1.12 Complex Integration Around Many Poles, The Cauchy-Goursat Theorem

So far we have calculated a contour integral around one pole (simple or multiple). Now we will integrate around more poles, either single or multiple ones.

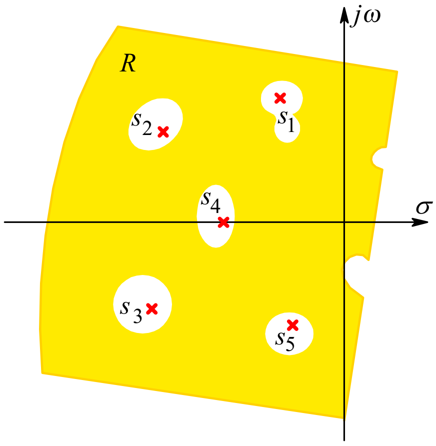


Fig. 1.12.1 : The cheese represents a regular (=analytical) domain R of a function which has one simple pole in every hole.

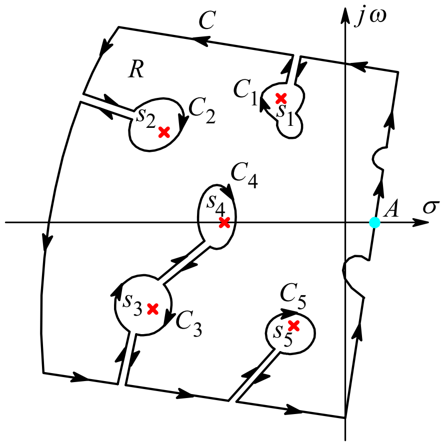


Fig. 1.12.2 : Encircling the poles by contours C, C_1, ..., C_5, so that the regular domain of the function is always on the left side.

Cheese is a regular part of French meals. So we may imagine that the great mathematician Cauchy observed a slice of Emmentaler cheese like that in Fig. 1.12.1 (the characteristic of this cheese is big holes) on his plate and reflected this way: Suppose all that is cheese is an analytical or as we also say, a regular domain R of a function $F(s)$. In the holes are the poles $s_1 \cdots s_5$. We are not interested in the domain outside the cheese. How could we "mathematically" encircle the cheese around the crust and around the rims of all the holes, so that the cheese is always on the left side of the contour?

Impossible? No! If we take a knife and make a cut from the crust toward each hole, without removing any cheese, we provide the necessary path for the suggested contour, as shown in Fig. 1.12.2.

Now we calculate a contour integral, starting from the point A in the suggested (counter-clockwise) direction until we come to the cut towards the first pole, s_1. We follow the cut towards contour C_1, following it around the pole and then go along the cut again, back to the crust. We continue around the crust up to the cut of the next pole and so on, until we arrive back to point A and close the contour. Since we did not remove any cheese by making the cuts, the path from the crust to the corresponding hole and back again cancel out in this integration path. As we have proved by Eq. 1.9.6 :

$$\int_a^b F(s)\,ds + \int_b^a F(s)\,ds = 0$$

Therefore, only the contour C around the crust and the small contours $C_1 \cdots C_5$ around the rims of the holes containing the poles are what we must consider in the integration around the contour in Fig. 1.12.2. From all these, the contour C was encircled **counter-clockwise**, while the contours $C_1 \cdots C_5$ were encircled **clockwise**.

We write down the complete integral :

$$\oint_{C^+} F(s)\,ds + \oint_{C_1^-} F(s)\,ds + \cdots + \oint_{C_5^-} F(s)\,ds = 0 \tag{1.12.1}$$

The value of the integral is zero, because along this circuitous integration contour we have had the regular domain always on the left side. By changing the sense of the encirclements of the contours $C_1 \cdots C_5$, we may write Eq. 1.12.1 also in the form :

$$\oint_{C^+} F(s)\,ds = \oint_{C_1^+} F(s)\,ds + \cdots + \oint_{C_5^+} F(s)\,ds \tag{1.12.2}$$

When we changed the sense of encirclements, we changed the sign of the integrals ; this allows us to put them on the right side with a positive sign. Now all the integrals have positive (counter-clockwise) encirclements. Therefore the integral encircling all the poles is equal to the sum of the integrals encircling each particular pole. By observing this equation we realize that the right side is the sum of residues for all the five poles, multiplied by $2\pi j$. Eq. 1.12.2 may also be written (for the general n-pole case) as :

$$\boxed{\oint_{C^+} F(s)\,ds = 2\pi j\,[\,res_1 + \cdots + res_n] = 2\pi j \sum_{i=1}^{n} res_i} \tag{1.12.3}$$

Eq. 1.12.2 and 1.12.3, which are essentially important for the inverse Laplace transform, are called the *Cauchy-Goursat theorem*.

