4.1 A Cascade of Equal, DC-Coupled, RC-loaded Amplifying Stages

A multistage amplifier with DC-coupled and RC-loaded stages is shown in Fig. 4.1.1. All stages are assumed to be equal. Junction field-effect transistors (JFETs) are being used as active devices, since we want to focus to essential design problems; with bipolar junction transistors (BJTs), we would have to consider the loading effect of a relatively complicated input impedance [Ref. 4.1].

At each stage load, the capacitance C represents the sum of all the stray-capacitances with the $C_{\mathrm{GS}}(1+A_k)$ equivalent capacitance (where C_{GS} is the gate-to-drain capacitance and A_k is the voltage gain of the individual stage). By doing so, we get a simple parallel RC-load in each stage. The input resistance of a JFET is many orders of magnitude higher than the loading resistor R, so we can neglect it. All loading resistors are equal and so are the mutual conductances g_{m} of all JFETs, consequently all individual gains A_k are equal as well. Therefore the half-power frequencies $\omega_{\mathrm{h}k}$ and all the risetimes $\tau_{\mathrm{r}k}$ of individual stages are also equal. In order to further simplify the circuit, we did not draw the power supply and the bias voltages.

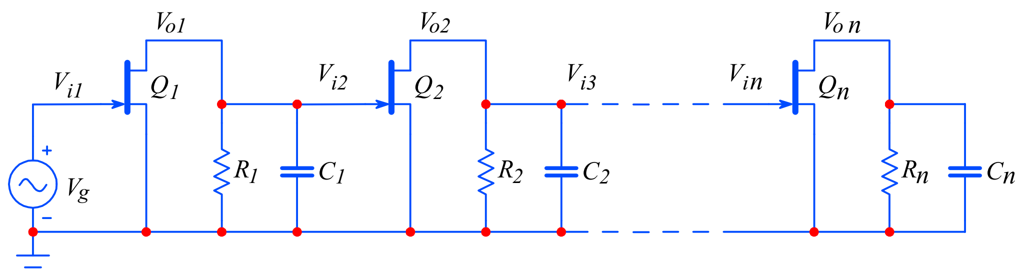

Fig. 4.1.1 : A multistage amplifier as a cascade of equal, DC-coupled, RC -loaded amplifying stages.

The voltage gain of an individual stage is:

$$A_k = g_{\mathrm{m}} R \, \frac{1}{1+j\omega RC} \tag{4.1.1}$$

with the magnitude:

$$|A_k| = \frac{g_{\mathrm{m}} R}{\sqrt{1+(\omega/\omega_{\mathrm{h}})^2}} \tag{4.1.2}$$

where:

g_{m} = mutual conductance of the JFET, [S] (siemens, [S] = [1/Ω])

$\omega_{\mathrm{h}} = 1/RC$ = upper half-power frequency of an individual stage, [rad/ s]

4.1.1 Frequency-Response and the Upper Half-Power Frequency

We have n equal stages with equal gains, so $A_1 = A_2 = \cdots = A_n = A_k$. The gain of the complete amplifier is then:

$$A = A_1 \cdot A_2 \cdot A_3 \cdots A_n = A_k^n = \left[\frac{g_{\mathrm{m}} R}{1+j\omega RC}\right]^n \tag{4.1.3}$$

The magnitude is :

$$|A| = \left[\frac{g_\mathrm{m}\, R}{\sqrt{1 + (\omega/\omega_\mathrm{h})^2}} \right]^n \tag{4.1.4}$$

To be able to compare the bandwidth of the multistage amplifier for different number of stages, we must normalize the magnitude by dividing Eq. 4.1.4 by the system DC-gain $(g_\mathrm{m}\, R)^n$. The plots are shown in Fig. 4.1.2. It is evident that the system bandwidth, ω_H , shrinks with each additional amplifying stage.

