cosylab

CONTROL SYSTEM LABORATORY

www.cosylab.com

LONGITUDINAL DYNAMICS

Tutorial exercises

FMF, 27.2.2007

Joël Le DuFF

Boris Vodopivec

summary

- Radio-Frequency Acceleration and Synchronism
- Properties of Radio-Frequency cavities
- Principle of Phase Stability and Consequences
- Synchronous linear accelerator
- The Synchrotron
- RF cavities for Synchrotron
- Energy-Phase Equations in a Synchrotron
- Phase space motions

Bibliography : Old Books

M. Stanley Livingston **High Energy Accelerators**
(Interscience Publishers, 1954)

J.J. Livingood **Principles of cyclic Particle Accelerators**
(D. Van Nostrand Co Ltd , 1961)

M. Stanley Livingston and J. B. Blewett **Particle Accelerators**
(Mc Graw Hill Book Company, Inc 1962)

K.G. Steffen **High Energy optics**
(Interscience Publisher, J. Wiley & sons, 1965)

H. Bruck **Accelerateurs circulaires de particules**
(PUF, Paris 1966)

M. Stanley Livingston (editor) The development of High Energy Accelerators
(Dover Publications, Inc, N. Y. 1966)

A.A. Kolomensky & A.W. Lebedev **Theory of cyclic Accelerators**
(North Holland Publihers Company, Amst. 1966)

E. Persico, E. Ferrari, S.E. Segre **Principles of Particles Accelerators**
(W.A. Benjamin, Inc. 1968)

P.M. Lapostolle & A.L. Septier **Linear Accelerators**
(North Holland Publihers Company, Amst. 1970)

A.D. Vlasov **Theory of Linear Accelerators**
(Programm for scientific translations, Jerusalem 1968)

Bibliography : New Books

M. Conte, W.W. Mac Kay **An Introduction to the Physics of particle Accelerators** (World Scientific, 1991)

P. J. Bryant and K. Johnsen **The Principles of Circular Accelerators and Storage Rings** (Cambridge University Press, 1993)

D. A. Edwards, M. J. Syphers **An Introduction to the Physics of High Energy Accelerators** (J. Wiley & sons, Inc, 1993)

H. Wiedemann **Particle Accelerator Physics** (Springer-Verlag, Berlin, 1993)

M. Reiser **Theory and Design of Charged Particles Beams** (J. Wiley & sons, 1994)

A. Chao, M. Tigner **Handbook of Accelerator Physics and Engineering** (World Scientific 1998)

K. Wille **The Physics of Particle Accelerators: An Introduction** (Oxford University Press, 2000)

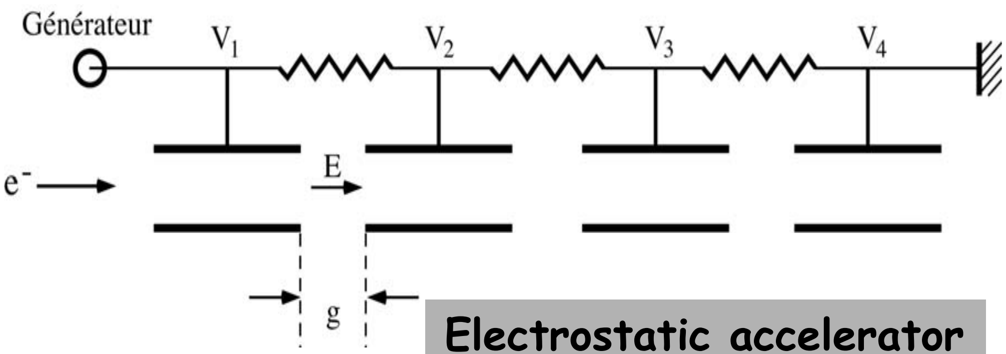
E.J.N. Wilson **An introduction to Particle Accelerators** (Oxford University Press, 2001)

