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Modeling of Stock Crashes in Analogy with Phase Transitions

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Abstract

There is a growing demand for physicists in financial circles. Their knowledge of complex systems can significantly contribute to understanding the market behaviour. Recent ideas of B. B. Mandelbrot connect fractals to price movements. The proof of such speculation should be self-similarity of graphs showing trading activity. As we do not consider that to be the case at all times, we rather restrict our observations to points close to crashes, which are correctly predicted by fractal structures. Self-similarity is the signature of systems undergoing continuous phase-transitions. The comparison between two-dimensional Ising spin model and market ordering is given. Based on similarities, we consider a crash really being a critical point in ordering. This allows us to treat the graphs as self-similar with correlation length going to infinity and thereby fractal, but only close to critical points. We can derive our basic laws solely from the requirements of self-similar solutions in agreement with renormalization group equations. This gives us power-law behaviour, and by letting the exponent being a complex number, we derive the log-periodic signature of data. Instead of the formal proof, application on real data is given. We try to fit the function with unknown parameters on stock indices in vicinity of the crash. As the fits perform reasonably well, we consider the theory to be correct. After shock signatures resemble those exhibited before the crash. With them future process of stocks was forecast and correct prediction in the case of Nikkei stock index was given. This makes the theory plausible.

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1 Introduction

Stock markets are places where fortunes are won and lost very easily, later being more usual. Fluctuations in prices follow almost unrecognizable paths and many efforts were made to predict at least to some extent the behaviour of future prices.

What is more, stock crashes often occur. By stock crashes we note sudden drops of stock prices. They can be limited only to one trading company on the stock exchange or can affect the market as a whole. Many people still recall the great American recession which followed the big stock crash in the early Thirties. Almost all of the stock indexes showed significant drops of as much as 30 % to the trading price. Why this big crashes occur, is still a mystery.

Now let us look at the theories most of the stock traders follow, also called portfolio theories. There are two basic ideas, on which stock markets are modeled. One is the assumption, that future prices are totally unpredictable. This is of course just an observation of the stock price variations, which are on the small scale rather un-correlated. Secondly one assumes, that the variation of the prices is somehow bound. Or more realistic, that the deviations of the price from some mean, smooth curve follow a normal, Gaussian or other bell-shaped curve. On one hand this curve stretches to infinity and thereby allows big variations, but on the other hand it gives them almost infinitesimal probability of occurrence. That means that stock crashes as big as the Thirties crash should occur only every 10^{12} years[1], but they are more frequent, they occur every decade at least. Remember the Wall Street crash of 1987 or continuous drops of Asian markets throughout this decade. Therefore this theories, usually refered to as portfolio theories, do not always confirm to real facts.

Whereas for economists such crashes are reasons for headache, they provide interesting deviation from serious science for many mathematicians as well as physicists. Such events are well detectable and they possess many resemblances with problems and processes described in other fields of science. Thus some solutions are readily available and they can, with some modifications, present serious alternative to established theories.

Our seminar will follow ideas recently developed by D. Sornette and A. Johansen (ref. [3]-[8]). They compare markets with systems close to critical point. The main feature of such systems is the self-similarity of solutions, which follows directly as correlation lengths tend to infinity. So we can appreciate their work to full extent, we should first reflect on fractals and their generation. Furthermore we will qualitatively inspect the system of spins as to get the feel of the system being driven to critical point. Only at that point we will use the simplified ideas of the renormalization group to state the model of log-periodic price oscillations. Considering its agreement with real data we will conclude the paper.

[1] Probability of a 5σ event according to normal distribution

2 Fractals and self-similarity

Fractals are functions, which are not aware of their scale. Their behaviour is the same, when we look at the global form of the oceans, setting typical distances to well over thousands of kilometers, or observe a single stone dipped into the water, where typical length stays well bellow one meter. With fractals we can thus talk of scale-invariance and in that view we define self-similarity as faithful representation of the same pattern on different scales. Nature itself offers many fractal structures. Most popular are already mentioned form of the coast when viewed at different length scales and growing of the plants.

