alpha << m(Z-2,A-4)$ one can neglect T_{Z-2} in the energy conservation equation.
The decay is energetically possible if
$T_\alpha = -|W(Z,A)| + |W_\alpha| + |W(Z-2,A-4)| \geq 0$
where we expressed the masses of nuclei by the bounding energy (as given by the semi-empirical mass formula, p. xx).

α decays

We can now estimate from which mass number A the α decay is possible by assuming A, Z >> 1, treating A and Z as continuous variables and re-writing the above condition as a differential:

$$T_\alpha = |W_\alpha| - \left(|W(Z,A)| - |W(Z-2,A-4)|\right) \approx$$

$$\approx |W_\alpha| - \left(\frac{\partial |W|}{\partial Z}\delta Z + \frac{\partial |W|}{\partial A}\delta A\right)$$

with $\delta Z=2$ and $\delta A=4$. Differentiating the semi-empirical mass formula yields

$$T_\alpha \approx |W_\alpha| - 4w_0 + \frac{8}{3}\frac{w_1}{A^{1/3}} + 4w_2\frac{Z}{A^{1/3}}(1 - Z/3A) -$$

$$-4w_3(1-2Z/A)^2 + 3w_4 A^{-7/4}\delta(Z,A) \geq 0$$

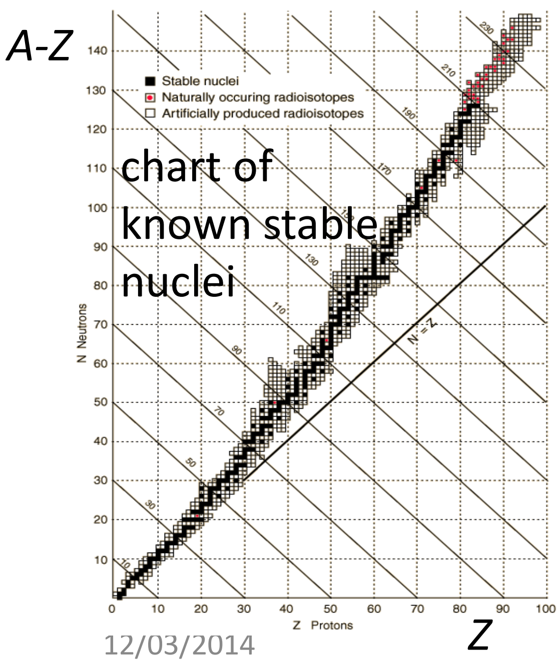
In the above condition we have two independent variables, Z and A. In order to obtain a condition expressed as a function of A only we make a further approximation, by including the relation between A and Z as valid for stable nuclei, i.e. the expression obtained from

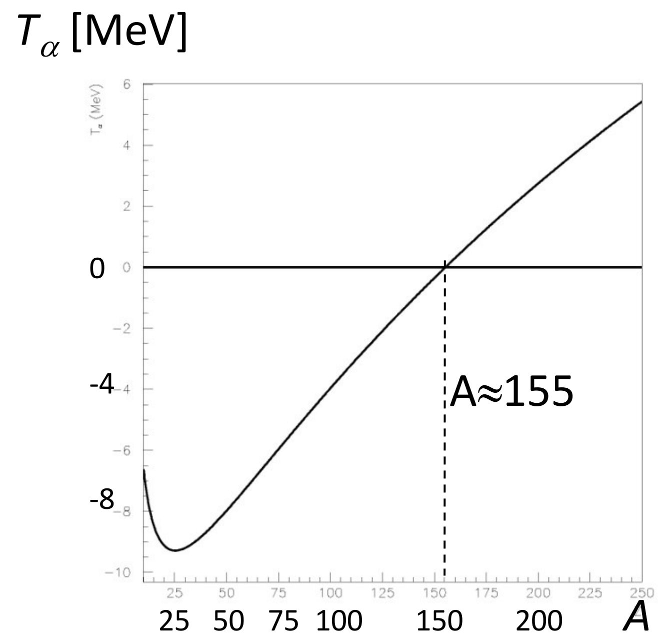
$$\frac{\partial |W|}{\partial Z} = 0 \Rightarrow Z = \frac{A}{2 + 0.015A^{2/3}}$$

α decays

Homework 8: check the precision of the *Z(A)* relation for various nuclei by comparing the calculated values with the chart of known stable nuclei.

This expression gives *Z* of nuclei which for a given *A* are most stable. Clearly this is not correct for the nuclei which undergo the α decay, but is sufficient to obtain a rough estimate. By inserting the above *Z(A)* relation into the condition for a decays following from the energy conservation one obtains a non-trivial expression that cannot be solved for *A* analytically but can be solved numerically, for example by drawing the dependence as a function of *A* taking into account the experimentally determined value $|W_\alpha|=28.3$ *MeV*.

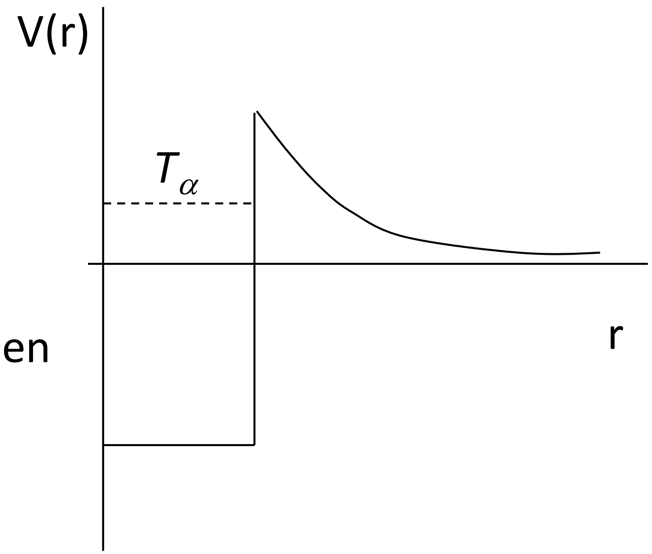




We can see that from the rough estimate one expects nuclei with A ≥ 155 to decay through an emission of α particles.

α decays

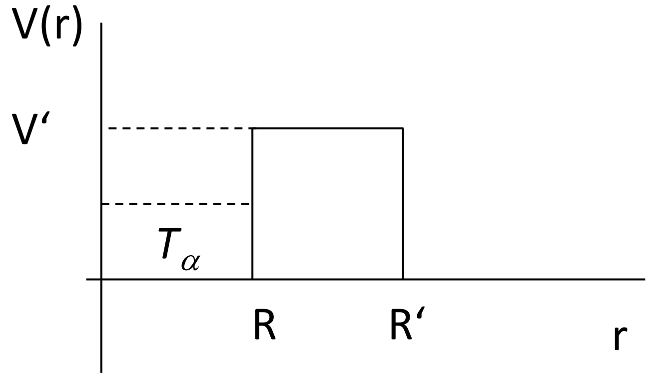
The estimate is not confirmed by experimental data. The latter proves α decays for nuclei heavier than $^{207}_{82}Pb$. This should not be surprising given the approximations we made. Moreover one needs to include quantum mechanical picture in order to understand at least qualitatively the lowest mass number from which the α decays are observed. If we imagine the initial nucleus composed of a He nucleus and the rest of the nucleons, the potential experienced by the a particle looks qualitatively as shown in the plot below:



At small distances between the α particle and the rest of the initial nuclei the potential is attractive (negative) due to the strong nuclear force binding the nucleons. At larger distances the repulsive (positive) Coulomb potential between the two positively charged entities becomes important. Hence the α particle with a positive kinetic energy T_α has to tunnellate through the positive potential barrier in order to escape from the initial nucleus.