Methods of Acceleration

1_ Electrostatic Field

Energy gain : $W=n.e(V_2-V_1)$

limitation : $V_{generator} = \Sigma\, V_i$



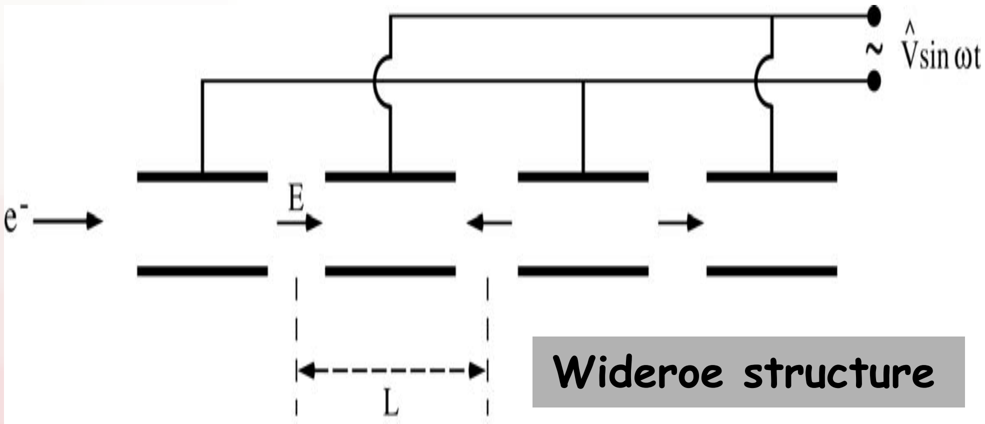
Electrostatic accelerator

2_ Radio-frequency Field

Synchronism : ➡ $L=vT/2$

v=particle velocity

T= RF period

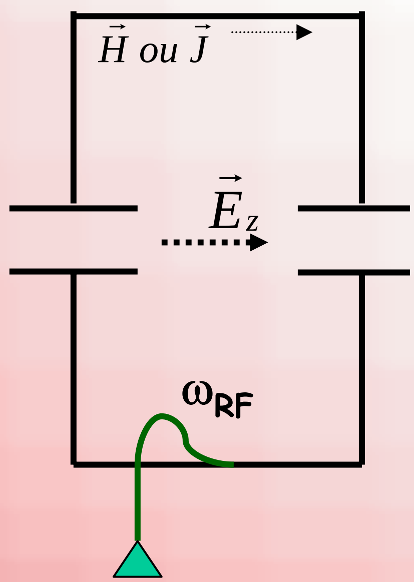


Wideroe structure

The advantage of Resonant Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one loses on the efficiency. The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency. The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.

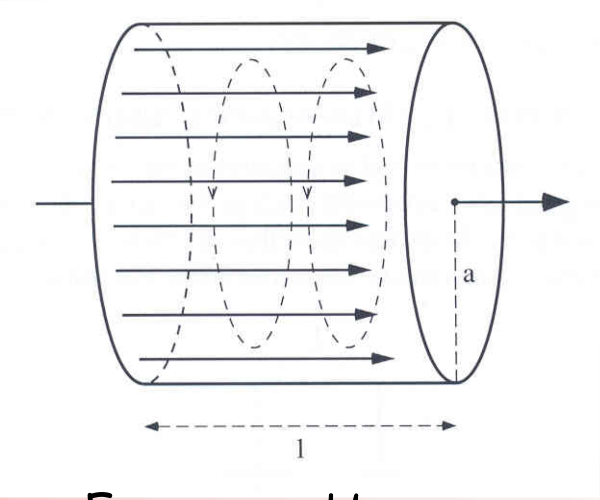


-Each such cavity can be independently powered from the RF generator.

- The electromagnetic power is now constrained in the resonant volume.

- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

The Pill Box Cavity

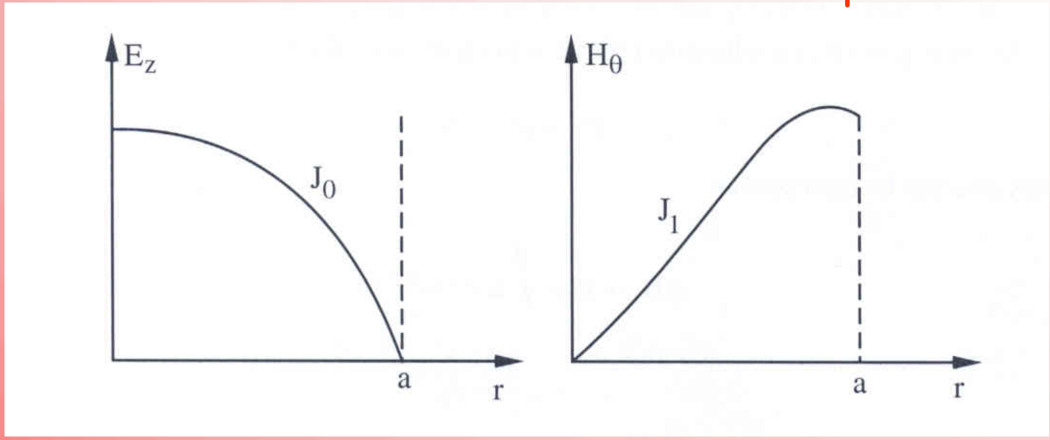


$\longrightarrow E_z$ ⋯⋯→ H_θ

From Maxwell's equations one can derive the wave equations :