There is no obvious reason that fractal structure governs market behaviour. But similarity of charts with different time scales gives us ground to introduce results of fractal geometry to market indices. And with special generators one can simulate the very thing portfolio theory does not give account for. The big crashes. They can be quite frequent when multifractals, special brand of fractals, try to model the market behaviour [2]. This is an interesting example of physical ideas being put to practice in economical field, not to mention its success in comparison with basic portfolio theory.

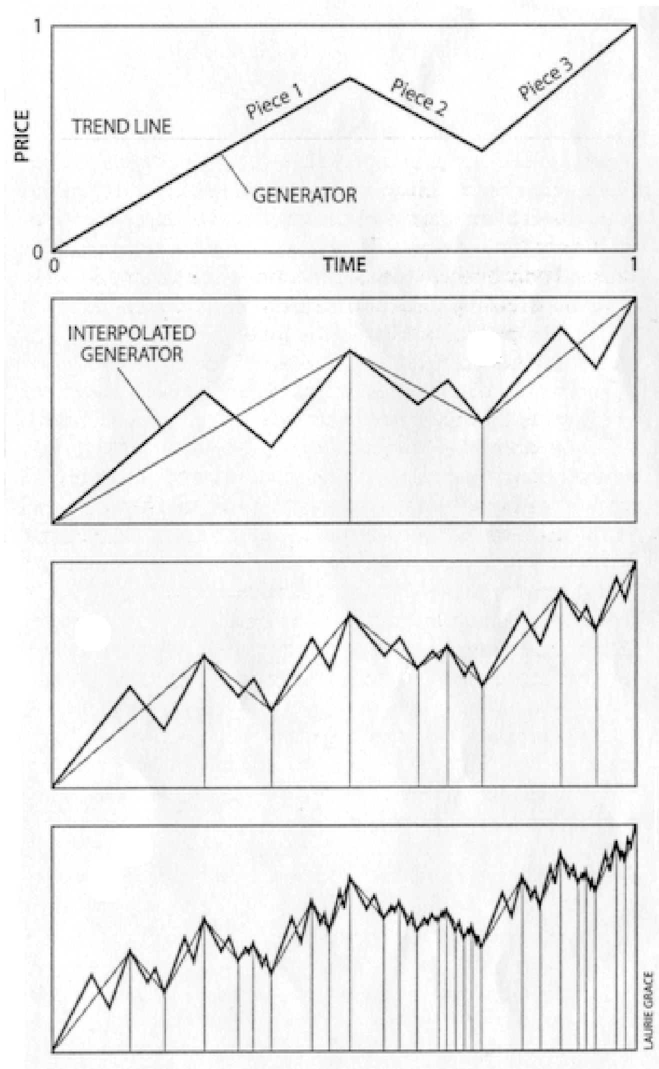

Figure 1: Generation of fractals, spanning from big scale and repeating itself on smaller and smaller scales. Ref. [2]

3 Phase transition and critical point

Observing figure 1 we can see that the idea of self-similarity handles the lively price fluctuations quite well. So we can develop our approach even further. Self-similarity is one of the basic characteristics of observable quantities in vicinity of the phase transition. We will take a look at the phase transitions and physical systems undergoing it and then we will try to find similarities in the progress of stock exchange. The next phase is to apply solutions of the physical system to the market, which will be done in a separate section.

For convenience reasons only we will examine the two-dimensional spin system. Most of the features discussed can be found in other phase changing systems as well. We can picture the two valued spins (up-down) sitting in a square lattice. We take the Ising model with Hamiltonian of the system defined as:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j - H \sum_i S_i \tag{1}$$

We defined sum indices $\langle ij \rangle$ as those of the nearest neighbours. Occupation probability is defined as the exponential:

$$\exp\left[\frac{\mathcal{H}}{k_B T}\right] \tag{2}$$

so we can change Hamiltonian to:

$$\mathcal{H} \rightarrow \frac{\mathcal{H}}{k_B T} \tag{3}$$

Let us first examine the case of no field present. Hamiltonian will be then of the form:

$$\mathcal{H} = -\frac{J}{k_B T} \sum_{\langle i,j \rangle} S_i S_j \tag{4}$$

There exists a critical temperature T_C, below which the constant J becomes powerful enough to prevail over thermal fluctuations. At that temperature and below it the system is strongly correlated, all spins point in the same direction (if the case of $J > 0$ is examined, as we pretend it is). Above the critical temperature spins point at very different directions.