α decays

We can estimate the tunneling probability by first considering tunneling through a potential barrier:



$$\left[\frac{\hat{p}^2}{2\mu}+V(r)\right]\psi = E\psi$$

$$\mu = \frac{4m_n(A-4)m_n}{Am_n} = 4\frac{A-4}{A}m_n$$

The first step is solving the Schrödinger equation for the two body problem (of α particle and the rest of the nuclei) with the reduced mass μ and momentum operator p^2 as written; the solution can be found using the ansatz involving the spherical function $Y_{l,m}$ (see p. xx). The solution for the radial part has three distinct intervals:

$$\hat{p}^2 = -\hbar^2\nabla^2 = -\hbar^2\left[\frac{\partial^2}{\partial r^2}+\frac{2\partial}{r\partial r}\right]+\frac{\hat{\ell}^2}{r^2}$$

$$\psi = \frac{u(r)}{r}Y_{l,m}(\vartheta,\varphi)$$

$$\psi = \begin{cases} u_I = Ae^{ikr}+Be^{-ikr} & r<R \\ u_{II} = Ce^{\kappa r}+Be^{-\kappa r} & R<r<R' \\ u_{III} = Ee^{ikr} & r>R' \end{cases}$$

where $$k^2 = \frac{2\mu}{\hbar^2}T_\alpha,\ \kappa^2 = \frac{2\mu}{\hbar^2}(V'-T_\alpha)$$

α decays

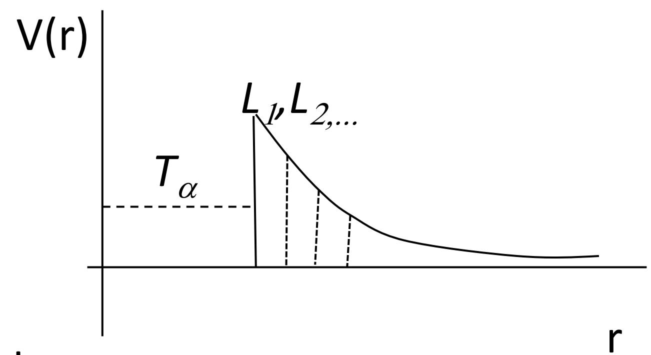
The boundary conditions from which one determines the unknown coefficients A,B,... are

$$u_I(R) = u_{II}(R), \left.\frac{\partial u_I}{\partial r}\right|_R = \left.\frac{\partial u_{II}}{\partial r}\right|_R$$

$$u_{II}(R') = u_{III}(R'), \left.\frac{\partial u_{II}}{\partial r}\right|_{R'} = \left.\frac{\partial u_{III}}{\partial r}\right|_{R'}$$

From these one obtains $\left|\frac{E}{A}\right|^2 = \left[\frac{4k\kappa}{k^2+\kappa^2}\right]^2 e^{-2\kappa(R'-R)}$ called also the barrier transmittancy.

In the case of α decay the barrier height is not constant; we can approximate this by dividing the Coulomb barrier into thin slices:



The transmittancy can then be written as

$$e^{-2\kappa L_1} e^{-2\kappa L_2} \cdots \rightarrow e^{-2\int \kappa dr} \equiv e^{-g}$$

Taking into account the Coulomb barrier dependence we get

α decays

Taking into account the Coulomb barrier shape we get

$$g = 2\sqrt{\frac{2\mu}{\hbar^2}}\int_R^{R'}\sqrt{V(r)-T_\alpha}\,dr,\ V(r)=\frac{2(Z-2)e^2}{4\pi\varepsilon_0 r}=\frac{2(Z-2)\alpha\hbar c}{r}$$

The above expression depends on *R'*; we estimate *R'* as the distance at which T_α becomes larger than *V(r)*, i.e. *V(R')* = T_α. The result of the integration is

$$g = 2\sqrt{\frac{2\mu}{\hbar^2}}\left[-R\sqrt{V(R)-T_\alpha}+R\frac{V(R)}{\sqrt{T_\alpha}}\arctan\sqrt{\frac{V(R)}{T_\alpha}-1}\right]$$

$$\underset{T_\alpha << V(R)}{\approx}\sqrt{\frac{2\mu}{\hbar^2}}\sqrt{V(R)}R\left[\pi\sqrt{\frac{V(R)}{T_\alpha}}-2\right]$$

homework 9: typical T_α for known decays is 5 MeV; compare this to the value of Coulomb potential at some typical nuclear distance.

The probability for tunneling

$$w \propto e^{-g},\ \ln w \propto -g \propto -k_1\frac{RV(R)}{\sqrt{T_\alpha}}+k_2R\sqrt{V(R)}$$

α decays

Since $V(R) \propto \frac{1}{R}$ we arrive at $\ln w \propto -k_3 \frac{1}{\sqrt{T_\alpha}} + k_4 \sqrt{R}$

We can read off two properties of the result:

$$w \propto e^{k_4 \sqrt{R}} = e^{k_5 A^{1/6}}$$

- probability for tunneling and thus for decay increases with the mass number A;

$$w \propto e^{-k_3 / \sqrt{T_\alpha}}, \tau \propto e^{k_3 / \sqrt{T_\alpha}}$$

- probability for α decay (inverse of the nuclei lifetime) depends specifically on T_α; the latter dependence is also known as the Geiger-Nuttal rule.

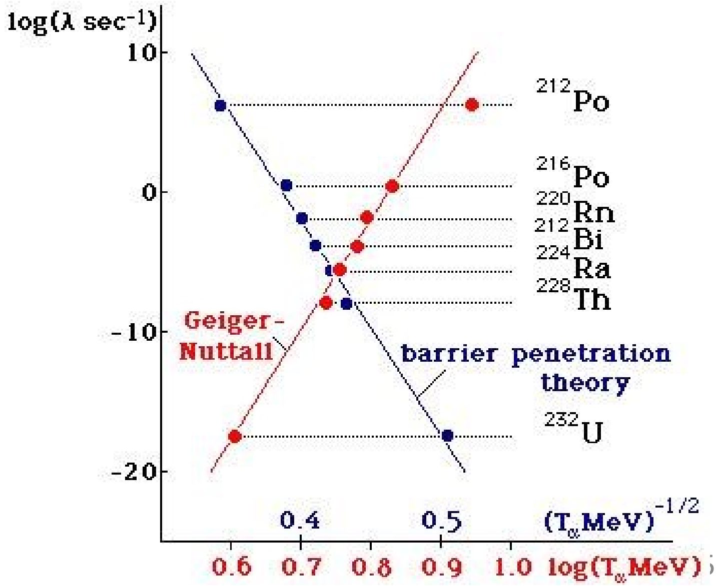
For α decays of nuclei with A ~ 155 (for which the decay becomes energetically possible) the tunneling probability is too low for decays to be observed; it becomes sizeable only for nuclei with A $\geq$ 207.