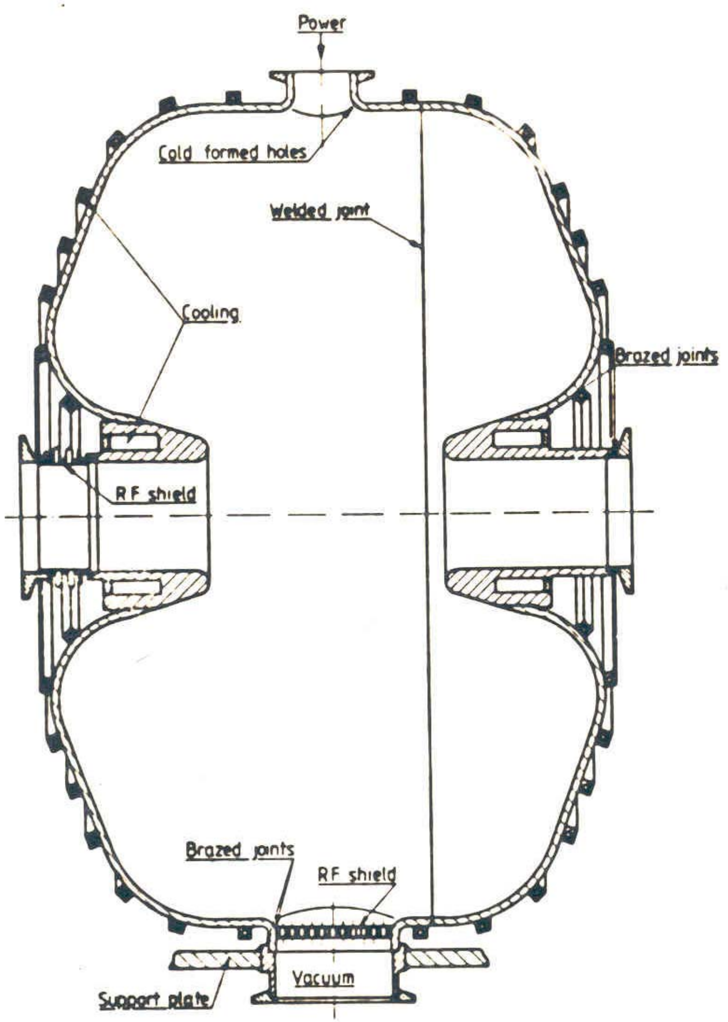
$$\nabla^2 A - \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \qquad (A = E \text{ ou } H)$$

Solutions for E and H are oscillating modes, at discrete frequencies, of types TM or TE. For l<2a the most simple mode, TM_{010}, has the lowest frequency ,and has only two field components:



$$\left.\begin{aligned} E_z &= J_0(kr) \\ H_\theta &= -\frac{j}{Z_0} J_1(kr) \end{aligned}\right\} e^{j\omega t}$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad \lambda = 2{,}62\,a \quad Z_0 = 377\Omega$$



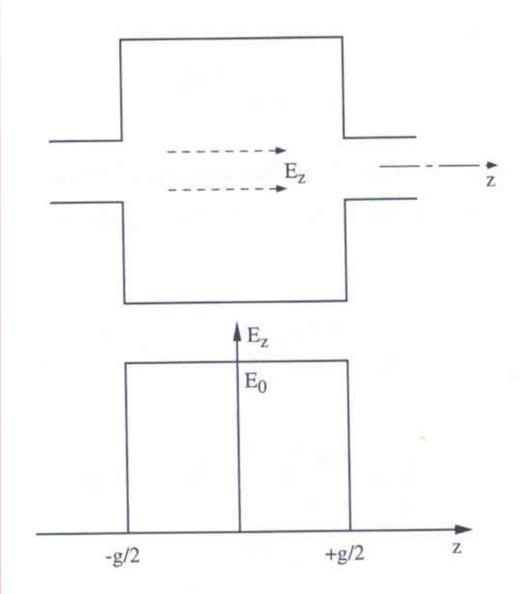
The design of a pill-box cavity can be sophisticated in order to improve its performances:

-A nose cone can be introduced in order to concentrate the electric field around the axis,

-Round shaping of the corners allows a better distribution of the magnetic field on the surface and a reduction of the Joule losses.

A good cavity is a cavity which efficiently transforms the RF power into accelerating voltage.