1.13 Equality of the Integrals $\oint F(s)\,\mathrm{e}^{st}\,ds$ and $\int\limits_{c-j\infty}^{c+j\infty} F(s)\,\mathrm{e}^{st}\,ds$

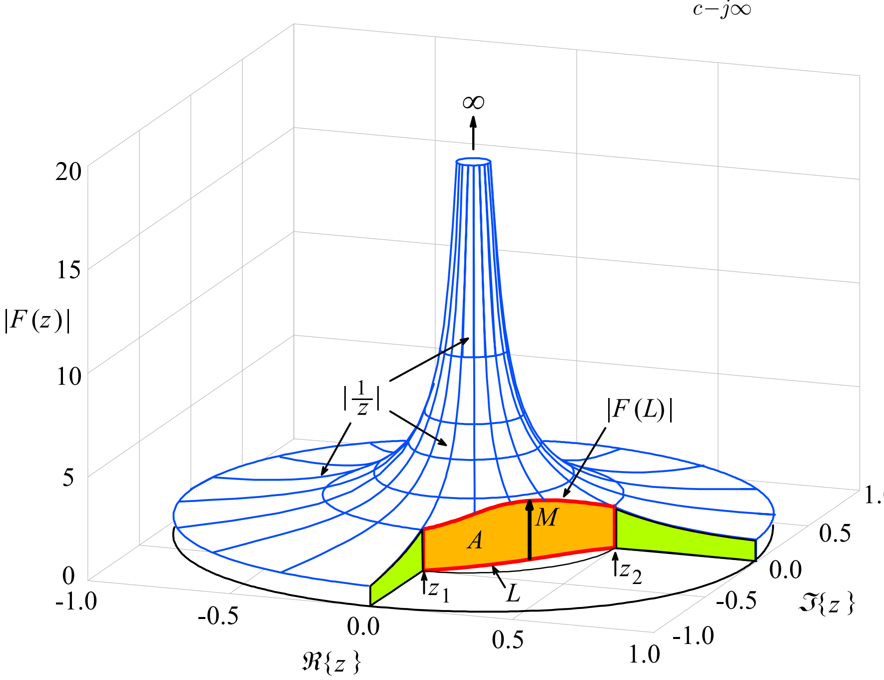
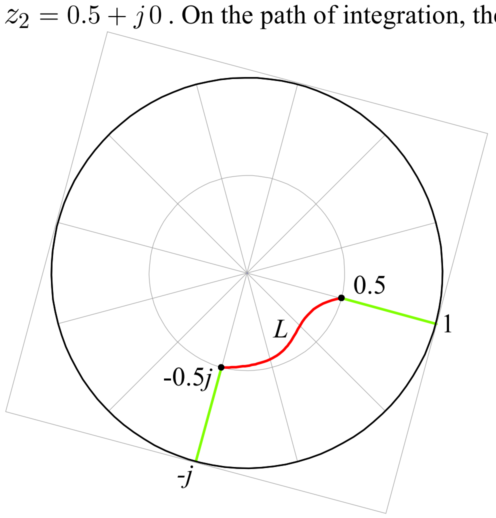


Fig. 1.13.1 : The complex function magnitude, $|F(z)| = |1/z|$. The resulting surface has been cut between $-j$ and $+1$ to expose an arbitrarily chosen integration path L , starting at $z_1 = 0 - j\,0.5$ and ending at $z_2 = 0.5 + j\,0$. On the path of integration, the function $|F(z)|$ has a maximum value M .



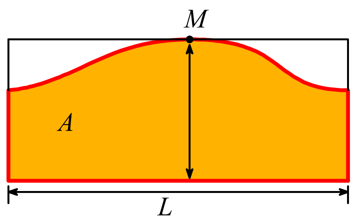


Fig. 1.13.2 : The complex domain of Fig. 1.12.1, showing the arbitrarily chosen integration path L , which starts at $z_1 = 0 - j\,0.5$ and ends at $z_2 = 0.5 + j\,0$.

Fig. 1.13.3 : The section between z_1 and z_2 of Fig. 1.12.1 is laid flat in order to show that the resulting integration area is smaller than the area of the rectangle $M \times L$.