α decays

Geiger-Nuttall's law:

$$\ln w \propto \frac{1}{\sqrt{T_\alpha}}$$

Red dots represent experimental data on probability for α decays of various nuclei. Line denoted „Geiger-Nuttal“ represents a fit with a straight line to the data.
Blue dots (blue horizontal scale) denotes the same data but in different scale to distinguish them from blue dots. The straight line through those dots represent the prediction obtained from the above described calculation (tunneling through a potential barrier) which represents a good description of experimental data.



http://www.shef.ac.uk/physics/teaching/phy303/phy303-4.html

1.2 Nuclear decays

1.2.2 β decays

β decays

In β decays the initial nucleus reduces its energy by emission of an electron and electronic anti-neutrino:

$(Z,A) \rightarrow (Z+1,A)\ e^- \bar{\nu}_e$

Neutrinos are particles discussed in more details in the elementary particles section. For now it should be sufficient to know that these particles interact only via the weak nuclear force and have mass almost equal to zero (anti-neutrino is an anti-particle of a neutrino, similarly as the positively charged electron, called positron, is an anti-particle of an electron).

The above decay is called a β^- decay due to a presence of the electron in the final state. At the level of nucleons

$\mathrm{n} \rightarrow \mathrm{p}\ e^- \bar{\nu}_e$

Since $(m_n - m_p)c^2$ = 1.29 MeV such a decay of a free neutron is actually possible. Neutrons are unstable and decays through a β^- decay with a lifetime of around 880 s.

β decays

The β^- decay is energetically possible if

$$m(Z,A)c^2 - m(Z+1,A)c^2 - m_e c^2 \geq 0$$

Taking into account the above neutron - proton rest energy difference and the rest energy of an electron, $m_e c^2 = 0.51\ MeV$, the condition reads

$$|W(Z+1,A)| - |W(Z,A)| \geq -0.78\ MeV$$

Another type of β decay is a β^+ decay:
$(Z,A) \rightarrow (Z-1,A)\ e^+\ \nu_e$

with a positron in the final state. At the nucleon level this is
$p \rightarrow n\ e^+\ \nu_e$

Due to the mass difference between the *p* and *n* such a decay of a free proton is not possible. It can only take place for the *p* and *n* which are bound inside the nuclei. The nuclear β^+ decay is energetically possible if

$$|W(Z-1,A)| - |W(Z,A)| \geq 1.79\ MeV$$

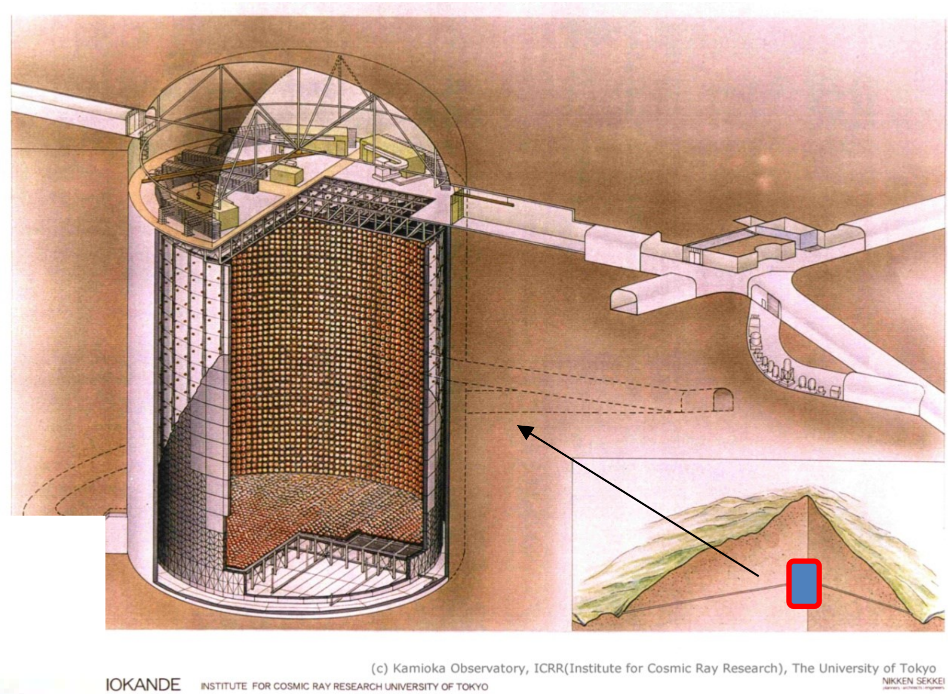
Some interesting facts about proton decay

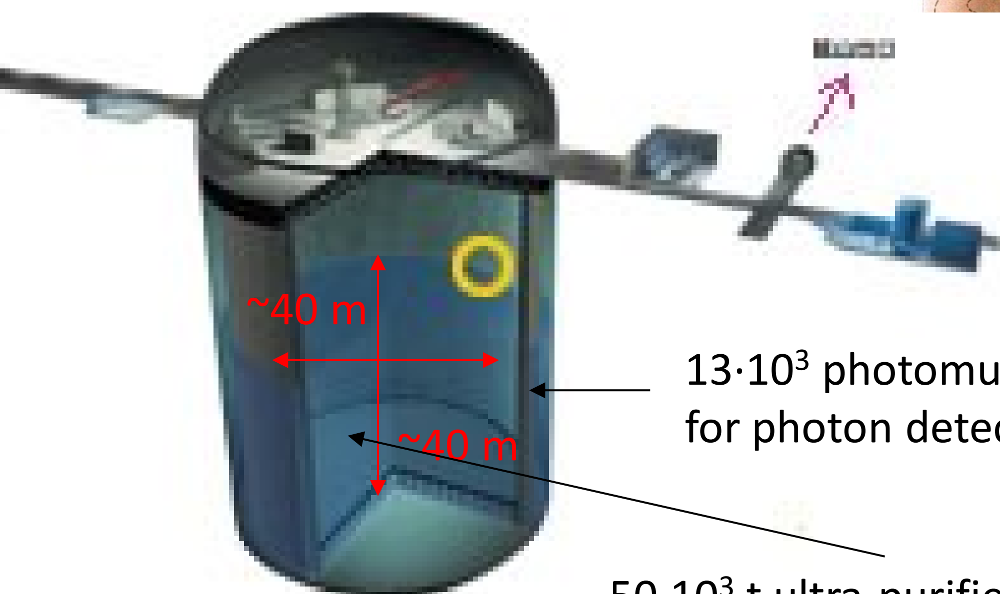
Proton is the lightest known baryon (particle composed of three quarks, see Part II, p. ???) Since the baryon number (see Part II, p. ???) is conserved in all so far known processes it follows that the proton does not decay. However, in order for the Universe to evolve from the Big Bang to the present state there are strong arguments that in the early stages of the Universe the baryon number conservation had to be violated (see Part 2, p. ???). Hence many theories which are experimentally not confirmed predict the decay of a p with lifetime $\tau_p \geq 10^{36}$ years. It is thus understandable that a search for possible *p* decays should be experimentally performed.

One of the most sensitive experiments in this search is the SuperKamiokande experiment in Kamioka in Japan.
It is placed in the cavern in the mountain to protect the experiment from various background sources such as the cosmic rays.

Some interesting facts about proton decay

The experiment consists of
a reservoir with $50 \cdot 10^3$ tons of
ultra-purified water, surrounded
by $13 \cdot 10^3$ photomultipliers
for detection of single photons.