Transit Time Factor



Oscillating field at frequency ω and which amplitude is assumed to be constant all along the gap:

$$E_z = E_0 \cos\omega t = \frac{V}{g} \cos\omega t$$

Consider a particle passing through the middle of the gap at time t=0 :

$$z = vt$$

The total energy gain is:

$$\Delta W = \frac{eV}{g} \int_{-g/2}^{g/2} \cos\omega \frac{z}{v} dz$$

$$\Delta W = eV \frac{\sin\theta/2}{\theta/2} = eVT$$

$$\theta = \frac{\omega g}{v} \quad \textit{transit angle}$$

$$T \quad \textit{transit time factor}$$

(0 < T < 1)

Transit Time Factor (2)

Consider the most general case and make use of complex notations:

$$\Delta W = e\Re_e \int_0^g E_z(z) e^{j\omega t} dz \qquad \omega t = \omega \frac{z}{v} - \psi_p$$

Ψ_p is the phase of the particle entering the gap with respect to the RF.

$$\Delta W = e\Re_e \left[e^{-j\psi_p} \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right]$$

$$\Delta W = e\Re_e \left[e^{-j\psi_p} e^{j\psi_i} \left| \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right| \right]$$

Introducing: $\phi = \psi_p - \psi_i$

$$\Delta W = e \left| \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right| \cos\phi$$

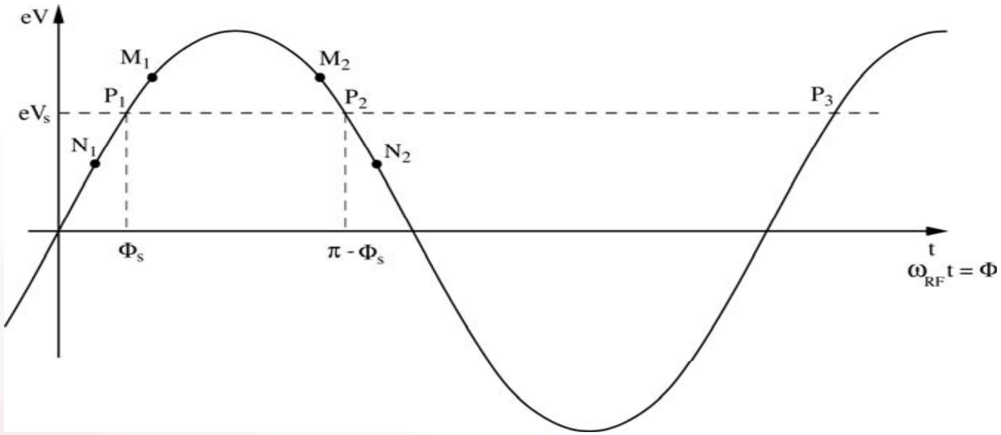
and considering the phase which yields the maximum energy gain:

$$T = \frac{\left| \int_0^g E_z(z) e^{j\omega t} dz \right|}{\int_0^g E_z(z) dz}$$

Principle of Phase Stability

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

For a 2π mode, the electric field is the same in all gaps at any given time.



$$eV_S = e\hat{V}\sin\Phi_S$$

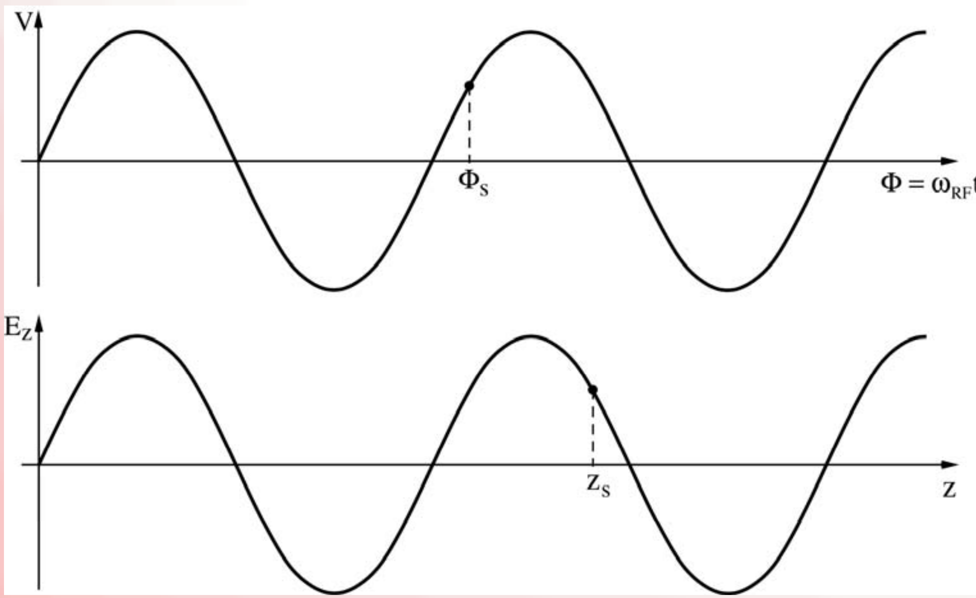
is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.

If an increase in energy is transferred into an increase in velocity, M_1 & N_1 will move towards P_1(stable), while M_2 & N_2 will go away from P_2 (unstable).