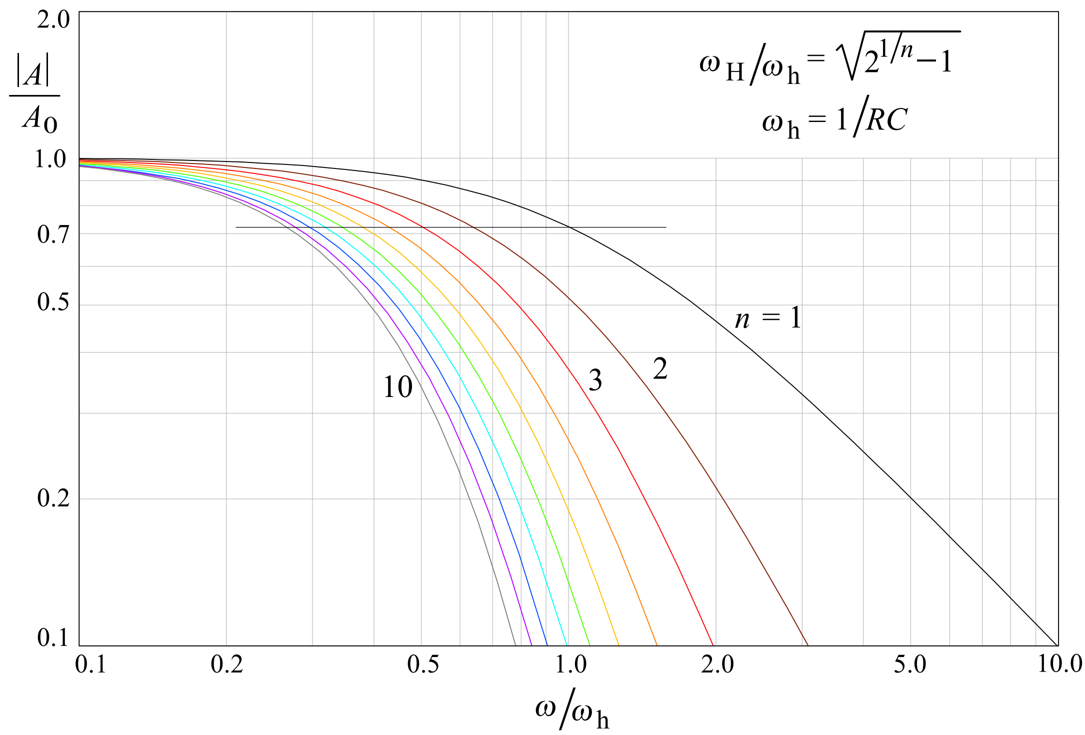

Fig. 4.1.2 : Frequency-response of a n-stage amplifier ($n = 1,\ 2,\ \cdots,\ 10$). To compare the bandwidth, the gain was normalized (divided by the system DC-gain, $(g_\mathrm{m}\, R)^n$). For each additional stage, the bandwidth (the crossing of the 0.707 level) shrinks by $\sqrt{2^{1/n} - 1}$.

The upper half-power frequency of the amplifier can be calculated by a simple relation :

$$\left[\frac{1}{\sqrt{1 + (\omega_\mathrm{H}/\omega_\mathrm{h})^2}} \right]^n = \frac{1}{\sqrt{2}} \tag{4.1.5}$$

Thus :

$$\left[1 + (\omega_\mathrm{H}/\omega_\mathrm{h})^2\right]^n = 2 \qquad \Rightarrow \qquad \left(\frac{\omega_\mathrm{H}}{\omega_\mathrm{h}}\right)^2 = 2^{1/n} - 1 \tag{4.1.6}$$

so, the upper half-power frequency of the complete amplifier is :

$$\boxed{\omega_\mathrm{H} = \omega_\mathrm{h}\, \sqrt{2^{1/n} - 1}} \tag{4.1.7}$$

At high frequencies, the first-stage response slope approaches the -6 dB/octave asymptote (= -20 dB/decade). The meaning of this slope is explained in Fig. 4.1.3. For the second stage, the slope is twice as steep and for the n^{th}-stage it is n times steeper.