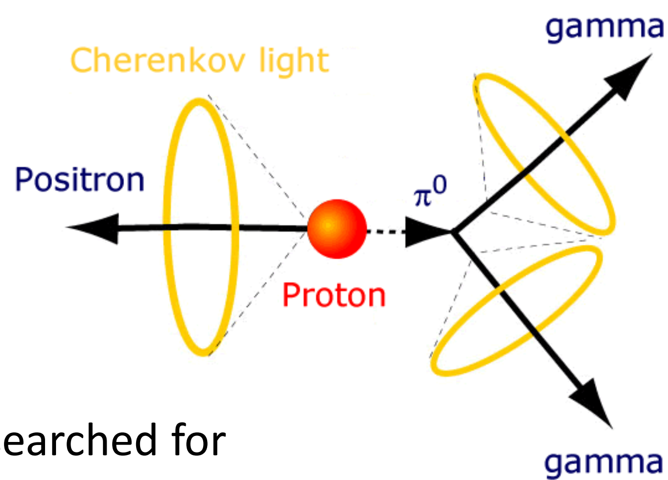


$13 \cdot 10^3$ photomultipliers
for photon detection

$50 \cdot 10^3$ t ultra-purified water

Some interesting facts about proton decay

One of potential (but unobserved) *p* decays *is* $p \rightarrow e^{+} \pi^{0}$, shown in the sketch below:



The decay can be searched for by detecting the photons of the Cherenkov light produced by the e^{+} in the water, and by detecting photons from the π^0 decays.

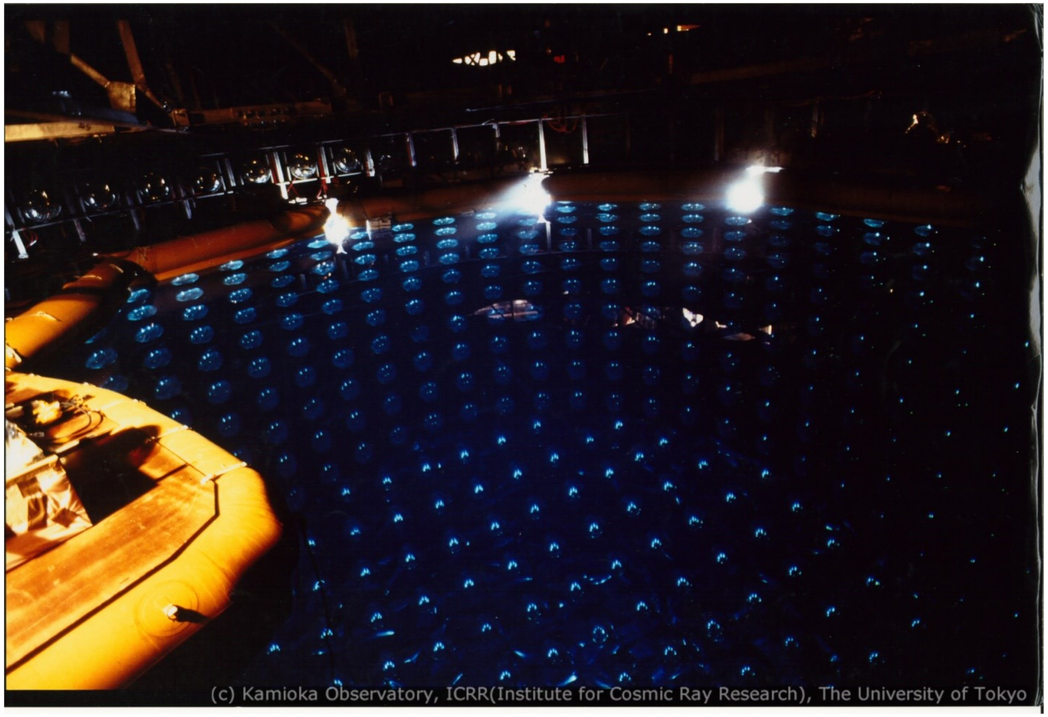
So far no significant signal of such decay was observed. The current limit on the *p* lifetime is $\tau_p > 10^{34}$ years. Of course this does not mean that the experiment needs to be operated for 10^{34} years. The *p* decays - if existing - would obey the exponential decay law, the number of *p* decaying in a time interval *[t, t+dt]* would be $dN/dt = N_0(1/\tau_p)\, e^{-t/\tau p}$, where N_0 is the number of all *p* being observed. Hence the large τ_p is compensated by the waste amount of *p* in $50 \cdot 10^3$ t of water.

Homework 9: estimate the number of expected *p* decays in $50 \cdot 10^3$ t of water in one year if the expected *p* lifetime is 10^{34} years.

Some interesting facts about proton decay

In 2001 the SuperKamiokande detector was hit by an accident: due to the implosion of few multipliers the shock-wave propagating through the water destroyed 6600 of photomultipliers (the price of each at the time was 3000 US $). With huge efforts the detector was repaired in 2006 (SuperKamiokande II).

The experiment is not dedicated only to a search for the *p* decay but also to the study of another interesting phenomena - oscillations of neutrinos.



Spectrum of $e^{\pm}$ in β decays

In 1934 Fermi gave the following description of the nuclear β decay, starting from the his golden rule (p. 14):

$$W_{fi} = \frac{2\pi}{\hbar}\left|V_{fi}\right|^2 \rho_f(E_i)$$

Since Fermi did not know any details about the (weak) interaction causing the decays he wrote

$$V_{fi} = G_F \int \psi_f^* \varphi_e^* \varphi_\nu^* \psi_i d^3 r$$

where obviously no details of potential are written and all the details of the interaction are described by a constant G_F, nowadays called the Fermi constant. ψ_f, ψ_i, ϕ_e and ϕ_ν are the wave functions of the initial and final nucleus, and of electron and neutrino, respectively. As an initial approximation one can use plane waves to describe the latter two,

$$\varphi_{e,\nu} = \frac{1}{\sqrt{V}} e^{i\vec{k}_{e,\nu}\vec{r}}$$, and the matrix element becomes $$V_{fi} = \frac{G_F}{V}\int \psi_f^* \psi_i e^{-i\vec{k}\vec{r}} d^3 r$$

with $\vec{k}=\vec{k}_e+\vec{k}_\nu$. Since typical $e^{\pm}$ energies in β decays are of the order of MeV, and electron rest energy is $m_e c^2 = 0.51$ MeV, one must use a relativistic relation between the momentum and energy:

$$k_e = \frac{p_e}{\hbar} = \frac{E_e - mc^2}{\hbar c}$$. The constant $\hbar c = 197\ MeV fm$ is a useful conversion constant worthwhile to remember.

Spectrum of $e^{\pm}$ in β decays

The integration in the matrix element is performed over full space, however since the nucleus wave functions are substantial only in the region of the nuclei, i.e. in the region with $R\sim$ few fm, we can see $kr \lesssim \frac{1\,MeV}{197\,MeVfm} 5\ fm \sim 0.25$. Hence we can make a Taylor series expansion of the exponential in the matrix element:

$$V_{fi} = \frac{G_F}{V}\int \psi_f^* \psi_i (1 - \vec{k}\vec{r} + \frac{(\vec{k}\vec{r})^2}{2!} + \ldots) d^3 r$$

. Considering that the exponential factor arises due to the e-ν wave function, and that the individual terms $(\vec{k}\vec{r})^l/l!$ have the angular dependence of a spherical harmonic $Y_{lm}(\theta,\phi)$, which is the eigenfunctions of the orbital angular momentum operator (see p. 26), we can associate each term in the expansion to a probability that the e and ν in the final state carry the orbital angular momentum $\hbar\sqrt{l(l+1)}$.