A Consequence of Phase Stability

Transverse Instability

Longitudinal phase stability means : $$\frac{\partial V}{\partial t} > 0 \Rightarrow \frac{\partial E_z}{\partial z} < 0$$

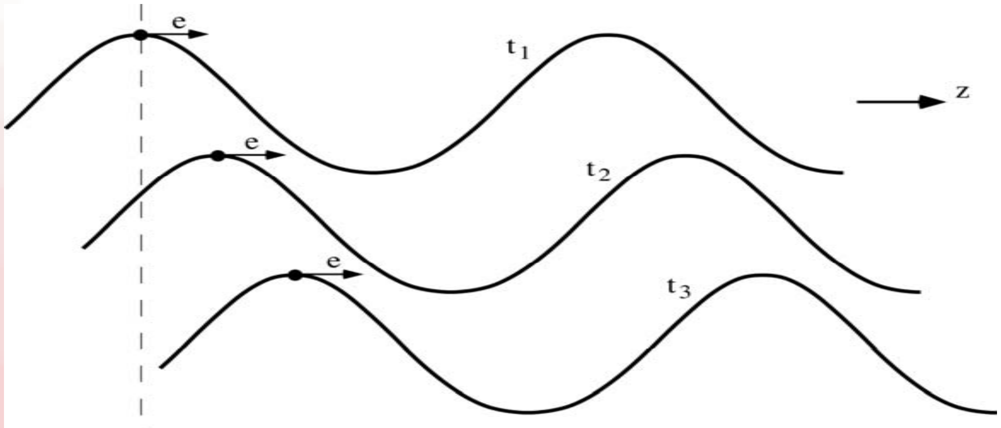


defocusing RF force

The divergence of the field is zero according to Maxwell : $$\nabla . \vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} > 0$$

External focusing (solenoid, quadrupole) is then necessary

The Traveling Wave Case



$$E_z = E_0 \cos\left(\omega_{RF} t - kz\right)$$

$$k = \frac{\omega_{RF}}{v_\varphi}$$

$$z = v\left(t - t_0\right)$$

$$v_\varphi = \textit{phase velocity}$$

$$v = \textit{particle velocity}$$

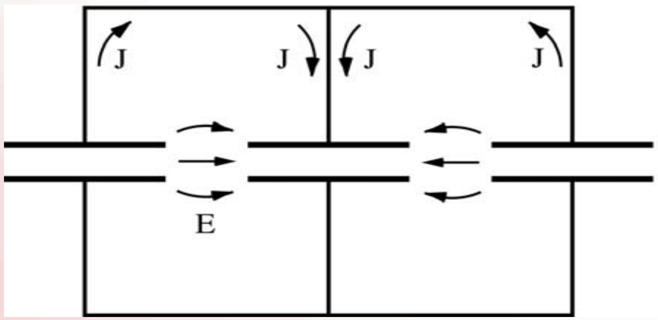
The particle travels along with the wave, and k represents the wave propagation factor.

$$E_z = E_0 \cos\left(\omega_{RF} t - \omega_{RF} \frac{v}{v_\varphi} t - \phi_0\right)$$

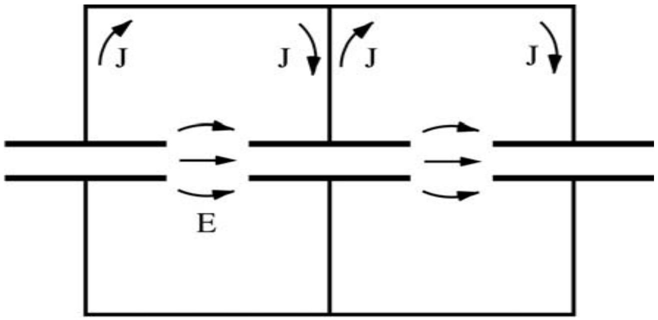
If synchronism satisfied: $v = v_\varphi \quad \textit{and} \quad E_z = E_0 \cos \phi_0$

where ϕ_0 is the RF phase seen by the particle.