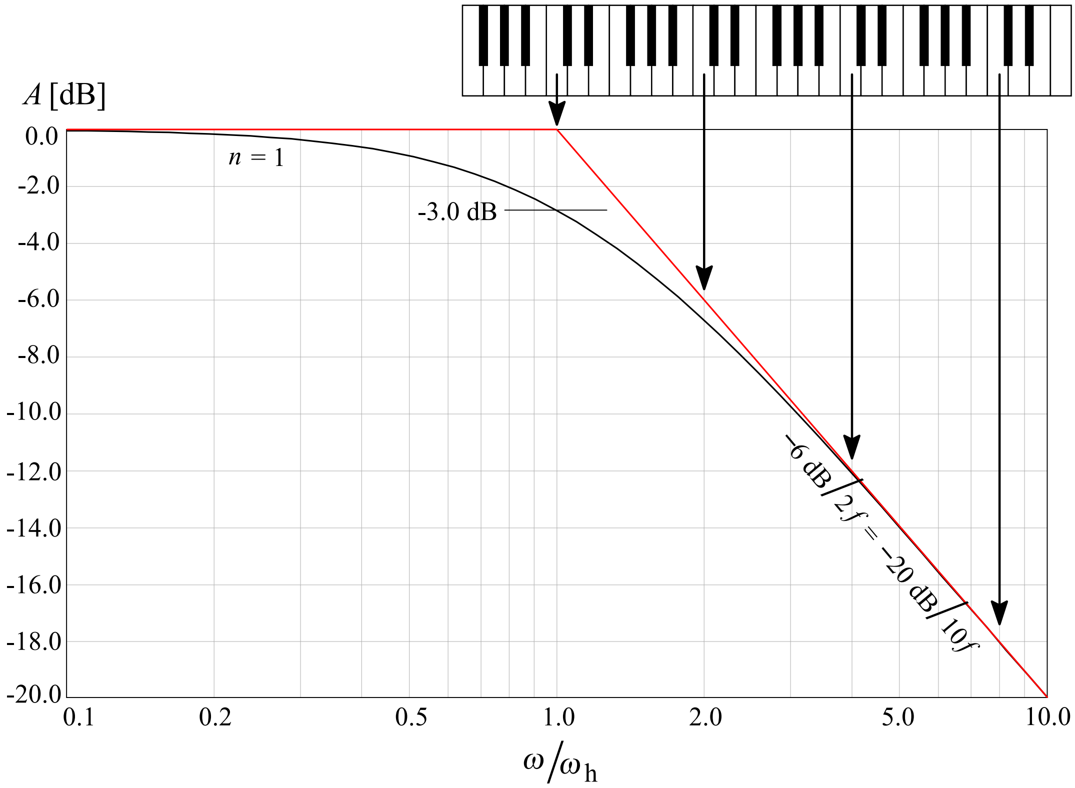

Fig. 4.1.3 : The first-order system response and its asymptotes. Below the cutoff, the asymptote is the level equal to the system gain at DC (normalized here to 0 dB). Above the cutoff, the slope is -6 dB/octave (an octave is a frequency span from f to $2f$), which is also equal to -20 dB/decade (a frequency decade is a span from f to $10\,f$).

The values ω_{H} for $n = 1 \cdots 10$ are reported in Table 4.1.1.

Table 4.1.1

n	1	2	3	4	5	6	7	8	9	10
ω_{H}	1.000	0.644	0.510	0.435	0.386	0.550	0.323	0.301	0.283	0.269

With ten equal stages connected in cascade, the bandwidth is reduced to a poor $0.269\,\omega_{\mathrm{h}}$; such an amplifier is definitively not very efficient for wideband amplification.