When cooling the system from well above the critical temperature, at first we see the spins following the thermal fluctuations and thus no order can be spotted. With approaching T_C the thermal energy $k_B T$ is lowered and order starts to emerge. At first only small portion of the lattice spins point in the same direction, but with getting closer and closer to the critical temperature, the areas of equal sign of spin spread.

The common result of such cooling below T_C with no field present is the separation of the whole area into “domains” of equally oriented spin. If we are to define correlation length ξ as typical size of such a “domain”, we could observe its temperature dependence $\xi(T)$. Exact calculations of parameters are beyond the scope of this seminar[2], but we can expect this quantity to go to infinity at the T_C. This statement goes well with the previous description of system going to T_C. With the correlation length close to infinity system becomes self-similar in the sense mentioned with fractal structures. And this is the center result of our observations of system going through phase transition.

Now we have to find grounds to insert this statement in the market behaviour. First we have to define a system to go through a phase transition. We can quickly imagine a crowd of stock brokers on the floor, shouting and phoning and hesitating and doing basically two things - buying or selling. So we actually have a similarity with the spin system previously mentioned.

[2] We can calculate the temperature dependence of the order parameter in the mean field theory, ref. [1], for 2-D exact solution was obtained by Önsager

Usually crashes end the periods of reasonably good market behaviour with prices achiving their all-time highs just before the crash. The stock actually becomes over-inflated with all the stock-holders expecting the price to grow even further. A growing stock is a popular one, so more and more people buy it, by means of that the price increases and new investors follow. This pre-crash process has a well recognizable pattern as we will see. The consequence of such trading is the over-pricing of the stock, forming a "bubble" of hope on which the price rests, which at some point jumps to its original value, thereby producing a crash. The "critical" point occurs as the people stop beliving that the price will increase further. If that belief is displayed by small amount ("domain") of traders, the bubble will not blow as still enough believers invest their money. But as the domains grow, crash becomes more and more probable. We can again define the correlation length as the diameter of the group of traders deciding in the same way. And the actual phase transition will occur as the whole market will be absorbed in a single domain. That of course does not happen, as smaller domains can blow the bubble and hence return the system to the state far from critical point.

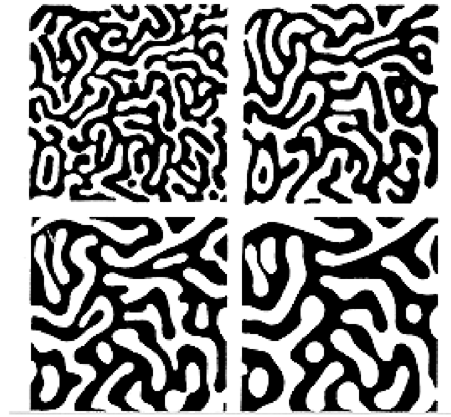

Figure 2: Cooling of the system. The domains of equally oriented spins grow, therewith the correlation length. Picture taken from [1].

As the crash appears in time, one thinks of it as the equivalent of temperature in spin-models. So we can set dimensionless measure of time as (compare (8)):

$$x = \frac{t_C - t}{t_C} \tag{5}$$

t_C being the critical moment. The other observable is of course the index, price, etc. It is actually the order parameter of our system, our dependable observable. We introduce the measure of it as:

$$F(x) = I_C - I(x), \tag{6}$$

with I_C being the highest price (or index), thus setting $F(0)$ to zero in the ordered phase.

Hence we found a good agreement between a physical problem and market activity. There are of course some modifications, i.e. with markets, correlation length varies with time and not with temperature, but overall pictures coincide quite well. This gives us ground to approach the problem with the self-similarity procedures, developed for phase transitions.

4 Renormalizability and self-similarity

As mentioned in the section above, the market close to crashes exhibits behaviour typical for the phase transitions with correlation length growing beyond any scale. Hence the price graph is self-similar in vicinity of the transitions. The rest we will leave to the theory of renormalization.