Multi-gaps Accelerating Structures:
A- Low Kinetic Energy Linac (protons,ions)

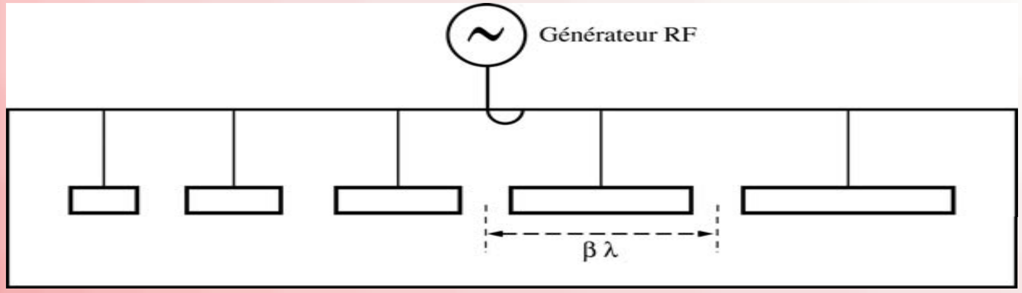


Mode π L= vT/2



Mode 2π L= vT = $\beta\lambda$

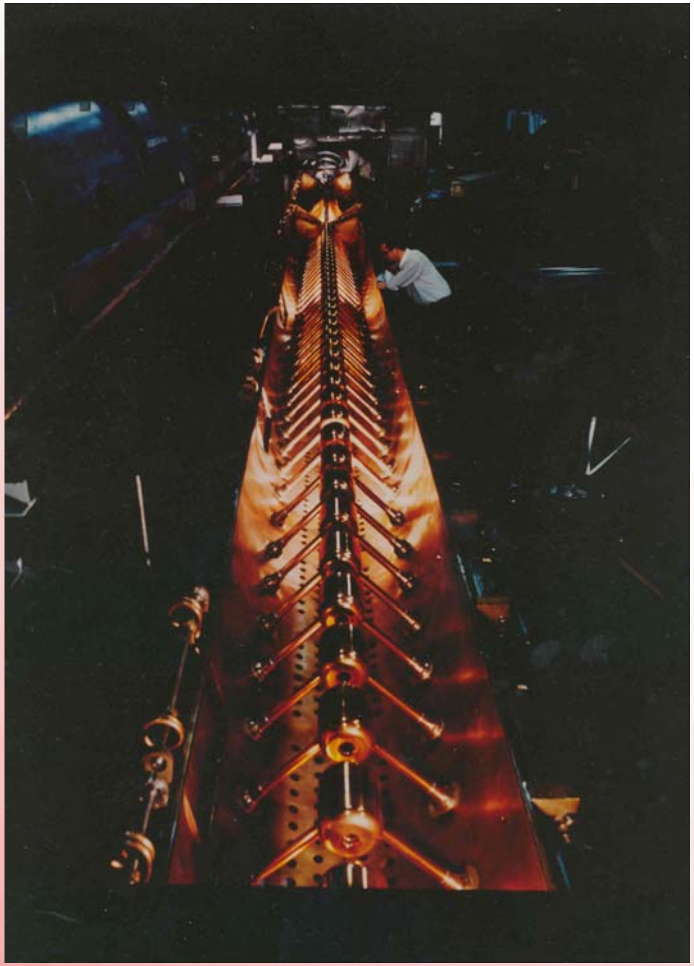
In « WIDEROE » structure radiated power $\propto \omega$ CV



ALVAREZ structure

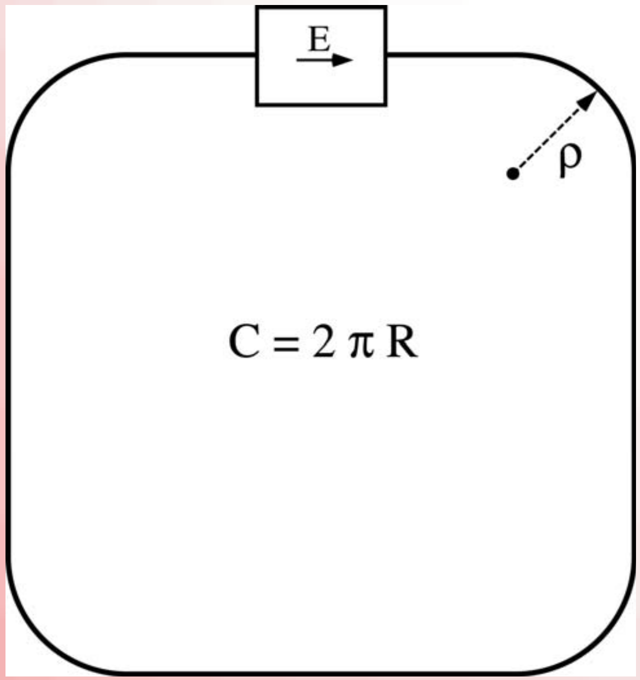
In order to reduce the radiated power the gap is enclosed in a resonant volume at the operating frequency. A common wall can be suppressed if no circulating current in it for the chosen mode.