The largest term in the matrix element is $(kr)^0$, if this term is non-zero. In the latter case the largest term would be $(kr)^1$, and so on. According to the exponent of the first non-zero term in the series we call the decays allowed ($(kr)^0$), once forbidden ($(kr)^1$), twice forbidden ($(kr)^2$), etc.
Moreover, for the allowed decays there is no dependence of the matrix element on the energy of e - ν system (since they involve $(kr)^0$). Hence the dependence of the decay probability on this energy arises only from the density of final states (see p. 15). For the forbidden decays

Spectrum of $e^{\pm}$ in β decays

there is some energy dependence in the matrix element, however it proves to be rather mild compared to the dependence arising from the density of final states. Contrary to the scattering discussed in 1.1.2., in β decays we have three final state particles. Due to the momentum and energy conservation only two are independent in terms of the momentum, and hence the density of final states can be written as

$$d^6 N_f = V^2 \frac{d^3 p_e d^3 p_\nu}{(2\pi\hbar)^6}, \quad d^5 \rho_f = \frac{d^6 N_f}{dE_f}, \quad d^2 N_f = 16\pi^2 V^2 \frac{p_e^2 dp_e p_\nu^2 dp_\nu}{(2\pi\hbar)^6}$$

where in the last step we integrated over the directions of both e and ν. Since ν's are difficult to detect the observable of interest is the $e^{\pm}$ energy distribution. We can replace the independent variable p_ν by the total energy of the electron-neutrino system E:

$$E_i = m(Z,A)c^2 = M(Z-1,A)c^2 + \underbrace{E_e + E_\nu}_{E} = E_f;\, E_e^2 = c^2 p_e^2 + m_e^2 c^4,\, E_\nu = cp_\nu,\, p_\nu = \frac{E - E_e}{c}$$

$$dp_\nu = \frac{dE}{c};\; d^2 N_f \propto p_e^2 dp_e (E - E_e)^2 dE$$

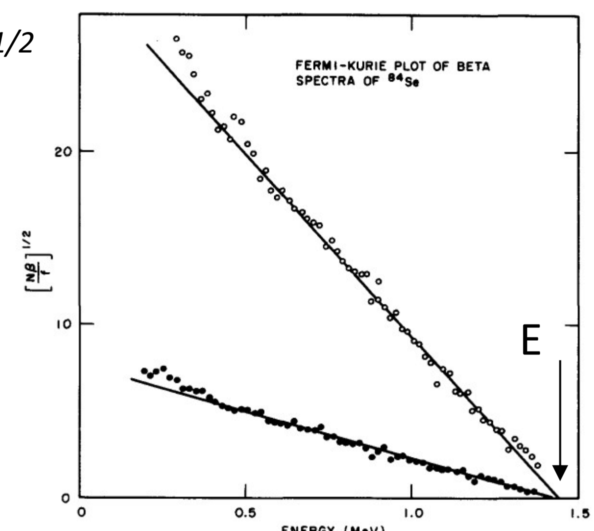
Spectrum of $e^\pm$ in β decays

The electron momentum distribution is $dW_{fi}/dp_e \propto d\rho_f/dp_e$

$$\frac{d\rho_f}{dp_e} = \frac{d^2 N_f}{dE_f dp_e} = \frac{d^2 N_f}{dE dp_e} \propto p_e^2 (E - E_e)^2$$

If one plots the $[(dW_{fi}/dp_e) / p_e^2]^{1/2}$ as a function of the electron energy (i.e. in observing decays of some β source, a sample of nuclei undergoing the β decay, we plot the number of detected $e^\pm$ with a given momentum, divided by this momentum squared, as a function of the $e^\pm$ energy) the dependence is linear. Such a plot is called the Fermi-Kurie plot.

$[N(p_e)/p_e^2]^{1/2}$



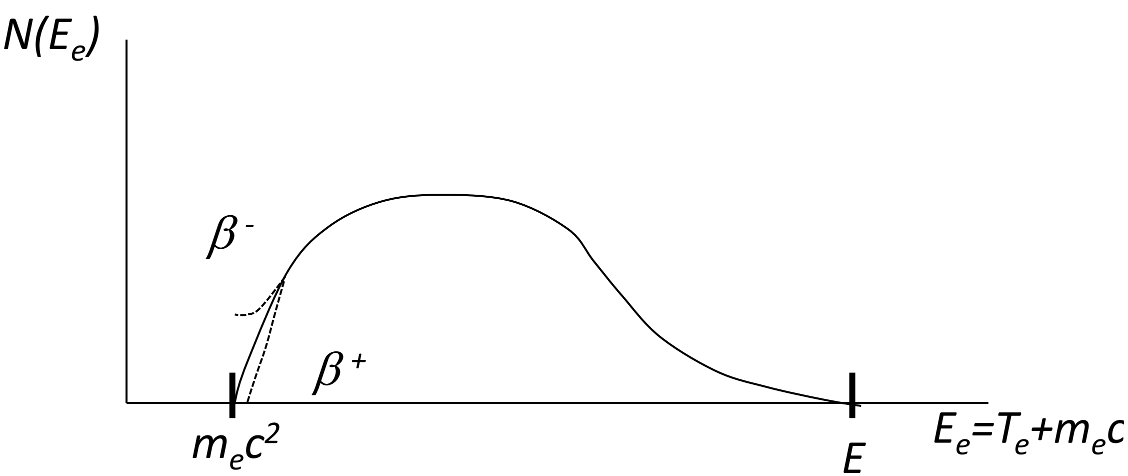
E_e [MeV]

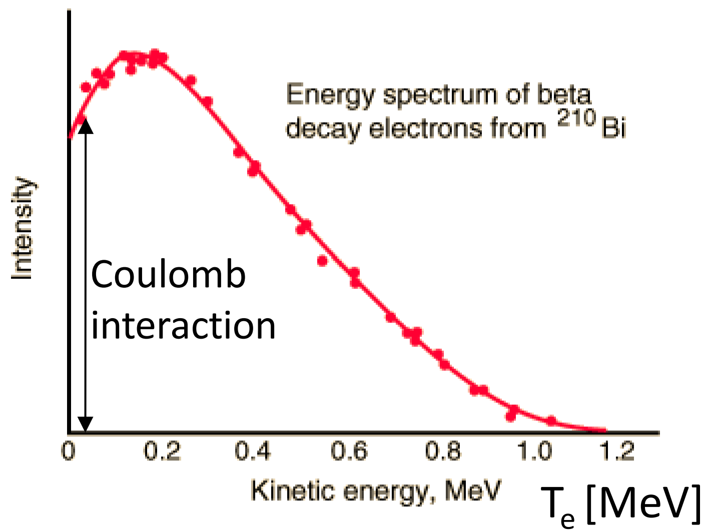
Spectrum of $e^{\pm}$ in β decays

The electron energy distribution is obtained by noting $E_e dE_e = p_e dp_e$, and

$$\frac{dW_{fi}}{dp_e} \propto p_e^2 (E - E_e)^2$$

$$\frac{dW_{fi}}{dE_e} = \frac{dW_{fi}}{dp_e}\frac{dp_e}{dE_e} \propto p_e E_e (E - E_e)^2 = \sqrt{E_e^2 - m_e^2 c^4}\, E_e (E - E_e)^2$$