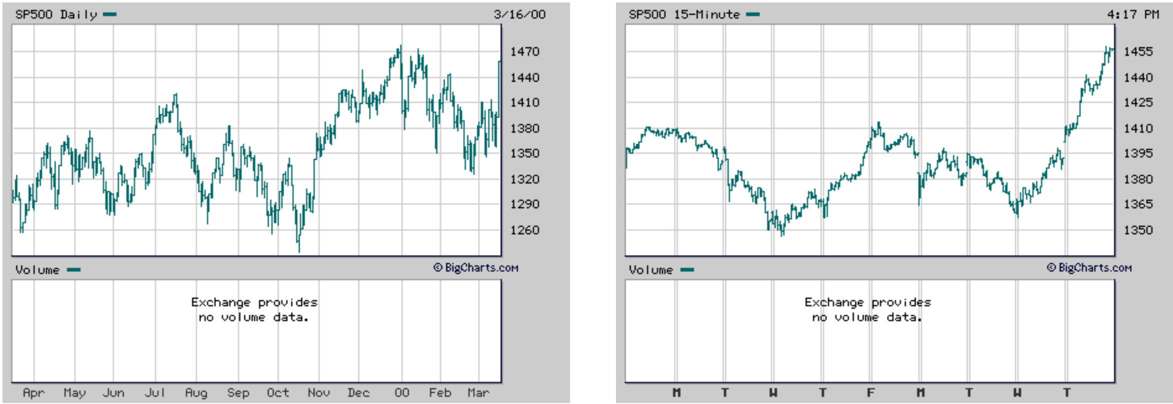

Figure 3: Self-similarity? Comparison in graphs for the SP&P500 index of Wall Street stock exchange, on 1 year time-scale (left) with daily values and 10 days time-scale (right) with 15-minutes updates. Fractal structure claims the two to be identical. We will let you be the judge of that.

Renormalization is technique of calculating the critical exponents of the system as approaching the critical point. Let us go back to the physical system described above, namely the Ising model. We can imagine a coarse-graining procedure which groups spins within a square with side ℓ into a new spin siting in the center of the square. Reasonable restraints for ℓ are the dimension of the original lattice a on the lower end and correlation length ξ on the upper end:

$$a \ll \ell \ll \xi \tag{7}$$

We can see, that renormalization is really a transformation $\mathcal{R}_\ell$, which changes all the parameters of the equation hence the state of the system itself. In the spin system, ℓ is the length of the lattice after the coarse-graining.

The original N spins were replaced by $N' = N(a/\ell)^2$. Hence we reduced the number of spins. Moreover, we changed the Hamiltonian as well. But we can still write it in the same form as before, only with changed coefficients ($J \to J'$) due to scaling. As the correlation length measured in ℓ is smaller by an amount of a/ℓ of the correlation length measured in the lattice units, we actually drove the Hamiltonian away from the critical point. So the scale renormalization affects the effective temperature of the system (T'), meassured in new units of length.

If we are to define:

$$\theta = \frac{T - T_c}{T_c} \tag{8}$$

we can always express ([1], p.250):