CERN Proton Linac



The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ Energy gain per turn

$\Phi = \Phi_s \longrightarrow$ Synchronous particle

$\omega_{RF} = h\omega_r \longrightarrow$ RF synchronism

$\rho, R \longrightarrow$ Constant orbit

$B\rho = P/e \Rightarrow B \longrightarrow$ Variable magnetic field

If v = c, ω_r hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron (2)

Energy ramping is simply obtained by varying the B field:

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\, B' \quad \Rightarrow (\Delta p)_{turn} = e\rho\, B' T_r = \frac{2\pi\, e\rho\, RB'}{v}$$

Since: $$E^2 = E_0^2 + p^2 c^2 \quad \Rightarrow \quad \Delta E = v \Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho RB' = e\hat{V}\sin\phi_s$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satifies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.

The Synchrotron (3)

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency : $\longrightarrow$ $\omega_r = \frac{\omega_{RF}}{h} = \omega(B, R_s)$

hence : $\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{e}{m} < B(t) > \quad \Rightarrow \quad \frac{f_{RF}(t)}{h} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{r}{R_s}B(t)$

Since $E^2 = m_0^2 c^4 + p^2 c^2$, the RF frequency must follow the variation of the B field with the law : $\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0 c^2 / ecr)^2 + B(t)^2}\right\}^{1/2}$ which asymptotically tends towards $f_r \to \frac{c}{2\pi R}$ when B becomes large compare to $(m_0 c^2 / 2\pi r)$ which corresponds to $v \longrightarrow c$ $(pc \gg m_0 c^2)$. In practice the B field can follow the law:

$$B(t) = \frac{\hat{B}}{2}(1 - \cos\omega t) = \hat{B}\sin^2\frac{\omega}{2}t$$

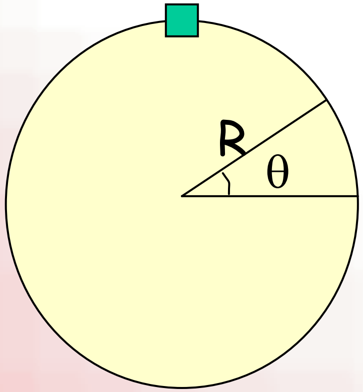
Longitudinal Dynamics

It is also often called " synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle. So let's introduce the following reduced variables:

revolution frequency	:	$\Delta f_r = f_r - f_{rs}$
particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_s$
particle energy	:	$\Delta E = E - E_s$
azimuth angle	:	$\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = h f_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h}\frac{d}{dt}(\Delta\phi) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s$ and $E^2 = E_0^2 + p^2c^2$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is: $$\frac{dE}{dt} = e\hat{V}\sin\phi \frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\,\Delta T_r + T_{rs}\,\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\,\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\,\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}} \dot{\phi}$$

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}} \frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time…………………

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi}+\frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi-\sin\phi_s\right)=0$$

with

$$\Omega_s^2=\frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi-\sin\phi_s=\sin\left(\phi_s+\Delta\phi\right)-\sin\phi_s\cong\cos\phi_s\Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi}+\Omega_s^2\Delta\phi=0$$

stable for $\Omega_s^2>0$ and Ω_s real

$\gamma < \gamma_{tr}$	$\eta > 0$	$0 < \phi_s < \pi/2$	$\sin\phi_s > 0$
$\gamma > \gamma_{tr}$	$\eta < 0$	$\pi/2 < \phi_s < \pi$	$\sin\phi_s > 0$

Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

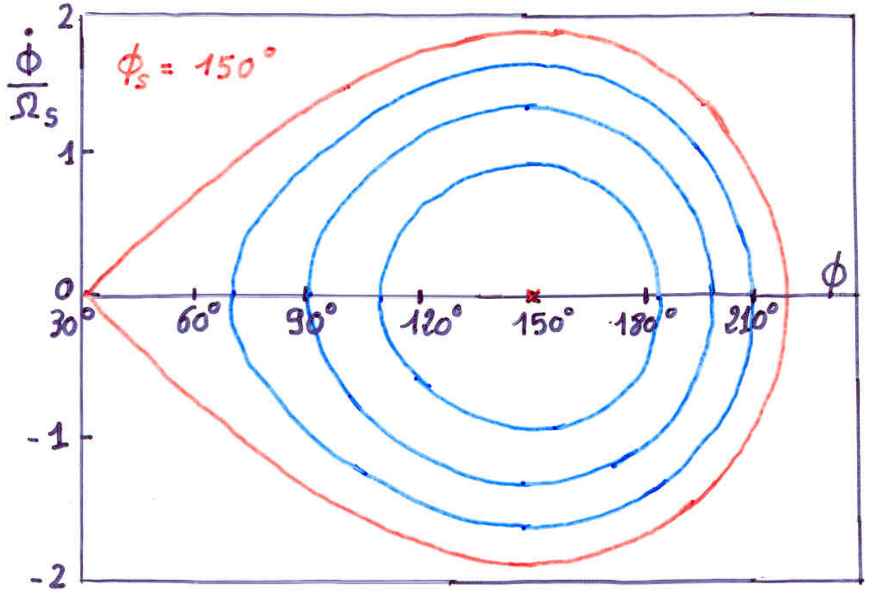
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{(\Delta\phi)^2}{2} = I$$

