

## Appendix 2.2

### Normalization of complex-frequency response functions

or

Why some expressions have strange signs

*Don't be afraid of mathematics !*  
*It's probably the only rational product of the human mind !*  
 (E.Margan)

A generalized expression of a non-normalized complex-frequency response function of an all-pole system (no zeros) of order  $N$  can be written in form of a product of poles of the characteristic polynomial :

$$F(s) = \frac{1}{\prod_{i=1}^N (s - s_i)} \quad (\text{A2.2.1})$$

At d.c., when  $s = 0$ , the system has a gain factor equal to the product of negated poles :

$$F(0) = \frac{1}{\prod_{i=1}^N (0 - s_i)} = \frac{1}{\prod_{i=1}^N (-s_i)} \quad (\text{A2.2.2})$$

We would like to be able to compare different systems on a fair basis. This is why we introduce parametric normalization in our equations. We have already seen how the frequency is normalized to  $\omega/\omega_H$ . This enabled us to compare the cut-off frequencies of systems which have some components, or even just one, equal, thus helping us to decide which circuit configuration is either better, simpler or more economical to build.

Obviously, the system gain is one such parameter which would influence the comparison in frequency if not taken into account. This situation is best resolved if we also normalize the d.c. gain of every system to some predefined value, preferably unity. Mathematically, we will have a normalized expression by simply dividing Eq. A.2.2.1 by Eq. A.2.2.2 :

$$F_n(s) = \frac{F(s)}{F(0)} = \frac{\frac{1}{\prod_{i=1}^N (s - s_i)}}{\frac{1}{\prod_{i=1}^N (-s_i)}} = \frac{\prod_{i=1}^N (-s_i)}{\prod_{i=1}^N (s - s_i)} \quad (\text{A2.2.3})$$

The numerator of the last term in Eq. A.2.2.3 can be written so that the signs are collected together in a separate product, defining the sign of the total :

$$F_n(s) = \frac{(-1)^N \prod_{i=1}^N s_i}{\prod_{i=1}^N (s - s_i)} \quad (\text{A2.2.3})$$

This means that **all odd-order functions must be multiplied by  $-1$**  in order to have a correctly normalized expression. But please, note that the sign-defining expression,  $(-1)^N$ , is **not** the consequence of the fact that all our poles lie in the left-half of the complex plane, as is sometimes wrongly explained in literature !

In Eq. A.2.2.3 the poles still retain their actual value, be it positive or negative. The term  $(-1)^N$  is just the consequence of the mathematical operation (subtraction) required by the function :  $s$  must acquire the exact value of  $s_i$ , **sign included**, if the function is to have a pole at  $s_i$  :

$$s - s_i = 0 \Big|_{s = s_i} \Rightarrow F(s_i) = \pm \infty \quad (\text{A2.2.4})$$

In some literature, the sign is usually neglected because we are all too often interested in the frequency-response magnitude, which is the absolute value of  $F(s)$ , or  $|F(s)|$ . However, as amplifier designers, we are interested mainly in the system time-domain performance and, if we calculate it by the inverse Laplace transform, we must have the correct sign of the transfer function and, consequently, the signs of the residues at each pole.

In addition, it is important to note that system with zeros must have the product of zeros normalized in the same way (even if some of the systems with zeros do not have a d.c. response, i.e. high-pass and band-pass systems). If our system has poles  $p_i$  and zeros  $z_k$ , the normalized transfer-function is :

$$F_n(s) = \frac{(-1)^N \prod_{i=1}^N p_i}{\prod_{i=1}^N (s - p_i)} \cdot \frac{\prod_{k=1}^M (s - z_k)}{(-1)^M \prod_{k=1}^M z_k} \quad (\text{A2.2.5})$$

From Eq. A2.2.5, the resulting sign factor is obviously :

$$(-1)^N \cdot (-1)^{-M} = (-1)^{N-M} \quad (\text{A2.2.6})$$

which is, incidentally, equal to  $(-1)^{N+M}$ , but there is nothing mystical about that, really.