$$\mathcal{R}_\ell \theta = \theta' = \theta \ell^{\dagger\tau}, \tag{9}$$

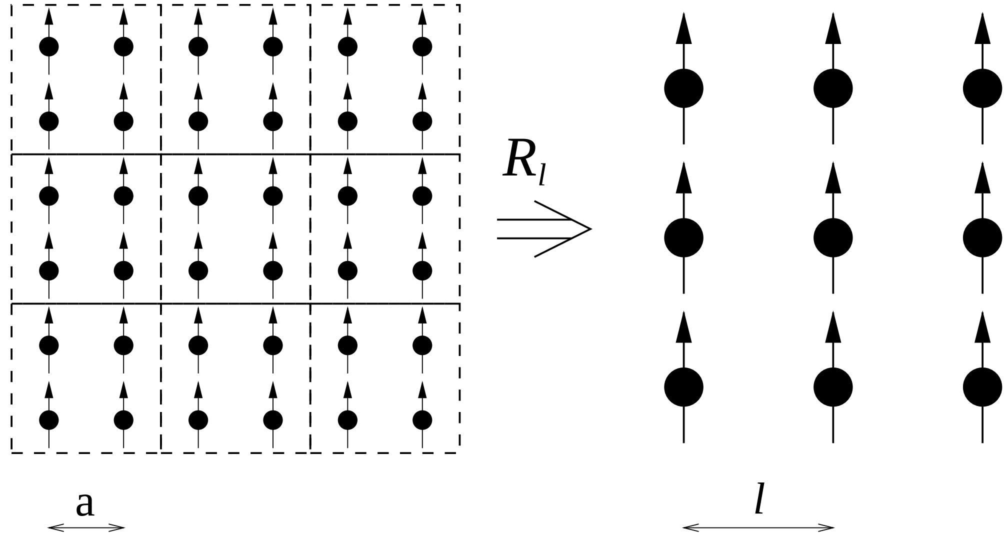

Figure 4: Coarse-grainnig of equally oriented spins within a domain. We replace spins in lattice with dimension a with spins in lattice with dimension ℓ. As the correlation length, measured in units of ℓ is smaller than the one, measured in units of a, coarse-grained system is further from critical point as the original one.

where y_T is the critical exponent for the temperature. As y_T is essentially a constant, we can define:

$$\theta_0 = \frac{1}{\ell^{y_T}}, \tag{10}$$

thereby producing a new parametrization of the renormalization transformation:

$$\mathcal{R}_\ell \to \mathcal{R}_{\theta}, \tag{11}$$

We are suddenly not coarse-graining the spins but rather scaling the temperature. We merely want to state that the effect of the both is actually the same.

Now we can select an arbitrary renormalization transformation to scale our market variables. Again we use our Ising model considerations, and according to (11) we can set our temperature-equivalent to be the renormalization transformation parameter. We select a convenient scaling of time axis with dimensionless constant τ:

$$\mathcal{R}_\tau x = \frac{x}{\tau} \tag{12}$$

There exists the transformation of the price $F(x)$ as well:

$$F'\left(\frac{x}{\tau}\right) = \mathcal{R}_\tau F(x). \tag{13}$$

By the function F'(x) we noted a new function of the scaled time. If we imagine scaling parameter τ to be smaller than 1, our transformation moves the solution $F'(t)$ closer to critical point.

Now we introduce the scaling of the function $F(x)$, by defining the transformation $\mathcal{R}_\tau$ for it. To prevent the function $F'(x)$ going sky-high, we introduce a scaling constant $\mu(\tau)$ and simply connect the original with its transform at an arbitrary point y:

$$F'\left(\frac{x}{\tau}\right) = \frac{1}{\mu(\tau)}F(x) \tag{14}$$

$$F'\left(\frac{y}{\tau}\right) = F(y) \tag{15}$$

There is one condition that our renormalization transformation should satisfy. In words, two consecutive transformations to scales τ_1, τ_2 should have the same effect as the combined transformation $\tau_1 \times \tau_2$. This is the consequence of $\mathcal{R}$ being a semi-group. Now let us look at he chain of two consecutive transformations:

$$F''\left(\frac{x}{\tau_1\tau_2}\right) = \frac{1}{\mu(\tau_2)}F'\left(\frac{x}{\tau_1}\right) \tag{16}$$

$$= \frac{1}{\mu(\tau_2)}\frac{1}{\mu(\tau_1)}F(x). \tag{17}$$

And on the other hand we have transformation $\mathcal{R}_{\tau_1\tau_2}$:

$$F'\left(\frac{x}{\tau_1\tau_2}\right) = \frac{1}{\mu(\tau_1\tau_2)}F(x). \tag{18}$$

By setting $F'(x/\tau_1\tau_2) = F''(x/\tau_1\tau_2)$ as the statement of semi-group, we get:

$$\mu(\tau_1)\mu(\tau_2) = \mu(\tau_1\tau_2) \tag{19}$$

With successive derivations with respect to τ_1 and τ_2 following relation is obtained:

$$\frac{d\ \ln\mu}{d\ \ln\tau} = \alpha \tag{20}$$

with α being an arbitrary constant. We can also imagine it being a complex number, hence $F(x)$ being a function in the complex plain. Our solution will be the real part of such function. We introduced a new parameter to the model.

We can stop now for a while to observe our achievement so far. In our graph of trading activities we first expanded the time-scale. This can be viewed in the sense of fractals as moving from one level to another. If the structure of the price fluctuations is indeed a fractal one, the new graph should at least resemble the old one. To get an exact copy we might have to shrink or expand the vertical axes. This is done with the factor μ. If the fractal structure is to be observed, our transformations should map the graph onto itself.