The reader is invited to examine Fig. 1.13.1, where the function $|F(z)| = |1/z|$, having one simple pole at the complex-plane origin, was plotted. The resulting surface was cut between $-j$ and 1 to expose an arbitrarily chosen integration path L between $z_1 = x_1 + j\,y_1 = 0 - j\,0.5$ and $z_2 = x_2 + j\,y_2 = 0.5 + j\,0$ (see the integration path in the z-domain plot in Fig. 1.13.2). Let's take a closer look at the area A between z_1 , z_2 , $|F(z_1)|$ and $|F(z_2)|$, shown in Fig 1.13.3. The area A corresponds to the integral of $F(z)$ from z_1 to z_2 and it can be shown that it is always smaller than, or at best equal to the rectangle $M L$:

$$\underbrace{\left|\int\limits_{z_1}^{z_2} F(z)\,dz\right| = \left|\int\limits_{z_1}^{z_2} \frac{dz}{z}\right| \le \int\limits_{z_1}^{z_2} \frac{|dz|}{|z|}}_{\text{along the path } L} \le \int\limits_{z_1}^{z_2} M\,|dz| = ML \tag{1.13.1}$$

Here M is the greatest value of $|F(z)|$ for this particular integration path L, as shown in Fig. 1.13.3, in which the resulting 3-D area between z_1, z_2, $|F(z_1)|$ and $|F(z_2)|$ was stretched flat. So :

$$\left|\int\limits_{z_1}^{z_2} F(z)\,dz\right| \le \int\limits_{z_1}^{z_2} |F(z)\,dz| \le ML \tag{1.13.2}$$

Eq. 1.13.2 is an essential tool in the proof of inverse $\mathcal{L}$-transform via the integral around the closed contour.

Let us now move to network analysis, where we have to deal with rational functions of complex variable $s = \sigma + j\,\omega$. These functions have a general form :

$$F(s) = \frac{s^m + b_{m-1}\,s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}\,s^{n-1} + \cdots + a_1 s + a_0} \tag{1.13.3}$$

where $m < n$ and both are positive and real. Since we can also express $s = R\,\mathrm{e}^{j\theta}$ (as can be derived from Fig. 1.13.4) , we may write Eq. 1.13.3 also in the form :

$$F(s) = \frac{R^m\,\mathrm{e}^{j\,m\,\theta} + b_{m-1}\,R^{m-1}\,\mathrm{e}^{j(m-1)\theta} + \cdots + b_1 R\,\mathrm{e}^{j\,\theta} + b_0}{R^n\,\mathrm{e}^{j\,n\,\theta} + a_{n-1}\,R^{n-1}\,\mathrm{e}^{j(n-1)\theta} + \cdots + a_1 R\,\mathrm{e}^{j\,\theta} + a_0} \tag{1.13.4}$$

According to Eq. 1.13.4, we have :

$$\big|F(s)\big| = \left|\frac{R^m\,\mathrm{e}^{j\,m\,\theta} + \cdots + b_0}{R^n\,\mathrm{e}^{j\,n\,\theta} + \cdots + a_0}\right| \le \frac{K}{R^{n-m}} = M \tag{1.13.5}$$

where K is a real constant and M is the maximum value of $\big|F(s)\big|$ within the integration interval, according to Fig. 1.13.1 and 1.12.3 (in [Ref. 1.10, p. 212] the interested reader can find the complete derivation of the constant K).

$$s_1 = \sigma_1 + j\,\omega_1 = R\,\mathrm{e}^{j\theta}$$

$$\sigma_1 = R\cos\theta \qquad \omega_1 = R\sin\theta$$

$$R = \sqrt{(\sigma_1 + j\,\omega_1)(\sigma_1 - j\,\omega_1)}$$

$$= \sqrt{\sigma_1^2 + \omega_1^2}$$

$$\theta = \arctan\frac{\omega_1}{\sigma_1}$$

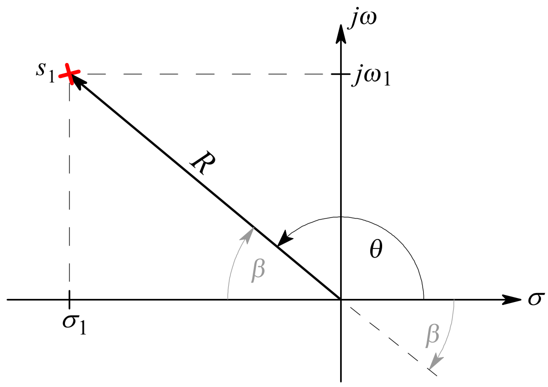


Fig. 1.13.4 : Cartesian and polar representations of a complex number (note that $\tan\theta$ is equal for ccw-defined θ from positive real axis and for cw-defined $\beta = \theta - \pi$) .