Alternatively, in order to preserve the bandwidth, a n-stage amplifier should have all its capacitors reduced by the same factor, $\sqrt{2^{1/n} - 1}$. But in wideband amplifiers we already strive to work with stray capacitances only, so this approach is not a solution.

Nevertheless, the amplifier in Fig. 4.1.1 is the basis for more efficient amplifier configurations, which we will discuss later.

4.1.2 Phase-Response

Each individual stage of the amplifier in Fig. 4.1.1 has a frequency-dependent phase-shift angle:

$$\varphi_k = \arctan \frac{\Im\{F(j\omega)\}}{\Re\{F(j\omega)\}} = \arctan(-\omega/\omega_{\mathrm{h}}) \tag{4.1.8}$$

where $F(j\omega)$ is taken from Eq. 4.1.1. For n equal stages the total phase-shift angle is simply n-times as much:

$$\boxed{\varphi_n = n \arctan(-\omega/\omega_{\mathrm{h}})} \tag{4.1.9}$$

The phase responses are plotted in Fig. 4.1.4. Note the high-frequency asymptotic phase shift increasing by $\pi/2$ (or 90°) for each n. Also note the shift at $\omega = \omega_{\mathrm{h}}$ being exactly $n\pi/4$, in spite of a reduced ω_{h} for each n.

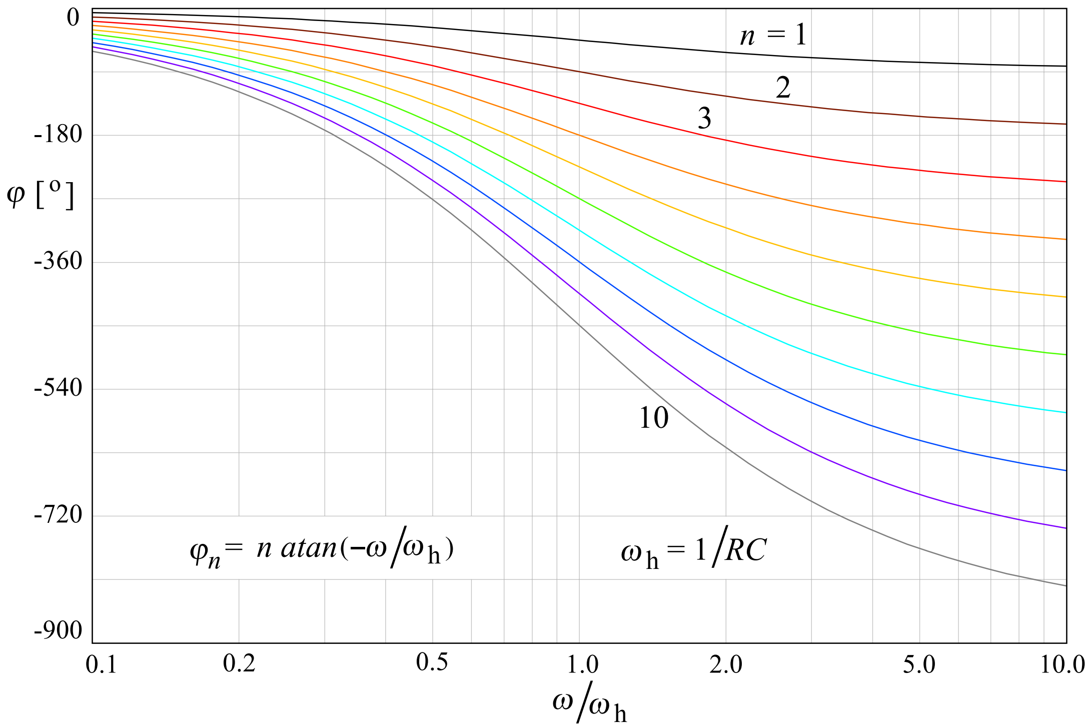

Fig. 4.1.4: Phase-angle of the amplifier in Fig. 4.1.1, for $n = 1 \cdots 10$ amplifying stages.