This is where we use to knowledge of our physical systems close to phase transitions. They all are self-similar, bearing fractal structure in vicinity of the critical point. So there exists a critical point τ^* for the transformation $\mathcal{R}_\tau$ where the transformation of the function remains the function itself:

$$F\left(\frac{x}{\tau^*}\right) = \mathcal{R}_{\tau^*}F(x) \tag{21}$$

in the vicinity of the critical point.

Thus the solution will be of the form (dropping the star from τ and using renormal;ization transformation (14)):

$$F\left(\frac{x}{\tau}\right) = \frac{1}{\tau^{\alpha}} F(x) \tag{22}$$

So the solution $F(x)$ is extracted by inverting (22):

$$F(x) = \tau^{\alpha} F\left(\frac{x}{\tau}\right) \tag{23}$$

Final solution should not be dependent of critical scale τ which was artificially introduced, therefore the following relation should be satisfied:

$$\frac{\partial F}{\partial \tau} = 0 \tag{24}$$

Differentiating (23) one obtains:

$$\tau^{\alpha-1}\left[\alpha F\left(\frac{x}{\tau}\right) - \frac{x}{\tau}\frac{dF}{d\left(\frac{x}{\tau}\right)}\right]. \tag{25}$$

Setting it to zero as required by (24), replacing x/τ with y we get our final result:

$$\frac{d\ \ln F(y)}{d\ \ln y} = \alpha \tag{26}$$

Equation is simply solved, we get a power-law solution of the form:

$$I(t) = A_1 + B_1\left(\frac{t_C - t}{t_C}\right)^{\alpha} \tag{27}$$

which diverges as t approaches t_C if α is to be less than one. This usually is the case as we observe divergent behaviour in vicinity of the critical point. We are not able to model the crash itself as it happens before the actual phase transition, but we can definitely limit the time of its occurrence. The question remains though, how far is it reasonable to try to fit the data with our approximation, as it is valid only close to the critical point.

We can even improve our solution without any theoretical framework. So far no limitations were stated to the nature of the critical exponent α. We can let it spread to imaginary axes[3], hence producing a new term in our solution. If α is to be written in the form $\alpha + i\omega$, we get the imaginary term as:

$$\exp(i\omega \ln\left[\frac{t_C - t}{t_C}\right]) \tag{28}$$

The preceding constant can be imaginary as well, hence producing an arbitrary additive phase φ in the argument of the exponent. Taking the real part of it, the new term is just:

$$C\left(\frac{t_C - t}{t_C}\right)\cos\left[\omega \ln\left(\frac{t_C - t}{t_C}\right) + \varphi\right] \tag{29}$$

[3]Some grounds are stated in section 4.1

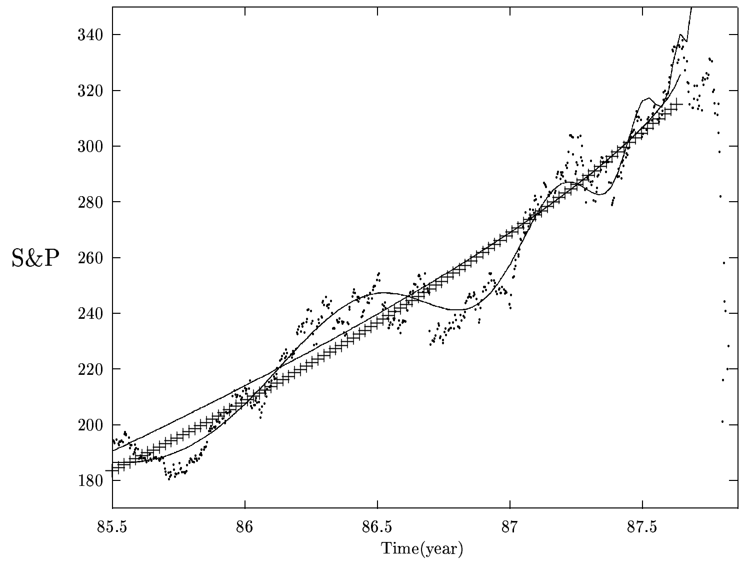

Figure 5: First success of log-periodicity. Fitting a log-periodic function to data prior to 1987 Wall Street crash. Time distance is approx. 2 years, the best fit gives $A_1 = 327$,$B_1 = -79$, $t_C = 87.65$, $\alpha = 0.7$, $\chi^2 = 107$ for the pure power-law (eq. (27)) and $A_2 = 412$, $B = -165$, $t_C = 87.74$, $C = 12$, $\omega = 7.4$, $T = 2.0$, $\alpha = 0.33$, $\chi^2 = 36$ for the log-periodic function (eq. (30)). The crash occured at 87.78. The + curve is the representation of a constant 30 % annual income rate (exponential function, $\chi^2 \approx 113$). Ref. [3]