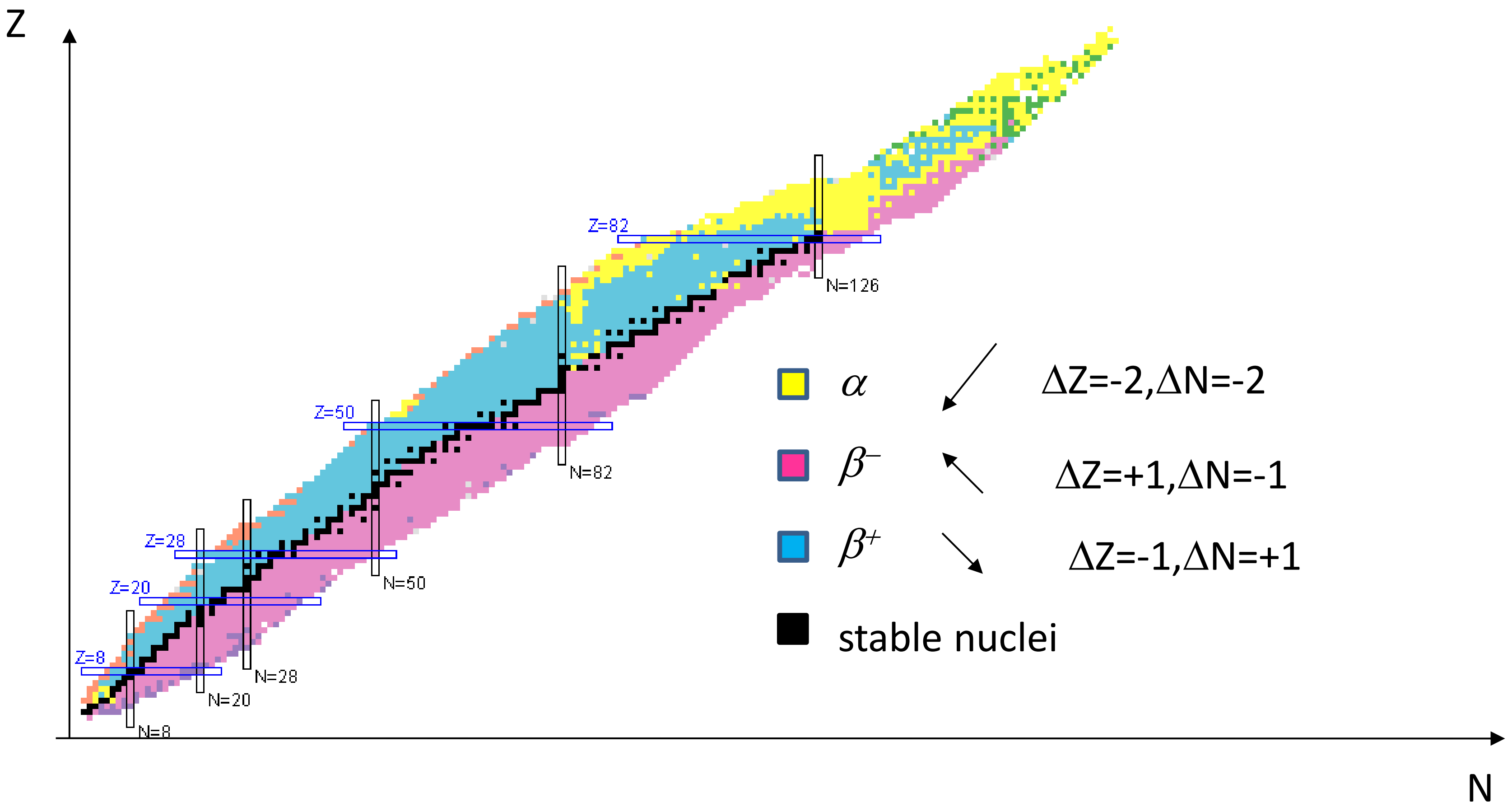




$e^{\pm}$ energy distribution in β decay; the deviations from the calculated shape are observed at low energies due to the Coulomb interaction between the $e^{\pm}$ and the final nucleus.

Spectrum of $e^{\pm}$ in β decays

The chart of nuclei with marked type of decays leading to the stable nuclei (denoted as black points) is shown below:



Selection rules for β decays

The conservation of the angular momentum requires

$$\vec{J}_f + \vec{J}_{e\nu} = \vec{J}_i$$

where $\vec{J}_f$ is the total angular momentum (spin) of the final nucleus, $\vec{J}_{e\nu}$ is the total angular momentum of the electron-neutrino system, and $\vec{J}_i$ is the the spin of the initial nucleus.

$\vec{J}_{e\nu}$ is composed of the orbital angular momentum relative to the final nucleus and the total spin of the electron-neutrino system:

$$\vec{J}_{e\nu} = \vec{s}_{e\nu} + \vec{\ell}$$

Since both electron and the neutrino are fermions of spin ½, $s_{e\nu}$ = 0 or 1. In the former case such decays are called Fermi decays (we will denote those with F), while in the latter they are called Gamow-Teller decays (GT). Hence in the case of Fermi decays $J_{e\nu} = \ell$, while in the case of Gamow-Teller decays $J_{e\nu} = \ell \pm 1$. The difference of spins between the initial and final nucleus, $\Delta\vec{J} = \vec{J}_i - \vec{J}_f$, equals the total angular momentum carried by the electron-neutrino system, $\Delta\vec{J} = \vec{J}_{e\nu}$.

Parity of the wave function describing a certain system determines the behaviour of the function under the reflection of space, $\vec{r} \leftrightarrow -\vec{r}$.

$$\hat{P}\psi(\vec{r}) = \psi(-\vec{r})$$

Selection rules for β decays

If we apply the parity operator on the wave function once more,

$$\hat{P}\hat{P}\psi(\vec{r}) = \hat{P}^2\psi(\vec{r}) = \hat{P}\psi(-\vec{r}) = \psi(\vec{r})$$

Hence

$$\hat{P}^2\psi(\vec{r}) = P^2\psi(\vec{r}) = \psi(\vec{r}) \Rightarrow P = \pm 1$$

where the eigenvalue of the parity operator $\hat{P}$ is denoted by P.

The parity in β decays is conserved.
This is not a trivial fact, which arises from experimental observations. When discussing the weak interaction that causes β decays we will see that some of the most intriguing properties of the weak interaction arise due to the non-conservation of parity. However, at the energies involved in the nuclear β decays, the parity is conserved despite the fact that in general the weak interaction does not conserve it. The parity of the final state thus equals the parity of the initial state,

$$P_i = P_{e\nu} \cdot P_f$$

As mentioned above the electron-neutrino system, carrying an orbital angular momentum ℓ, is described by a spherical harmonics Y_{lm}. The property of spherical harmonics is

$$\hat{P}Y_{lm} = (-1)^l$$

Selection rules for β decays

The parity of the electron-neutrino system with the orbital angular momentum ℓ is thus $(-1)^l$. This represents the change in the parity between the initial and final nucleus, $\Delta P = P_i/P_f = (-1)^l$.

From the discussion above we can determine a set of selection rules for the angular momentum and parity in β decays. For example, if the spins of the initial and final nucleus are equal, $\Delta J = 0$, the decay can be either a Fermi decay with $l = 0$, or Gamow-Teller decay with $l = 1$, which together with $s_{e\nu} = 1$ can give $J_{e\nu} = 0$.
If furthermore the parity of the initial and final nucleus are equal, this has to be a Fermi decay since in this case the change of parity $\Delta P = (-1)^l = 1$.