4.1.3 Envelope-Delay

For a single amplifying stage ($n = 1$), the envelope-delay is the frequency-derivative of the phase, $\tau_{\mathrm{e}n} = d\varphi_n/d\omega$ (where φ_n is given by Eq. 4.1.9). The normalized single-stage envelope-delay is:

$$\tau_{\mathrm{e}}\,\omega_{\mathrm{h}} = \frac{-1}{1 + (\omega/\omega_{\mathrm{h}})^2} \tag{4.1.10}$$

and for n equal stages:

$$\boxed{\tau_{en}\,\omega_h = \frac{-\,n}{1+(\omega/\omega_h)^2}} \tag{4.1.11}$$

Fig. 4.1.5 shows the frequency-dependent envelope-delay for $n = 1 \cdots 10$. Note the delay at $\omega = \omega_h$ being exactly $1/2$ of the low-frequency asymptotic value.

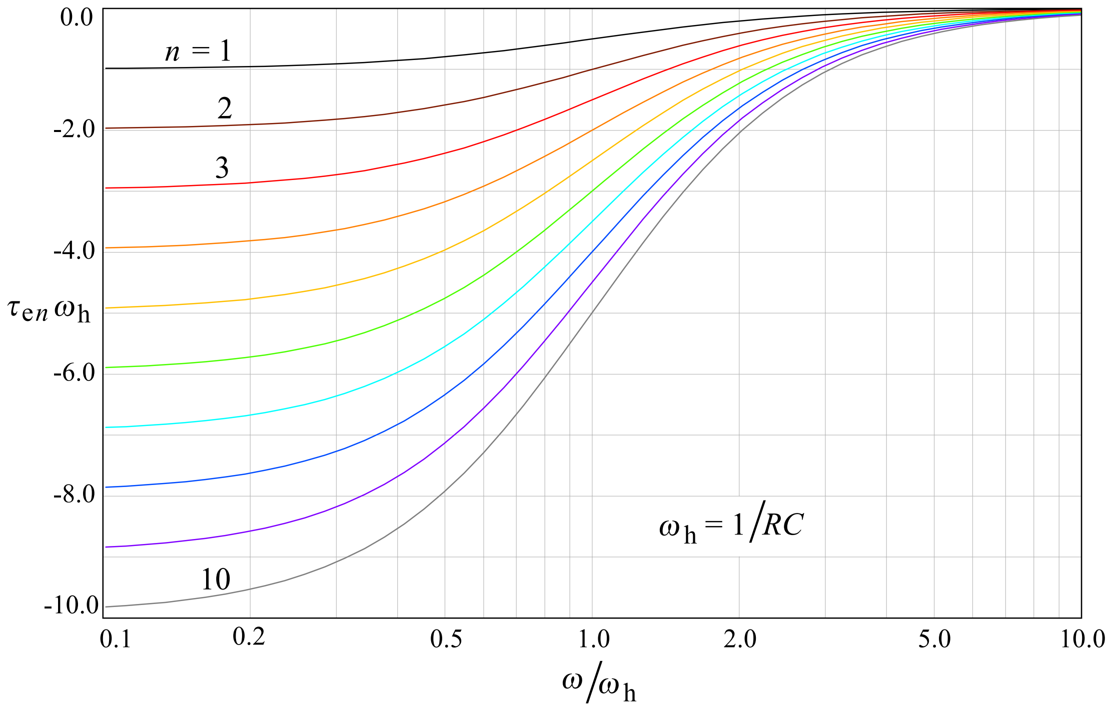

Fig. 4.1.5 : Envelope-delay of the amplifier in Fig. 4.1.1, for $n = 1 \cdots 10$ amplifying stages. The delay at $\omega = \omega_h$ is $1/2$ of the low-frequency asymptotic value. Note that if we were using f/f_h for the abscissa, we would have to divide the τ_e scale by 2π .