We can always set the arbitrary constant φ as a time scale by defining $\varphi = \omega \ln T$, which gives us the combined result as:

$$I(t) = A_2 + B_2 \left(\frac{t_C - t}{T}\right)^{\alpha} \left(1 + C \cos\left[\omega \ln\left(\frac{t_C - t}{T}\right)\right]\right) \tag{30}$$

We have four non-linear parameters and three linear. To fit the data we need a parameter of the type χ^2, that is the measure of the fit. As the variance of the points is not a well-defined parameter with the prices of stocks it is a natural alternative to take simple minimum squares function, given with:

$$\chi^2 = \frac{1}{N-n} \sum_{i=1}^{N} (I_i - I(t_i))^2 \tag{31}$$

So we can compare different models among themselves. The later model, gives its variance χ^2 as much as three times smaller as the pure power-law (27) or naive exponent function. Hence the implication of imaginary exponents as the power-law solutions can be justified by the practice itself.

One of the most striking feature of our approximations is the correct account for the appearance of the shortening waves in the prices as prices rise. This so called log-periodicity of the curve was observed with other phenomena as well. The measurement of concentration of sulfur-oxide (SO_4^-) in mineral water close to the epicenter of 1995 Kobe earthquake showed the same log-periodic signature. That was the confirmation of application of similar theory to earthquakes. Thus we are dealing with a natural phenomena.

We can quantitatively observe the length of the waves and see it diminish as approaching the critical point. Separation between maxima is namely connected with the logarithm. By setting periodicity condition:

$$\omega \ln \frac{t_n}{T} = 2\pi n \tag{32}$$

we can calculate the time of the $n-$th maximum:

$$t_n = T e^{2\pi\omega n} \tag{33}$$

The difference between consecutive maxima

$$\Delta t_n = t_n - t_{n-1} \tag{34}$$

follows a geometric series:

$$\frac{\Delta t_n}{\Delta t_{n-1}} = \lambda = e^{2\pi/\omega} \tag{35}$$

This is the most recognizable feature of log-periodicity and can be well observed.

4.1 Discrete versus continuous scale-invariance

The complex exponents are signatures of special kind of scale-invariance, called discrete scale-invariance. Scale-invariance is a term used to denote self-similarity in view of scale-transformations. Discrete scale-invariance is observed, when renormalization transformation maps onto itself only at some discrete values of its parameter [9]. This is the approach we used with stock indices with introduction of τ^*. On the other hand, Ising model (in 2-D, as solved by Onsager) does not show log-periodic behaviour when approaching critical point.

We can say, that log-periodic oscilations, as they are observed with price fluctuations, denote the existance of a special time scale in approaching crash. Such scales can spontaneously appear in other complex systems as well, one of them being the growth of cracks leading to earthquakes. Explanation of these events is rather obscure [9], so it will not be given in the seminar. We will only extract, that such scales *can* arise and that they can be explained. The Ising model, on the other hand, is to simple to bear such structures. But slight complication, such as intoduction of special constant for next nearest neighbours, already breaks the unitary symetry and log-periodic oscilations are to be observed ([9], p.14).

5 Discussion

There is much discussion going on whether stock markets can be modeled by physical systems. The mere thought of comparing an investor possessing brains with a system of brainless spins, plausible as it might be, is far from credible in financial circles. But there is some point in investigating these considerations even further.

It is quite natural to assume physical system tending towards minimum energy[4]. We can easily picture particles organizing themselves as to spend as much energy as possible even though they do not posses a kind of a thinking device to guide them. We merely speculate them to feel the forces pushing them blindly towards minimum of energy.