(the variable is $\Delta\phi$ and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring. Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extremum when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2\{2 + (2\phi_s - \pi)\tan\phi_s\}$$

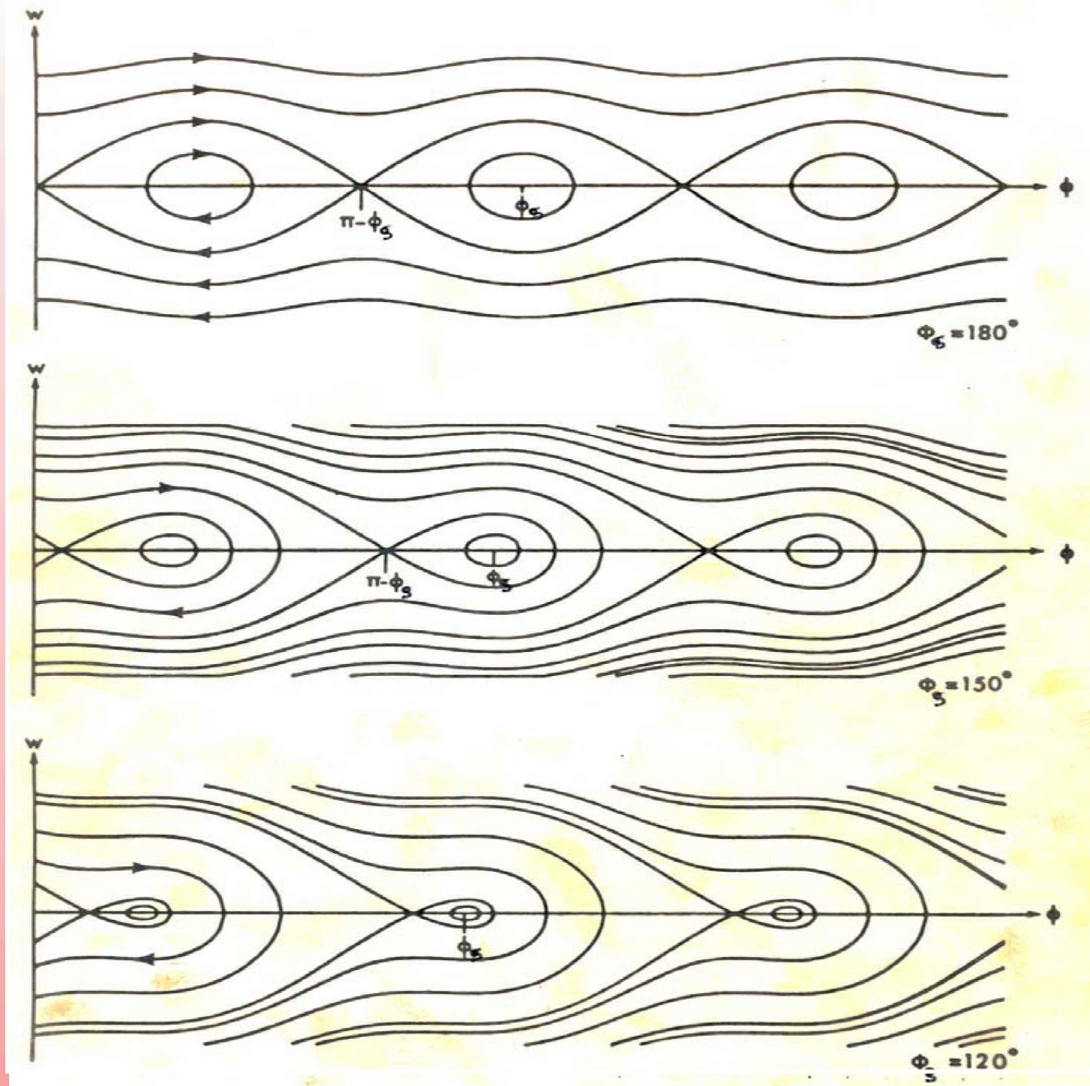
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s}\right)_{\max} = \mp\beta\left\{-\frac{e\hat{V}}{\pi h \eta E_s}G(\phi_s)\right\}^{\frac{1}{2}}$$

$$G(\phi_s) = [2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s]$$

This “RF acceptance” depends strongly on ϕ_s and plays an important role for the electron capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



As the synchronous phase gets closer to 90° the area of stable motion (closed trajectories) gets smaller. These areas are often called "BUCKET".

The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.