4.1.4 Step-Response

To get the step-response, the amplifier in Fig. 4.1.1 must be driven with the unit-step function :

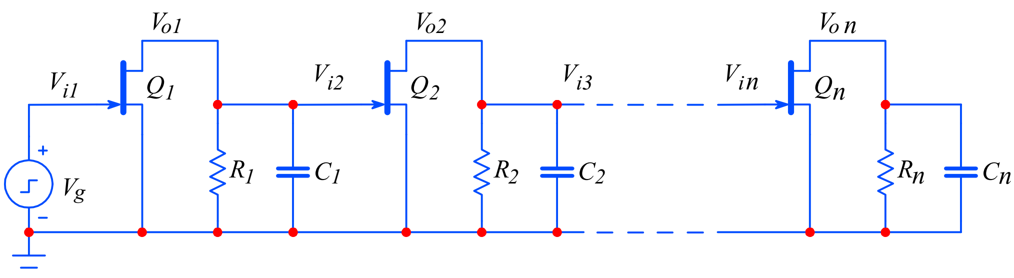

Fig. 4.1.6 : Amplifier with n equal DC coupled stages, excited by an unit-step pulse

We can derive the step-response expression from Eq. 4.1.1 and Eq. 4.1.3. In order to simplify and generalize the expression, we will normalize the magnitude by dividing the transfer function by the d.c. gain, $g_m R$, and normalize the frequency by setting $\omega_h = 1/RC = 1$. Since we will use the $\mathcal{L}^{-1}$-transform, we will replace the variable $j\omega$ by the complex variable $s = \sigma + j\omega$. With all these changes we get:

$$F(s) = \frac{1}{(1+s)^n} \tag{4.1.12}$$

The amplifier input is excited by the unit-step, therefore we must multiply the above formula by the unit-step operator $1/s$:

$$G(s) = \frac{1}{s\,(1+s)^n} \tag{4.1.13}$$

The corresponding function in the time-domain is:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum res \frac{e^{st}}{s\,(1+s)^n} \tag{41.14}$$

We have two residues. The first one does not depend of n:

$$res_0 = \lim_{s\to 0} s\left[\frac{e^{st}}{s\,(1+s)^n}\right] = 1$$

while the second is:

$$res_1 = \lim_{s\to 1} \frac{1}{(n-1)!}\cdot\frac{d^{(n-1)}}{ds^{(n-1)}}\left[(1+s)^n\,\frac{e^{st}}{s\,(1+s)^n}\right] =$$

$$= \lim_{s\to 1} \frac{1}{(n-1)!}\cdot\frac{d^{(n-1)}}{ds^{(n-1)}}\left(\frac{e^{st}}{s}\right) \tag{4.1.15}$$

Since res_1 depends on n, for $n = 1$ we get:

$$res_1\Big|_{n=1} = -\,e^{-t} \tag{4.1.16}$$

for $n = 2$:

$$res_1\Big|_{n=2} = -\,e^{-t}\,(1+t) \tag{4.1.17}$$

for $n = 3$:

$$res_1\Big|_{n=3} = -\,e^{-t}\left(1+t+\frac{t^2}{2}\right) \tag{4.1.18}$$

... etc. The general expression for the step-response for any n is:

$$\boxed{g_n(t) = \mathcal{L}^{-1}\{\mathrm{G}(s)\} = res_0 + res_1 = 1 - e^{-t}\sum_{k=1}^{n}\frac{t^{k-1}}{(k-1)!}} \tag{4.1.19}$$

Here we must consider that $0! = 1$, by definition.

As an example, by inserting $n = 5$ into Eq. 4.1.19, we would get:

$$g_5(t) = 1 - e^{-t}\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!}\right) \tag{4.1.20}$$

The step-response plots for $n = 1 \cdots 10$, calculated by Eq. 4.1.19, are displayed in Fig. 4.1.7. We note that there is **no overshoot** in any of the curves. Unfortunately, the efficiency of this kind of amplifier in the "bandwidth per number of stages" sense is poor, since it has no peaking networks that would prevent bandwidth decrease with n.