Let us continue with the comparison. We could without any problem design an energy functional describing stock exchange. We can set it to be the profit. Furthermore, we can almost extract humanity out of the system by imaging brokers to be interested in money only. By that we implicitly assume all their actions oriented towards one goal and one goal only. Namely increasing profit. Therefore they take only those actions which increase the profit and avoid doing otherwise. Almost like the spins feeling the existence of the field surrounding them.

We should leave some freedom as to allow small fluctuations in the market due to unwise decisions and deficit of data available (like spins not feeling the external forces). Then again when big fluctuations occur, crashes as we have called them, its' magnitude provides enough information so we can presume the whole market is aware of it. Hence every single trader is influenced by its raise and furthermore, he is bound to take actions to increase his profit. We can well see the field forming and influencing the whole system of particles in it.

6 Conclusion

In conclusions we will state some facts which support the theory, or rather, support its derivations. This paper gives namely only basic ideas, whether complicated structures are to be found already resting on the same mental approach. First we will give some proofs that the theory exhibits some confidence level. Finally we will present its application to post-crash price evolution and its success.

Our considerations so far showed periods of ordering phases in otherwise disordered market. Getting to this point we must mention that many unsuccessful fits were done in periods with no significant crashes, therefore proving that crashes are consequences of cooperative behaviour. By unsuccessful we mean that their variation disagreed with variation of the fit close to crash for several units. There is another point to these crashes. As more and more fits were done with equation (30) similar values for parameters ω and α were obtained close to the crashes [7]. And the parameters for fits at other times on real data as well as numerical generated data according to null-hypothesis scheme showed significant deviation from those values [6], giving 95% confidence of log-periodicity for one event.

Let us reflect on the possible use of this revelations in day-to-day trading. And we might just give up. The log-periodic patterns are hard to follow in the noisy markets of the present time. Coexistence of several minima due to this noisy oscillations prevents us to recognize true minima of the log-oscillations. Moreover

[4]With different thermodynamical systems different functions are minimized, e.g. free enthalpy, entrophy,...But one is, and we will call it simply the energy of the system

the recognized pattern does not give us credibility to predict any crash let alone foretell the form and moment of its appearance.

Our power-law solution is fortunately only the basic stone in development of the approach. By including corrections of higher order in $F(t)$ one can derive complicated forms which take account for the behaviour after the critical point as well. After-shock signatures are the most useful for the traders. With crash occurring we have a fixed point to start our fit and extending it to future in agreement with intervals in fit in pre-crash market. With this weapon the growth of Nikkei stock market index in the year 1999 up to 2000 was predicted [5] and confirmed [8].

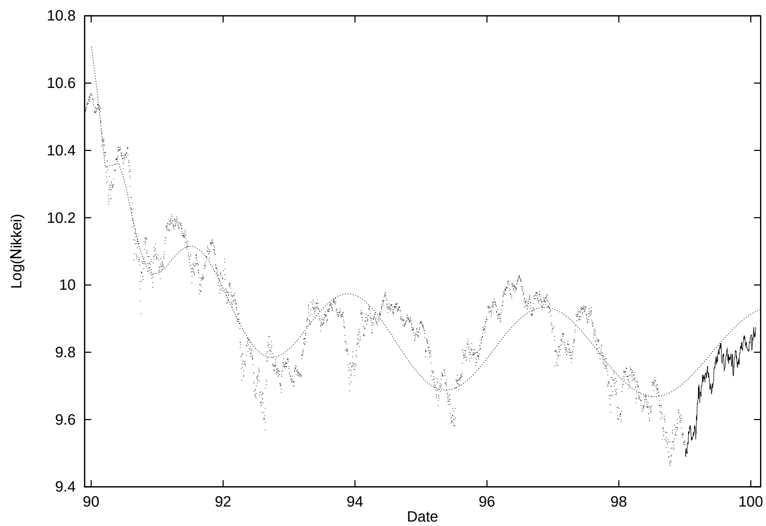

Figure 6: Correct prediction of the behaviour of Nikkei index. The fitted interval goes from the spike in 1990 and finishes in the year 1999. Nevertheless the assumption of rise in the price of the index is correct! Note that the y-axis, the price, is in logarithmic scale. This prediciton gives log-periodicity credibility and makes it one of the tools of the future.

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