Following similar arguments we can build a table of selection rules shown in the next page.

Selection rules for β decays

$\|\Delta J\|=J_{e\nu}$	ΔP	Type
0	no	F0
0	yes	GT1
1	yes	F1
1	no	GT0
2	no	F2
2	yes	GT1
2	no	GT2
:	:	:

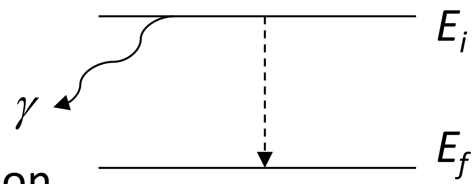
In the above table F and GT denotes Fermi and Gamow-Teller decays, respectively. The number following this notation represents ℓ, the orbital angular momentum of the electron-neutrino system.

1.2 Nuclear decays

1.2.3 γ decays

γ decays

In γ decays a nucleus emits electro-magnetic radiation (photon = γ particle) and by this de-excites from a higher to a lower, perhaps a ground energy level.



Such decays are caused by the electro-magnetic interaction.
In the decay the charge distribution as well as the distribution of the magnetic dipole moments of nucleons is changed.

The electro-magnetic (EM) field around a system of moving charges (nucleons in the nuclei) can be described by expanding it into the multipole series (see p. xx). For example, an oscillating classical electric dipole radiates a dipole field, with an average radiated power of

$$\overline{P} = \frac{\omega^4 p_{e0}^2}{3\pi\varepsilon_0 c^3}$$

where ω is the oscillation frequency and p_{e0} is the amplitude of the electric dipole, $p_{e0}=er_0$.

Higher multipoles radiate a corresponding higher multipole radiation. If the average radiated power is interpreted in terms of photons, the probability of radiating a photon of energy per unit of time is

$$w_e = \frac{\overline{P}}{\hbar\omega} = \frac{k^3 p_{e0}^2}{3\pi\varepsilon_0 \hbar}$$

γ decays

Multipole series:

any function of angles ϕ and θ in sperical coordinates can be written as

$$f(\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}c_{\ell,m}Y_{\ell,m}(\theta,\varphi)$$

where $Y_{\ell,m}(\theta,\phi)$ are spherical harmonics.

Wigner-Eckart theorem:

irreducible tensor operator of rank *k* is defined as any set of *2k+1* quantities that transform as the spherical harmonics $Y_{k,q}$ under rotations;

example: operator $\hat{\vec{r}}$ (tensor of rank *k=1*) can be defined in terms of $Y_{1,q}$ through

$$Y_{1,0}=\sqrt{\frac{3}{4\pi}}\frac{z}{r};\ Y_{1,\pm1}=\mp\sqrt{\frac{3}{4\pi}}\frac{x\pm iy}{\sqrt{2}r}$$

The Wigner-Eckart theorem states that the expectation value of any component of an irreducible tensor operator T^k_q ($q=-k, -k+1,\ldots,k$) between two angular momentum states $|j',m'\rangle$ and $|j,m\rangle$ can be factored as $\left\langle j,m\middle|\hat{T}^k_q\middle|j',m'\right\rangle=C^{jm}_{kqj'm'}\left\langle j\middle\|\hat{T}^k\middle\|j'\right\rangle$, where

γ decays

$\left\langle j \middle\| \hat{T}^k \middle\| j' \right\rangle$ is the reduced matrix element and does not depend on *q, m* and *m‘*.

$C_{kqj'm'}^{jm}$ are the appropriate Clebsch-Gordan coefficients. These coefficients equal zero unless $m = m' + q$ and $|j' - k| \leq j \leq j' + k$!

Hence also $\left\langle j, m \middle| Y_{\ell,m} \middle| j', m' \right\rangle = 0$, unless $|j' - \ell| \leq j \leq j' + \ell$

γ decays

Quantum-mechanically the expression is analogous with the dipole moment defined as

$$\vec{p}_e = \int \psi_f^* e\vec{r}\,\psi_i d^3 r$$

A similar expression holds for the radiation of a magnetic dipole,

$$w_m = \frac{\mu_0 k^3 p_{m0}^2}{3\pi\hbar} \qquad \vec{p}_m = \mu_N \int \psi_f^* (g_l \vec{\ell} + g_s \vec{s}) \psi_i d^3 r$$

(see sect. 1.1.4. for definitions of μ_N, g_l and g_s). Expressions for higher multipoles are not easy to write in a simple vector format. It suffices for the moment to know that the corresponding electric multipole operators have an angular dependence of $Y_{lm}(\theta,\phi)$ and the magnetic multipole operators that of $\vec{\ell}\,\vec{\nabla}\, Y_{lm}(\theta,\phi)$. Why is this important? The Wigner-Eckart theorem (see p. xx) tells us that in order for the expected value of an operator,

$$\langle O \rangle = \int \psi_f^* \hat{O} \psi_i d^3 r \neq 0$$

, the angular momenta of the initial and final state (J_i, J_f), and the multipolness of the operator (l) should obey the triangular relation $J_f + l \geq J_i \geq | J_f - l |$. From this one can derive the selection rules for the angular momentum in γ decays.

Selection rules for γ decays

The matrix element for a general γ decay can thus be schematically written as

$$V_{fi} = \int \psi_f^* \sum_l (\hat{O}_l^{(m)} + \hat{O}_l^{(e)}) \psi_i d^3 r$$

The superscripts *(m)* and *(e)* denote the nature of the operator - magnetic and electric, respectively. For example, $O_1^{(e)}$ is just $e\vec{r}$.

Based on the triangular relation above the difference between the spins of the initial and final nuclei is related to the multipolness of the operator causing the transition, $|J_f - J_i| = l$.
The notation of various transitions is based on the *l* of the lowest multipole operator in the matrix element resulting in a non-zero V_{fi}, and on the nature of operator as either electric (E) or magnetic (M). Thus a transition caused by the electric dipole operator is denoted as E1, the transition caused by the magnetic quadrupole operator as M2, etc.

For a given *l* the probability for a magnetic transition is significantly lower than the probability for the electric transition. This can be easily estimated for the dipole operators :

Selection rules for γ decays

$$\frac{w_m}{w_e} = \frac{\mu_0 k^3 p_{m0}^2}{3\pi\hbar} \frac{3\pi\varepsilon_0\hbar}{k^3 p_{e0}^2} = \frac{1}{c^2} \frac{p_{m0}^2}{p_{e0}^2}$$

As an approximation for the magnitude of the electric and magnetic dipole we take *e R* (with R being the "radius" of a nuclei) and μ_N (the nuclear magneton). We get

$$\frac{w_m}{w_e} = \frac{1}{c^2} \frac{\mu_N^2}{e^2 R^2} = \frac{(\hbar c)^2}{(2m_N c^2 R)^2} \sim \frac{(200\ MeVfm)^2}{(2 \cdot 900\ MeV\ 5\ fm)^2} \sim 10^{-3}$$

The ratio is even lower for higher multipole transitions.