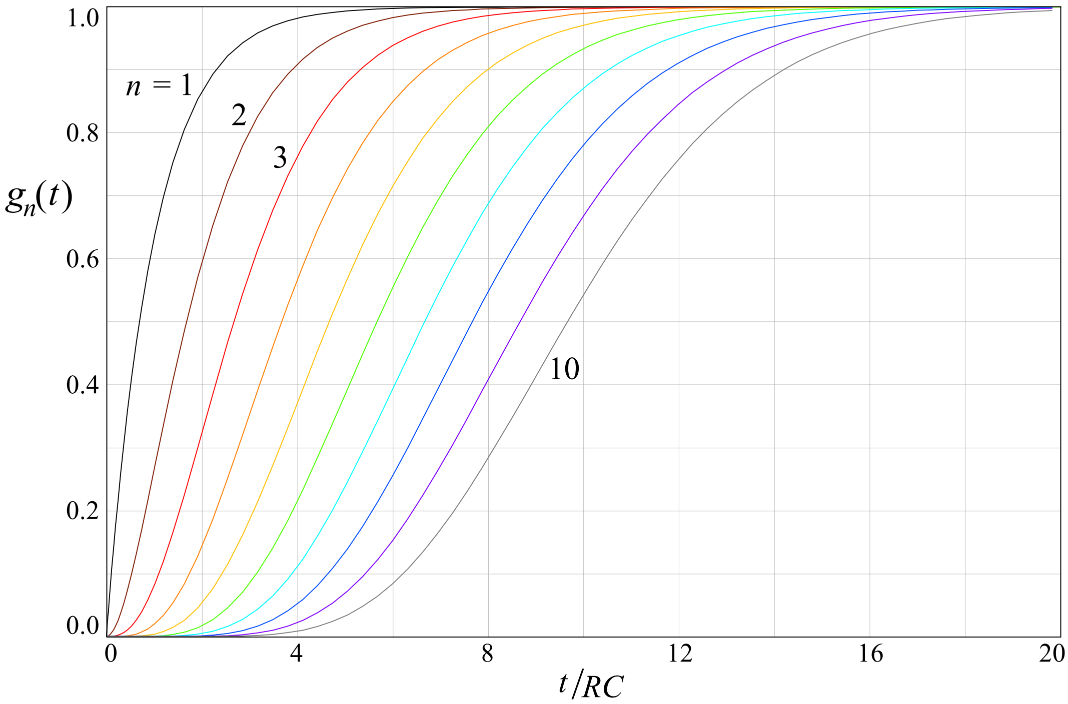

Fig. 4.1.7 : Step-response of the amplifier in Fig. 4.1.6, for $n = 1 \cdots 10$ amplifying stages

4.1.5 Risetime Calculation

In case of a multistage amplifier, where each particular stage has its respective risetime, τ_{r1} , τ_{r2} , $\cdots$, τ_{rn} , we calculate the system risetime [Ref. 4.2] as :

$$\tau_r = \sqrt{\tau_{r1}^2 + \tau_{r2}^2 + \tau_{r3}^2 + \cdots + \tau_{rn}^2} \tag{4.1.21}$$

In Part 2, Sec. 2.1.1, Eq. 2.1.1, we have calculated the risetime of an amplifier with a simple RC load to be $\tau_{r1} = 2.20\,RC$. Since we have here n equal stages, the risetime of the complete amplifier is then :

$$\boxed{\tau_r = \tau_{r1}\,\sqrt{n} = 2.20\,RC\,\sqrt{n}} \tag{4.1.22}$$

Table 4.1.2 shows the risetime increasing with the number of stages :

Table 4.1.2

n	1	2	3	4	5	6	7	8	9	10
τ_{rn}/τ_{r1}	1.00	1.41	1.73	2.00	2.24	2.45	2.65	2.83	3.00	3.16