Electromagnetic interaction preserves the parity. For the electric transitions the relation $P_f \cdot P_{Ylm} = P_i$ thus holds, where the P_{Ylm} is the parity of the Y_{lm} spherical harmonic (since this is the angular dependence of the electric multipole operator), $(-1)^l$. The selection rule regarding the parity in electric transitions is thus $P_f \cdot (-1)^l = P_i$.
For the magnetic operators the angular dependence is $\nabla Y_{lm}(\theta,\phi)$ and hence the parity is $(-1)^{l+1}$ (the derivatives in also change sign when space is reflected). The selection rule for parity in the magnetic transitions is $P_f \cdot (-1)^{l+1} = P_i$. A summary table of selection rules for γ decays is given on the next page.

Selection rules for γ decays

l	J_i	J_f	Parity change	Transition type
1	0	1	No	M1
			Yes	E1
	1	0,1,2	No	M1
			Yes	E1
	2	1,2,3	No	M1
			Yes	E1
	3	2,3,4	No	M1
			Yes	E1
	:	:	:	:
2	0	2	No	E2
			Yes	M2
	1	1,2,3	No	E2
			Yes	M2
	2	0,1,2,3,4	No	E2
			Yes	M2
	3	1,2,3,4,5	No	E2
			Yes	M2
	:	:	:	:
:	:	:	:	:

Examples of decay schemes

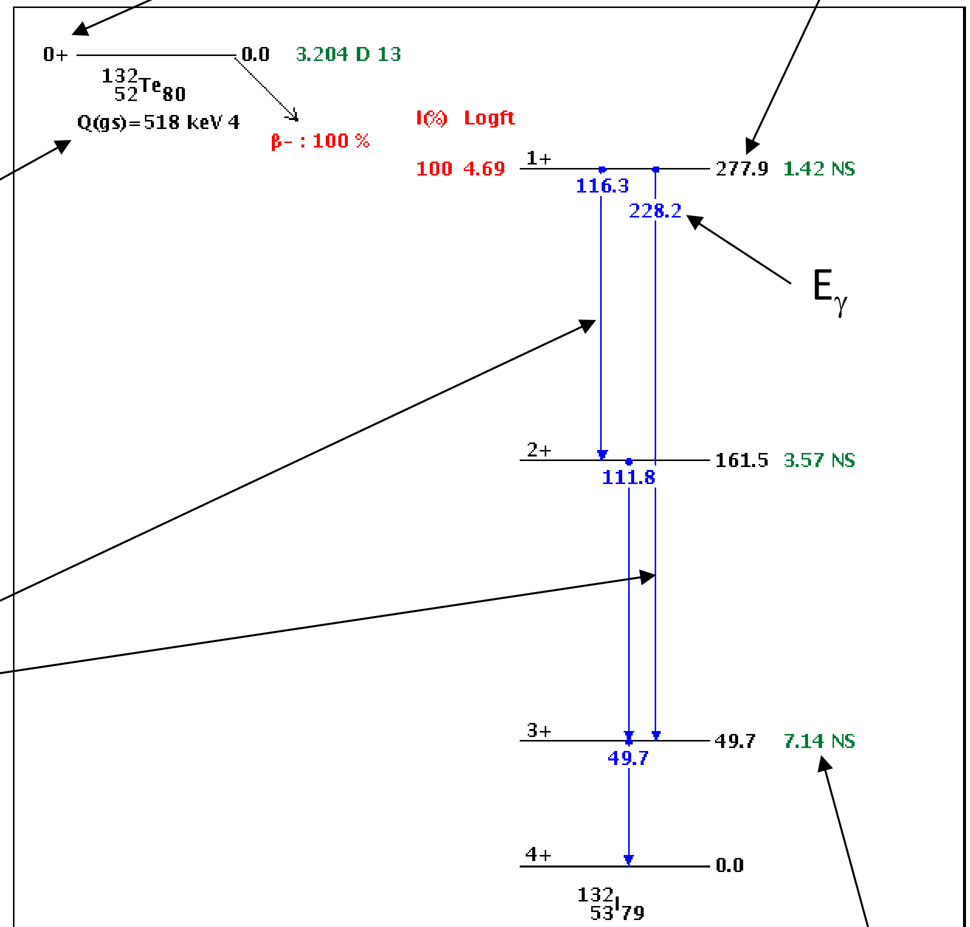
β^- decay ${}^{132}_{52}Te \rightarrow {}^{132}_{53}I\ e^-\ \nu_e$

J P(=±)

energy of nuclear state
(relative to the ground state)

maximal energy (E)

E_γ



Homework 10: based on the selection rules for γ decays determine which operators are responsible for these transitions!

decay half-time
$t_{1/2} = \tau \ln 2$

Examples of decay schemes

In some cases the γ decay from an initial to a final energy state can only proceed through high multiple transitions. In this case the lifetime of such a state is long and the state can be treated as quasi-stable. Such states are called isomers.

Homework 11:
determine which operators are responsible for the transition shown!

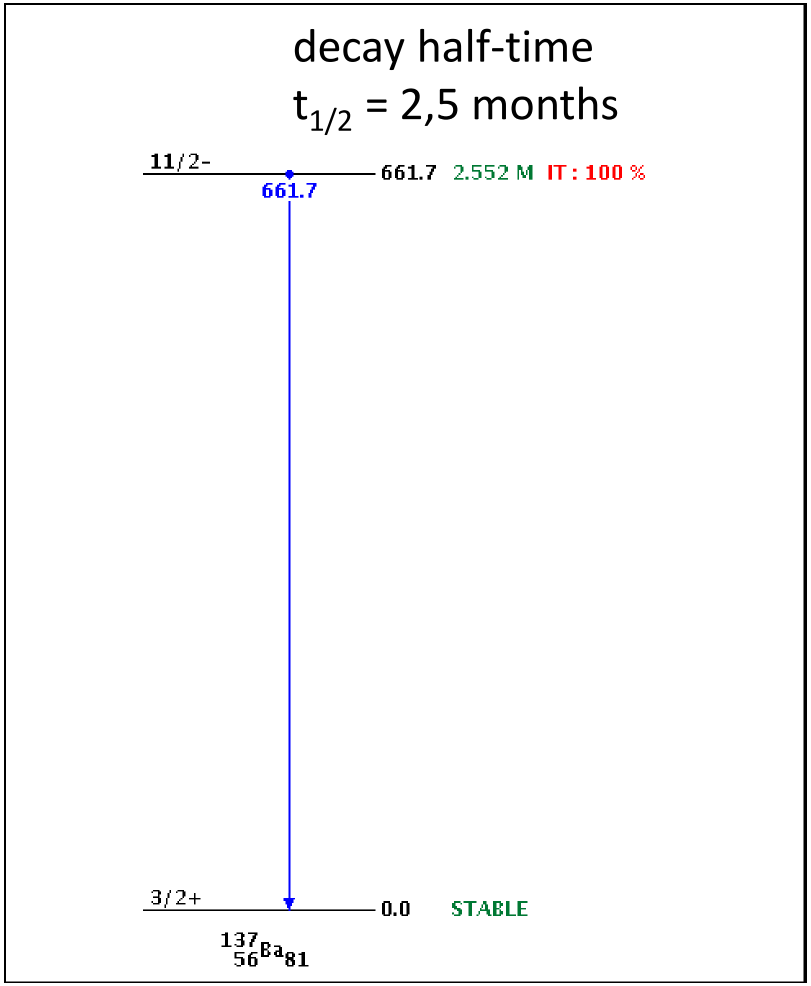


Table of Contents:

Index: