Solution of the Kondo problem

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This review covers in great detail the Bethe-ansatz approach to the solution of various versions of the Kondo problem.

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I. INTRODUCTION

In the Kondo problem (Kondo, 1964) one studies the low-temperature behavior of a system consisting of magnetic impurities, dissolved at a very low concentration in a nonmagnetic metal. The impurities are represented by localized spins that couple to the conduction-band electrons via a spin exchange interaction.

At sufficiently low impurity density we may concentrate on a single impurity—localized, say, at $x=0$—and study how its properties are modified due to its coupling with the electrons.

As we are interested in the low-temperature properties of the model, we shall consider only excitations close to the surface of the Fermi sphere (Fig. 1).

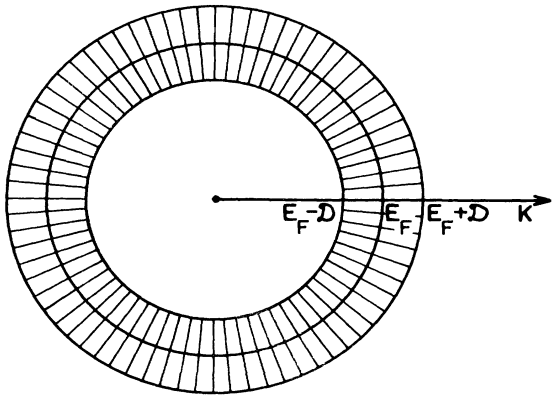

FIG. 1. The region in momentum space relevant to the formulation of the Kondo problem. As we are interested only in low-temperature phenomena, we may linearize the spectrum around E_F and impose a cutoff $\mathscr{D}$. We also keep only the s-wave component. We thus build the Hamiltonian out of operators c_{klm} with $|K-E_K| \leq \mathscr{D}$ and $l=m=0$.

Thus linearizing the spectrum around $E_F=0$ and imposing a momentum cut-off $\mathscr{D}$, $-\mathscr{D}<k<\mathscr{D}$, one is led to consider the following Hamiltonian (summation over a,b is implied):

$$\mathscr{H}_{\text{Kondo}}=\sum_k kc^*_{ka}c_{ka}+\frac{J}{2\pi}\sum_{kk'}\mathbf{S}c^*_{ka}\sigma_{ab}c_{k'b}\ , \qquad -\mathscr{D}<k\ ,\ k'<\mathscr{D}\ , \quad (1.1)$$

where only the s-wave states (around the impurity) are kept, since higher partial waves do not couple to the impurity. The operator c_{ka} destroys an electron with momentum k and spin component a ($a=\pm\frac{1}{2}$). $\mathbf{S}$ is the impurity spin operator localized at $\mathbf{x}=0$. It is important to note that the Hamiltonian, although one dimensional, represents the physics in three (or any) dimensions. This is due to the spherical symmetry of the contact interaction and the localized nature of the impurity.

The linearization procedure is valid only so long as $T\ll\mathscr{D}$. For temperatures of the order of the cutoff the linearization may break down, and details of the band structure (reflected in the cutoff procedure) become relevant.

We, however, shall consider only those quantities that characterize low-temperature properties of the model (Fig. 2) and are independent of the cutoff scheme. These quantities we shall call universal.

But so long as we consider only universal quantities we do not have to insist on a particular cutoff scheme. Various schemes may be employed to give the same universal quantities, although they would differ if pushed beyond the realm of relevance. If we want to analyze properties at $T\sim\mathscr{D}$, much more care must be taken in the construction of the model, and there may be only one physically acceptable cutoff scheme.

Thus instead of Hamiltonian (1.1) we shall consider

$$\mathscr{H}=-i\int\phi^*_a(x)\partial_x\phi_a(x)dx+J\mathbf{S}\phi^*_a(0)\sigma_{ab}\phi_b(0)\ , \quad (1.2)$$

where $\phi_a(x)$ is an electron field and $\phi^*_a(0)\sigma_{ab}\phi_b(0)$ represents the electronic spin density interacting with the impurity at $x=0$. The two Hamiltonians are identical in the absence of cutoffs. We shall construct the Hamiltonian (1.2) by using a cutoff scheme K different from the $\mathscr{D}$ scheme. The universal results, however, are the same (see below).

The above observations mean that we may apply the considerations and methods of conventional quantum field theory to the problem. Thus, as the coupling constant J is dimensionless, the Hamiltonian is renormalizable. This means that due to infinities occurring in the calculations, the coupling constant must be properly defined within a particular scheme. There is no meaning to the statement $J=0.001$ unless we also specify how J was defined, and J may have different values in different schemes, yet lead to the same results in the universal regime.

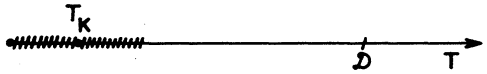

FIG. 2. The temperature axis. We are interested only in the region $T\ll\mathscr{D}$ (shaded), which is referred to as the scaling or universal regime. The dynamic scale T_k divides this region into a high-temperature region $T_k\ll T$ (still $T\ll\mathscr{D}$) and a low-temperature region $T\ll T_k$.

At this point one may think that there remains no scale in the problem, as the coupling constant is dimensionless and the cutoff is considered infinite if we restrict ourselves to the universal regime ($T\ll\mathscr{D}$). One of the fundamental properties of the model (Anderson *et al.*, 1970), however, is that it is able to generate a dynamical scale T_0 (to be defined later) which uniquely determines the low-temperature physics. This scale depends on the cutoff $\mathscr{D}$ and the coupling constant J in the following generic way: $T_0\mathscr{D}\exp[-\alpha/\lambda(J)]$ where $\lambda(J)\to J$ as $J\to 0$. The explicit form of $\lambda(J)$ depends on the scheme used. In the momentum cutoff scheme, for example, one finds

$$T_0=\mathscr{D}e^{-(\pi/g)+(1/2)\ln g+\cdots}\ ,$$

while in another scheme which we choose to employ, the D scheme, one finds

$$T_0=De^{-\pi/2J}\ .$$

Here g and J are the coupling constants defined in the $\mathscr{D}$ and D schemes, respectively. Still, both constructions may be characterized by a choice of, say, $T_0=0.0007$ eV. This value is the only relevant scale in the scaling (universal) regime which defines the low-temperature and low magnetic field properties of the model. In this region the free energy F takes the form

$$\frac{F}{T}(T,H;D,J)\underset{H,T\ll D}{\to}f\left[\frac{T}{T_0},\frac{H}{T}\right]\ , \quad (1.3)$$

where the function f is universal in the sense that it is independent of the particular scheme used to define the model. The cutoff and coupling constant enter only in the combination determining T_0. Also, any other scale must be related to T_0 by pure numbers that are directly calculable. These numbers are universal.

That part of the scaling region where $T\gg T_0$ will be called the high-temperature region (still $T\ll\mathscr{D}$). As we shall see, this is the weak coupling regime (Anderson *et al.*, 1970), where the effective coupling constant is small and perturbation theory relevant. The low-temperature region ($T\ll T_0$), however, is a strong coupling regime and not accessible to expansion in the coupling constant. The crossover in behavior from the strong coupling regime to the weak coupling is the essence of the Kondo problem.

Let us quantify and characterize this crossover. We concentrate on the impurity susceptibility χ^i, which is the term in the susceptibility left over after we subtract from the total susceptibility $\chi=\partial M/\partial H$ the contribution of the electrons. (We take electrons and impurity to have the same g factor.)

As pointed out earlier (and will be proven later), the

high-temperature region is a weak coupling, perturbative regime. The higher the temperature, the weaker the effective coupling becomes. This is a statement of asymptotic freedom. The effective (running) coupling constant $J(T)$, which is obtained by summing logarithms via the renormalization group, is vanishing logarithmically $J(T)\sim 1/[\ln(T/T_0)]$. For high temperatures it has the same leading behavior as the bare coupling J, which vanishes as a function of the cutoff $J\sim 1/[\ln(D/T_0)]$. Applying perturbation theory, one finds that the impurity susceptibility attains its free value $\chi^i=\mu^2/T$ (Curie law) up to corrections that vanish logarithmically at high temperatures:

$$\chi^i \underset{T\gg T_0}{\rightarrow} \frac{\mu^2}{T}\left[1-\frac{1}{\ln\frac{T}{T_k}}-\frac{1}{2}\frac{\ln\ln\frac{T}{T_k}}{\ln^2\frac{T}{T_k}}+\left[\frac{1}{\ln\frac{T}{T_k}}\right]^3\right], \tag{1.4}$$

where a new scale T_k has been defined by the requirement that the $1/[\ln^2(T/T_k)]$ term be absent. This is equivalent to a normalization condition on T_k—the high-temperature or perturbative scale, which is conventionally referred to as the Kondo temperature.

While the high-temperature region is thus accessible to perturbation theory, the system enters a strong coupling regime at low temperature and its properties change dramatically.

Consider the Curie law $\chi^i=\mu^2/T$. Its divergence at $T=0$ indicates a net impurity spin. However, due to the strong interaction with the electrons the impurity spin will be quenched (screened) leading to a finite susceptibility at zero temperature. Thus let us define

$$\chi_0^i=\frac{\mu^2}{\pi T_0},$$

where T_0 is the scale that characterizes the low-temperature regime (see Fig. 3).

The ratio

$$W=\frac{T_k}{T_0} \tag{1.5}$$

is a universal number. It characterizes the crossover in properties from the weak coupling, asymptotically free region which is perturbatively accessible to the strong coupling regime that has to be constructed nonperturbatively. This was achieved numerically by Wilson (1975). Below, using an exact diagonalization of the Hamiltonian, we shall find an analytic expression for W. This is the main result of our calculations.

The crossover occurs also as a function of the magnetic field H. (We shall also consider $H\ll\mathscr{D}$ so as to be in the universal regime.) Consider now the impurity magnetization $\mathscr{M}^i$. For a free spin, at $T=0$,

$$\mathscr{M}^i=\mu\,\mathrm{sgn}(H)$$

and this value will be approached asymptotically at high fields ($H\gg T_0$)

$$\mathscr{M}^i\rightarrow\mu\left[1-\frac{1}{2}\frac{1}{\ln\frac{H}{T_H}}-\frac{1}{4}\frac{\ln\ln\frac{H}{T_H}}{\ln^2\frac{H}{T_H}}+O\left[\frac{1}{\ln\left[\frac{H}{T_H}\right]}\right]^3\right] \tag{1.6}$$

where again a scale T_H has been introduced by legislating away the $\{1/[\ln(H/T_H)]\}^2$ term. This scale characterizes the weakly coupled perturbatively accessible high-field region [the expression (1.6) will be deduced later].

At low fields there will be a crossover in the properties and $\mathscr{M}^i$ will have a finite slope (χ^i finite) at zero field (see Fig. 4).

Again, one defines a universal number W', which characterizes the crossover in magnetic field H

$$W'=\frac{T_H}{T_0}.$$

Both W' and W require a complete solution of the problem for their evaluation. However, W' is more accessible (to put it mildly). The reason is simple. The magnetic field excites from the ground state only excitations

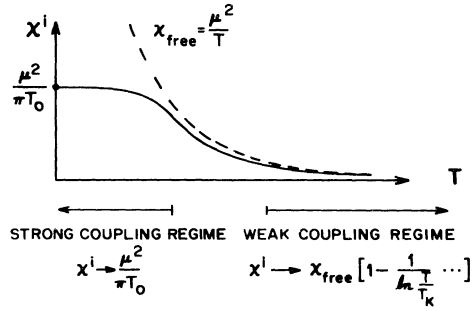

FIG. 3. The impurity susceptibility is plotted and compared with the free-spin susceptibility χ_{free}. At high temperatures χ^i approaches χ_{free} logarithmically on a scale set by T_k. As the temperature is lowered, it crosses over to a *finite* value at $T=0$, indicative of a screened spin.

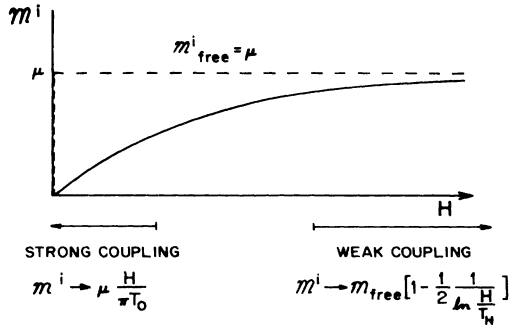

FIG. 4. The impurity magnetization as a function of the magnetic field H. For large magnetic fields it approaches the free value $\mathscr{M}^i_{\text{free}}=\mu$ logarithmically on a scale set by T_H. As the field, is lowered, it crosses over to yield a finite slope. (The convention here is that the magnetic moment is μ.)

which have maximum spin for a given energy. The temperature, on the other hand, excites all of them. Thus when one sums over states to form the free energy, the construction of the temperature dependence is by far more difficult.

As pointed out before, the ratio of any two scales is a pure, calculable (and universal) number. So, in particular, is $U=T_k/T_H$; in contradistinction to W or W', however, it does not relate a strong coupling to a weak coupling regime, but rather relates two regions both of which are accessible by perturbation theory so that U can be calculated without having to solve the theory (see Fig. 5).

Wilson (1975) also calculated another universal number $U'=C_v^i/T\chi^i|_{T=0}$, where C_v^i is the impurity contribution to the specific heat. Again, this number, being defined in the strong coupling regime only, does not characterize the crossover and can be calculated without having to solve the theory [see, for example, Nozières (1976)].

A complete characterization of the crossover is of course furnished by the free energy $F(H,T)$ and its various derivates—the specific heat, magnetization, and susceptibility. We shall discuss below a set of equations determining F and deduce scaling and crossover properties. In particular, we shall present recently obtained computer results exhibiting the thermodynamics in the full H-T plane. The Kondo problem has been, since its inception, interesting both in its own right and as a testing ground for many other ideas and techniques. It has been attacked by various methods: perturbation theory, various resummation techniques, S-matrix formalism, dispersion relation, renormalization-group techniques of the first and second kind, and more. For reviews see Kondo (1969), Grüner and Zawadowski (1974), Wilson (1975), and Nozières (1975).

This article follows a different line and approaches the problem via an exact diagonalization of the Hamiltonian using the Bethe ansatz (Bethe, 1931).

Models that admit a consistent Bethe-ansatz solution are very special although numerous. They enjoy the existence of an infinite number of conservation laws which severely restrict the dynamics and allow the wave functions to be expressible as linear combinations of single-particle wave functions. The conservation laws guarantee that the interactions will only generate (generalized) phase shifts without requiring the addition of new terms in the wave functions (all this will be made clear in Sec. II). This property of no production (or no diffraction) renders the wave functions explicitly constructible. In a sense these models are generalized free field theories.

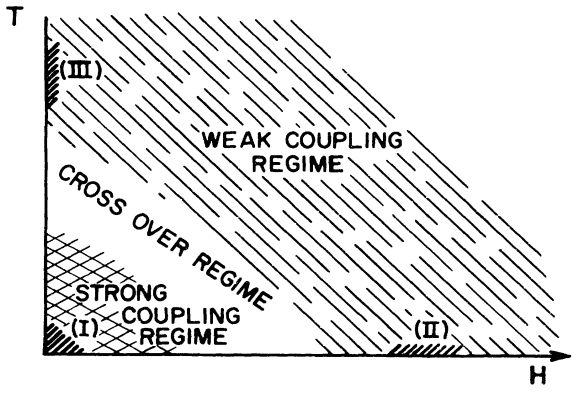

FIG. 5. Plot of the H-T plane. The weak coupling regime is characterized by either high-temperature or high magnetic field values. The regions (II) and (III), parametrized, respectively, by T_H and T_k, are in the weak coupling regime, while region (I), parametrized by T_0, is in the strong coupling regime.

The conservation laws led recently to a very interesting development where the full S matrix of various one-space-dimensional models was determined without explicitly solving them [see Zamolochikov and Zamolochikov (1978) for a review]. The key observation is that due to the simple dynamics, enforced by the conservation laws, the full S matrix would factorize and be expressible as a product of the two-body S matrices. This property can, of course, be deduced from the scattering eigenstates once they have been determined. Thus far, many more models, however, have succumbed to the S-matrix treatment than to direct diagonalization.

The Kondo model belongs to this distinguished class of exactly soluble models. This was shown by using the analogy to the soluble chiral Gross-Neveu model [Andrei (1980), or also directly Wiegman (1980)]. The chiral Gross-Neveu (or backscattering) model describes particles interacting via spin exchange and differs from the Kondo model only in that some of the particles are left-moving electrons rather than stationary impurities (see Sec. II).

This similarity allows one to take over the formalism developed in the diagonalization of the Gross-Neveu model [Andrei and Lowenstein (1979); partly developed also by Belavin (1979)] and apply it with only minor modification to the Kondo Hamiltonian. Indeed, the backscattering model, the Kondo model, and also the Heisenberg model will be shown to be very similar from the Bethe-ansatz point of view, all being spin exchange models differing only in the kinetic properties of their constituents.

Having an explicit representation for the eigenstates, one may proceed to construction of the thermodynamics. By methods developed earlier (Yang and Yang, 1969; Gaudin, 1971; Takahashi, 1971) it is easy to formulate a set of coupled nonlinear integral equations which determine the free energy (Filyov *et al.*, 1981; Andrei and Lowenstein, 1981).

Although these equations have not yet yielded an explicit solution, they can be used not only to prove various properties such as scaling or screening but can be cast into a form that provides a convenient iteration scheme leading to their numerical solution (Rajan *et al.*, 1982).

Furthermore, it is possible to extract from the equation and from universality considerations (see Sec. VI) an analytic expression for $W=T_k/T_0$ (Andrei and Lowenstein, 1981). This universal number was found to be in good agreement with the numerical value found by Wilson (1975), a fact of great importance. The reason is that in the diagonalization of the chiral Gross-Neveu model (Andrei and Lowenstein, 1979) (and subsequently in the diag-

onalization of the Kondo model) the authors had to use a new cutoff scheme, referred to before as the D scheme, which is *not* analytically related to the conventional momentum or lattice cutoff schemes. While it was felt (Andrei, 1980) that due to the renormalizability of the model, cutoff effects should be irrevlevant in the scaling regime, only the explicit construction of $W=T_k/T_0$ within the D scheme allayed doubts that the new method might have changed the model's universality class. The D scheme, therefore, challenges some accepted ideas about universality classes, and in particular about the role of analyticity as a necessary criterion. The question as to what, indeed, does determine the extent of a universality class should be further investigated.

Thus far we have discussed only the simplest Kondo system—that of a spin $S=\frac{1}{2}$ impurity coupled to the s-wave component of the conduction band. More common, however, is the situation when the impurity spin is higher.

If we restrict the interaction of impurity to be only with the s-wave component of the band, then we have to consider a Hamiltonian of the form (1.2) where **S**, however, represents a higher spin operator. In other words, the matrices S^i, $i=1,2,3$ form a higher representation of the spin group SU(2).

While the diagonalization proceeds along lines similar to the spin-$\frac{1}{2}$ case (Fateev and Wiegmann, 1981a, 1981b; also Furuya and Lowenstein, 1982), there are also basic differences in the results. It is found, for example, that while there is a crossover from a weak coupling to a strong coupling regime, the screening of the impurity is incomplete (Mattis, 1967). The ground state is characterized by a spin S-$\frac{1}{2}$, and the impurity susceptibility diverges accordingly. At high temperature, on the other hand, we approach the asymptotically free regime, and the susceptibility reveals a spin-S behavior.

A different way for a spin-S impurity to interact with an electronic band is provided by the rare-earth impurities where the angular momentum is unquenched. Now one has to expand the electronic wave function in total angular momentum eigenstates around the impurity and allow transition between the different states (Coqblin and Schrieffer, 1969). One is thus led to consider

$$\mathscr{H}_{\text{Kondo}}^{\text{SU}(2S+1)}=\sum_{k,m} kC_{km}^{*}C_{km}-J\sum_{kk'}\sum_{mm'} C_{k'm'}^{*}C_{km}C_{m}^{*}C_{m'}\ , \tag{1.7}$$

where C_{km} (C_m) destroys an electron (an impurity) with m being the z component of the angular momentum, $-S\le m\le S$. This is therefore an SU($2S+1$) generalization of the basic Kondo problem. The Hamiltonian is invariant under the transformation $C_{km}\to U_{mm'}C_{km'}$, $C_m\to U_{mm'}C_{m'}$, where U is a $(2S+1)$-dimensional unitary matrix. It is important to note the difference between this case and the preceding one, where the symmetry is still SU(2) but not in the fundamental spin-$\frac{1}{2}$ representation. The physics differ accordingly, and this time one finds a complete screening and finite susceptibility at $T=0$.

The diagonalization of the SU(N) version of the backscattering model (Andrei and Lowenstein, 1980) was carried out using a nested Behte-ansatz method (Sutherland, 1967). The results carry over to the corresponding SU(N) Kondo model, whose thermodynamics were recently considered (Tsvelick and Wiegmann, 1981).

We have discussed up to now the static properties of some variants of the Kondo model. The transport properties, on the other hand, are less accessible, and their determination is a major open problem. These quantities are expressible in terms of time-dependent correlation functions whose calculation even in the static case is yet unachieved [although some progress has been made in the case of the nonlinear Schrödinger model (for a review see Thacker, 1981)].

An important transport quantity is the resistivity which is given by a retarded two-point function of the current. Still, due to the simple structure of the Kondo exchange interaction, this quantity is expressible in terms of the on-shell scattering T matrix, which allows its determination from the spectrum. This was carried out at $T=0$ for an arbitrary magnetic field (Andrei, 1981) and can be generalized to arbitrary temperatures.

The plan of this review is as follows: In Sec. II we discuss the diagonalization of the spin-$\frac{1}{2}$ Hamiltonian. The method we follow is due to Yang (1967), and we present it in great detail. A different approach was presented by Gaudin (1967) in a shorter paper. These ideas have recently been reformulated by Faddeev *et al.* (1980). In Sec. III we present the spectrum of the model and also introduce the string hypothesis. The zero-temperature magnetization is derived in Sec. IV and $W'=T_H/T_0$ determined. The thermodynamics are formulated in Sec. V, and various asymptotic regions are discussed and scaling displayed. Section VI contains a discussion of the universality of our results, comparison to the conventional scheme, and in particular calculation of $W=T_k/T_0$. We then accelerate our pace and in Sec. VII discuss the spin-S Kondo model and present the magnetization curve and the thermodynamics. Section VIII contains an analogous discussion for the SU($2S+1$) Coqblin and Schrieffer version of the spin-S impurity imbedded in a metal, and in Sec. IX we calculate the zero-temperature magnetoresistance.

II. DIAGONALIZATION OF THE KONDO HAMILTONIAN

In this section we shall show how to construct the eigenstates of the Kondo Hamiltonian

$$H_{\text{Kondo}}=(-i)\int \phi_a^{*}(x)\partial_x\phi_a(x)+J\sigma\phi_a^{*}(0)\sigma_{ab}\phi_b(0)$$
$$+J'\phi_a^{*}(0)\phi_a(0)\ , \tag{2.1}$$

where we replace **S**, the localized spin operator by σ to denote spin-$\frac{1}{2}$. The case of higher spin will be discussed later. Additionally, we include a potential scattering term whose effect, we shall see, is merely to renormalize the coupling constant J.

First, we shall rewrite the Hamiltonian in a more symmetric way which will exhibit its relation to another

well-known model, the backscattering model. Then passing to first-quantized language, we derive a Schrödinger equation for the eigenfunctions.

Expressing the wave functions in the Bethe form, we show in great detail that the ansatz is indeed consistent. After identifying the permutation symmetries and imposing periodic boundary conditions, we proceed, employing the brilliant method of Yang (1967) and Gaudin (1967) to solve the ansatz. An alternative way of solving the ansatz is to use Baxter-type transfer matrix methods developed by Faddeev *et al.* and used in this case by Wiegmann (1980).

A. Rewriting the Hamiltonian

We would like to rewrite the Hamiltonian in a way that is more symmetric between electrons and impurities. Let us introduce an impurity field $\chi_a(x)$ analogous to the electron field $\phi_a(x)$. We may then rewrite the Hamiltonian as follows:

$$\mathscr{H}_{\text{Kondo}}=-i\int\phi_a^*(x)\partial_x\phi_a(x)dx+J\int\chi_a^*(x)\sigma_{ab}^i\chi_b(x)\phi_{a'}^*(x)\sigma_{a'b'}^i\phi_{b'}(x)dx+J'\int\chi_a^*(x)\chi_a(x)\phi_b^*(x)\phi_b(x)dx\ . \tag{2.2}$$

Two remarks are in order. First, the impurity number operator $N^i=\int\chi_a^*\chi_a(x)dx$ is conserved, so we may choose $N^i=1$ as before. Second, it is important to note that the field $\chi_a(x)$ has no kinetic energy associated with it. One may therefore form localized wave packets for the impurity which do not disperse with time. This formalism, then, is equivalent to the other way of writing the Hamiltonian, but allows also a discussion of the case when the impurity is spread.

It is convenient now to introduce a field $\psi_a(x)$, which contains impurity and electron fields

$$\psi_a(x)=\begin{bmatrix}\phi_a(x)\\ \chi_a(x)\end{bmatrix}\begin{matrix}\alpha=1\\ \alpha=0\ .\end{matrix}$$

This is pure cosmetics. It allows us to handle electrons and impurities on equal footing. We call α the purity index and give it the values 1 for an electron and 0 for an impurity. The fields $\psi_{a\alpha}$ are assumed to have canonical anticommutation relations,

$$\begin{aligned}&\{\psi_{a\alpha}(x),\psi_{b\beta}(y)\}=0\ ,\\ &\{\psi_{a\alpha}(x),\psi_{b\beta}^*(y)\}=\delta_{ab}\delta_{\alpha\beta}\delta(x-y)\ .\end{aligned} \tag{2.3}$$

The fact that the electron field $\psi_{a1}=\phi_a$ and the impurity field $\psi_{a0}=\chi_a$ are assumed to anticommute is merely a convention and has no bearing on the physics.

We can now write the Hamiltonian in the following form:

$$\begin{aligned}\mathscr{H}_{\text{Kondo}}=&-i\sum_{\beta=0,1}\int dx\,\psi_{a\beta}^*(x)\beta\partial_x\psi_{a\beta}(x)\\ &+J\int dx\,\psi_{a0}^*(x)\sigma_{ab}\psi_{b0}(x)\psi_{a'1}^*(x)\sigma_{a'b'}\psi_{b'1}(x)\\ &+J'\int dx\,\psi_{a0}^*(x)\psi_{a0}(x)\psi_{b1}^*(x)\psi_{b1}(x)\ .\end{aligned} \tag{2.4}$$

Note that for the impurity $\beta=0$ and that the impurity has no contribution in the kinetic energy.

In this form, we can immediately see the connection with the backscattering model, which describes left- and right-moving electrons interacting via a spin exchange. It is of the same form as $\mathscr{H}_{\text{Kondo}}$ with the only difference that $\beta=\pm1$ indicating left and right movers rather than $\beta=0$ or 1, with $\beta=1$ indicating a right-moving electron and $\beta=0$ indicating a stationary particle, an impurity.

Since the backscattering model (AKA the chiral Gross-Neveu) was solved by a Bethe-ansatz method (Andrei and Lowenstein, 1979; partly also Belavin, 1979), it is obvious that the Kondo model is also a member of the class of exactly soluble models. The discussion that follows applies, then, to both models, although we will use the Kondo terminology.

B. Reduction to first-quantization formalism

Let us consider the Hilbert space on which $\mathscr{H}_{\text{Kondo}}$ acts. Obviously, since the number of electrons N^e and the number of impurities, $N^i=1$, are separately conserved, we could use these quantities to label our space. Thus the most general state is given by ($N=N^e+N^i$ is the number of particles)

$$|\mathscr{F}\rangle=\sum_{a_i\beta_i}\int\pi\,dx_i\mathscr{F}(x_i\beta_ia_i)\prod_{i=1}^N\psi_{a_i\beta_i}^*(x_i)\,|0\rangle\ , \tag{2.5}$$

where $|0\rangle$ is the vacuum, $\psi_{a\alpha}(x)\,|0\rangle=0$, and N^e of the purity indices have the value $\beta=1$ and N^i of them the value $\beta=0$. The wave function $\mathscr{F}$ thus depends on the space coordinates x, the spin coordinates a, and the purity coordinates β, which keep track of whether we mean an electron or an impurity.

We now want to find $\mathscr{F}$ such that

$$\mathscr{H}_{\text{Kondo}}\,|\mathscr{F}\rangle=E\,|\mathscr{F}\rangle\ . \tag{2.6}$$

Using the canonical commutation relations, we find that the wave function must satisfy a first-quantized Schrödinger equation

$$h\mathscr{F}=E\mathscr{F}\ ,$$

where

$$\begin{aligned}h=&-i\sum_{i=1}^N\beta_i\partial_i+J\sum_{i,j=1}^N\delta(x_i-x_j)\sigma_i\cdot\sigma_j(\beta_i-\beta_j)^2\\ &+J'\sum_{i,j=1}^N\delta(x_i-x_j)(\beta_i-\beta_j)^2\ .\end{aligned} \tag{2.7a}$$

Let us note some features:

(1) The kinetic energy is linear in the electron momen-

tum, reflecting the fact that the model describes a physical situation in which only electrons very close to the Fermi surface are relevant.

(2) The impurities make no contribution to the kinetic energy. Although the Hamiltonian is translationally invariant and the Bethe-ansatz wave functions, to be constructed below, are labeled by impurity as well as electron momenta, one may, if one wishes, superpose such eigenstates to obtain states in which the impurity is arbitrarily well localized.

Indeed, we may observe that if one sets $\beta_i=1, i=1,\ldots,N^e$, $\beta_N=0$, and $X_N=0$, one finds

$$h=\sum_{i=1}^{N^e}-i\frac{\partial}{\partial x^i}+2J\sum_{i=1}^{N^e}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_N\delta(x_i)+2J'\sum_{i=1}^{N^e}\delta(x_i)\ , \tag{2.7b}$$

which is the first-quantized Hamiltonian corresponding to Eq. (2.1).

(3) The factor $(\beta_i-\beta_j)^2$ limits, as it should, the interaction to that of electrons with impurities only.

The Hamiltonian h acts on the position variable x, on the spin variable a, and the purity variable β. We may simplify matters by observing that we can write h as

$$h=-i\sum\beta_i\partial_i+\sum\delta(x_i-x_j)(\beta_i-\beta_j)^2\times[2J\tfrac{1}{2}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)+(J'-J)]\ ,$$

where we have formed the combination $P_\sigma^{ij}=\frac{1}{2}(1+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)$, which is a spin exchange operator: $P_\sigma^{ij}\mathcal{F}(\ldots a_i\ldots a_j\ldots)=\mathcal{F}(\ldots a_j\ldots a_i\ldots)$.

However, due to the complete antisymmetry requirement of $\mathcal{F}$,

$$\mathcal{F}(\ldots(a_ix_i\beta_i)\ldots(a_jx_j\beta_j)\ldots)=-\mathcal{F}(\ldots(a_jx_j\beta_j)\ldots(a_ix_i\beta_i)\ldots)\ ,$$

we can replace $\delta(x_i-x_j)P_\sigma^{ij}$ by $(-)\delta(x_i-x_j)P_\beta^{ij}$, where P_β^{ij} is a purity exchange operator.

We have finally

$$h=(-i)\sum\beta_i\partial_i+\sum\delta(x_i-x_j)(\beta_i-\beta_j)^2\times[-2JP_\beta^{ij}+(J'-J)]\ . \tag{2.8}$$

This form is also the starting point of the discussion of the SU(N) symmetry generalization (Andrei and Lowenstein, 1980) of the backscattering and Kondo models (see Sec. VIII).

C. Construction of the eigenfunctions: the Bethe ansatz and its consistency

Since the Hamiltonian h in (2.8) no longer depends on the spin variables, we may split the wave function as

$$\mathcal{F}(x\beta a)=F(x\beta)t(a)\ ,$$

where $t(a)$ is the spin-wave function, and we wish to determine the part of the wave function, $F(x,\beta)$, which depends on spatial and purity coordinates. It satisfies the Schrödinger equation

$$hF(x\beta)=EF(x\beta)\ .$$

We shall also require that the wave functions satisfy periodic boundary conditions

$$F(\ldots(x_i=0,\beta_i)\ldots)=F(\ldots(x_i=L,\beta_i)\ldots)\ .$$

To construct $F(x,\beta)$ we divide configuration space into $N!$ regions, labeled by permutations $Q\in S_N$ according to the ordering of the particles on the line. Thus, for example, the region where

$$0\le x_3<x_1<x_4<x_2<\cdots\le L$$

will be denoted by

$$Q=\begin{bmatrix}1&2&3&4&\cdots\\3&1&4&2&\cdots\end{bmatrix}\in S_N\ .$$

As the interaction is local it becomes operative only on the boundaries of the regions Q, namely, when two coordinates coincide: $x_{Qa}=x_{Q(a+1)}$.

In the interior of each region, the wave function is given by a linear combination of plane waves which satisfy the free Hamiltonian.

Let us then consider the following form of the wave function labeled by momenta $k_1\cdots k_N$ and purities $\alpha_1\cdots\alpha_N$:

$$F^{k,\alpha}(x,\beta)=\sum_{P,Q\in S_N}\xi_{Q,P}\theta(x_Q)\exp\left[i\sum_j k_{Pj}x_{Qj}\right]\prod_l\delta_{\beta_{Ql}}^{\alpha_{Pl}}\ , \tag{2.9}$$

where $\theta(x_Q)=\theta(x_{Q1}<x_{Q2}<\cdots<x_{QN})$ is a product of step functions which is equal to unity for $x_{Q1}<x_{Q2}<\cdots<x_{QN}$ and which vanishes outside that region. $\xi_{Q,P}$ are numerical coefficients that form a $N!\times N!$ matrix. The summation is over permutations P and Q of the symbols $1,\ldots,N$. In what follows, a particular permutation will be specified by listing, without commas, the respective numerals into which $1,2,\ldots,N$ are mapped, e.g.,

$$Q=Q1\,Q2\ldots QN\ .$$

The function $F^{k,\alpha}(x,\beta)$ obviously satisfies the Schrödinger equation away from the boundaries, and we find

$$hF^{k,\alpha}=EF^{k,\alpha}\ ,\quad E=\sum_{j=1}^{N}\alpha_jk_j=\sum_{j=1}^{N^e}k_j^{(e)}\ . \tag{2.10a}$$

$$PF^{k,\alpha}=PF^{k,\alpha}\ ,\quad P=\sum_{j=1}^{N}k_j=\sum_{j=1}^{N^e}k_j^{(e)}+k^{(\mathrm{imp})}\ . \tag{2.10b}$$

Thus the energy E is given as the sum of the electronic momenta only and is independent of the momentum $k^{(\mathrm{imp})}$ associated with the motion of the impurity. This allows us to sum over the latter and localize it.

Obviously, the core of the solution is to find the coefficients $\xi_{P,Q}$ and momenta k_i so as to satisfy the equation in the full configuration space. To enumerate a complete

basis, we will impose periodic boundary conditions.

The form of the wave function is particularly simple in that it is labeled in all sections by the *same* set of momenta k_i. (We call them momenta with an abuse of the language, since, due to the interaction, individual particle momenta are not conserved.) In general, starting in a region with momenta $\{k_i\}$, we discover that crossing the boundary to another region produces an interaction which will generate new modes. The fact that this is not the case here is due to the existence of the conservation laws which restrict the dynamics. This question will be discussed below when we show that the ansatz is indeed consistent.

We proceed to determine the allowed momenta k_i and coefficients $\xi_{Q,P}$.

In order for the potential to become operative, two particles must cross each other. Take them to be $i=Qa$ and $j=Q(a+1)$. We want to relate region Q to region Q', where $Q'a=Q(a+1)$, $Q'(a+1)=Qa$, and $Qj=Q'j$, $j\neq a,a+1$. When h is applied to F, with the convention $\delta(x)\theta(x)=r\delta(x)$, $0<r\leq 1$, there arise singularities of the type

$$\delta(x_{Qa}-x_{Q(a+1)})e^{ix_{Qa}(k_{Pa}+k_{P(a+1)})}\prod_{j\neq a,b}e^{ik_{Pj}x_{Qj}}\theta(x_{Q1}<\cdots<x_{Qa}<x_{Q(a+2)}<\cdots<x_{QN})\,. \tag{2.11}$$

For a given P and Q, only four terms in the double summation over permutations produce singularities with the precise x dependence (2.11), namely, those labeled by the pairs $(Q,P),(Q'P),(QP'),(Q'P')$, where P' is related to P the same way as Q' is related to Q. Writing out the relevant parts of these terms, we have

$$\begin{aligned}F=&\cdots+\xi_{QP}\theta(x_{Q'a}-x_{Qa})e^{i(k_{Pa}x_{Qa}+k_{P'a}x_{Q'a})}\delta_{\beta_{Qa}}^{\alpha_{Pa}}\delta_{\beta_{Q'a}}^{\alpha_{P'a}}\cdots+\xi_{Q'P}\theta(x_{Qa}-x_{Q'a})e^{i(k_{Pa}x_{Q'a}+k_{P'a}x_{Qa})}\delta_{\beta_{Q'a}}^{\alpha_{Pa}}\delta_{\beta_{Qa}}^{\alpha_{P'a}}\cdots\\&+\xi_{QP'}\theta(x_{Q'a}-x_{Qa})e^{i(k_{P'a}x_{Qa}+k_{Pa}x_{Q'a}}\delta_{\beta_{Qa}}^{\alpha_{P'a}}\delta_{\beta_{Q'a}}^{\alpha_{Pa}}\cdots+\xi_{Q'P'}\theta(x_{Qa}-x_{Q'a})e^{i(k_{P'a}x_{Q'a}+k_{Pa}x_{Qa})}\delta_{\beta_{Q'a}}^{\alpha_{P'a}}\delta_{\beta_{Qa}}^{\alpha_{Pa}}\cdots+\cdots\,.\end{aligned}$$

Applying to these terms $(h-E)$ we have

$$\begin{aligned}0=(h-E)F=\cdots+\delta(x_{Q'a}x_{Qa})e^{i(k_{Pa}+k_{P'a})x_{Qa}}\{&-i(\beta_{Q'a}-\beta_{Qa})[(\xi_{QP}-\xi_{Q'P'})\delta_{\beta_{Qa}}^{\alpha_{Pa}}\delta_{\beta_{Q'a}}^{\alpha_{P'a}}+(\xi_{QP'}-\xi_{Q'P})\delta_{\beta_{Q'a}}^{\alpha_{Pa}}\delta_{\beta_{Qa}}^{\alpha_{P'a}}]\\&-2rJ(\beta_{Q'a}-\beta_{Qa})^2[(\xi_{QP}+\xi_{Q'P'})\delta_{\beta_{Q'a}}^{\alpha_{Pa}}\delta_{\beta_{Qa}}^{\alpha_{P'a}}+(\xi_{QP'}+\xi_{Q'P})\delta_{\beta_{Qa}}^{\alpha_{Pa}}\delta_{\beta_{Q'a}}^{\alpha_{P'a}}]\\&-rJ'(\beta_{Q'a}-\beta_{Qa})^2[(\xi_{QP}+\xi_{Q'P'})\delta_{\beta_{Qa}}^{\alpha_{Pa}}\delta_{\beta_{Q'a}}^{\alpha_{P'a}}+(\xi_{QP'}+\xi_{Q'P})\delta_{\beta_{Q'a}}^{\alpha_{Pa}}\delta_{\beta_{Qa}}^{\alpha_{P'a}}]\}\\&+\cdots\,.\end{aligned}\tag{2.12}$$

At this point, we make the choice $r=\frac{1}{2}$, noting that any other positive value gives merely a rescaling of the coupling constant. (This reflects the fact that a four-fermion operator needs to be properly defined and that its meaning depends on the definition.)

Consider now the case where $\alpha_{Pa}\neq\alpha_{P(a+1)}$. We must choose $\xi_{Q'P'}$, $\xi_{QP'}$, $\xi_{Q'P}$, and ξ_{QP} to satisfy (we set $J''=J'-J$)

$$\begin{aligned}&-i(\alpha_{P'a}-\alpha_{Pa})(\xi_{QP}-\xi_{Q'P'})-J(\xi_{QP'}+\xi_{Q'P})\\&\qquad+\tfrac{1}{2}J''(\xi_{QP}+\xi_{Q'P'})=0\,,\end{aligned}\tag{2.13}$$

$$\begin{aligned}&+i(\alpha_{P'a}-\alpha_{Pa})(\xi_{QP'}-\xi_{Q'P})-J(\xi_{QP}+\xi_{Q'P'})\\&\qquad+\tfrac{1}{2}J''(\xi_{QP'}+\xi_{Q'P})=0\,.\end{aligned}$$

Solving these linear equations for $\xi_{QP'}$ and $\xi_{Q'P'}$ in terms of ξ_{QP} and $\xi_{Q'P}$, we may express the result as a matrix relation linking the neighboring columns of ξ, labeled by permutations P and P' (we denote the corresponding columns by ξ_P and $\xi_{P'}$),

$$\xi_{P'}=Y_{Pa,P(a+1)}^{a,a+1}\xi_P\,, \tag{2.14}$$

where

$$\begin{aligned}Y_{ij}^{a,b}=&\frac{2i(\alpha_i-\alpha_j)J}{1+J^2-\frac{1}{4}J''^2+i(\alpha_i-\alpha_j)J''}\mathbb{1}\\&+\frac{1-J^2+\frac{1}{4}J''^2}{1+J^2-\frac{1}{4}J''^2+i(\alpha_i-\alpha_j)J''}\mathscr{P}^{ab}\\\equiv&a_{ij}\mathscr{P}^{ab}+b_{ij}\mathbb{1}\,.\end{aligned}\tag{2.15}$$

Here, $\mathscr{P}^{ab}$ is defined by $(\mathscr{P}^{ab}\xi_P)_Q=\xi_{Q(a,b),P}$, with (a,b) the transposition that exchanges a and b.

There is no restriction imposed by the Hamiltonian in the case $\alpha_{Pa}=\alpha_{P(a+1)}$. We must remember, however, that we want to construct a consistent set of eigenfunctions. It is this consistency requirement that will impose restrictions on the Y matrix in the equal purity case.

Indeed, the Y matrix allows us to relate all of the columns of the coefficient matrix to one another. In particular, if the $N!$-component column vector $\xi_0\equiv\xi_{12\ldots N}$ is chosen arbitrarily, any other ξ_P can be obtained by applying an appropriate product of Y matrices on ξ_0 (here we make use of the fact that any P can be written as a product of transpositions of neighboring symbols). For example,

$$\xi_{4321}=Y^{12}_{34}\xi_{3412}=Y^{12}_{34}Y^{23}_{14}\xi_{3142}=Y^{12}_{34}Y^{23}_{14}Y^{34}_{24}\xi_{3124}$$
$$=Y^{12}_{34}Y^{23}_{14}Y^{34}_{24}Y^{12}_{13}\xi_{1324}=Y^{12}_{34}Y^{23}_{14}Y^{34}_{24}Y^{12}_{13}Y^{23}_{23}\xi_{1234}\ .$$

Of course, the succession of permutations connecting a given P to the identity is not unique. If the determination of ξ_P is to be self-consistent, however, it must not depend on the path. The following are necessary and sufficient conditions which the Y matrices must satisfy to generate path independence (Zinn-Justin and Brezin, 1966; Yang, 1967):

$$Y^{ab}_{ij}Y^{ab}_{ji}=1\ , \tag{2.16a}$$

$$Y^{ab}_{ij}Y^{cd}_{kl}=Y^{cd}_{kl}Y^{ab}_{ij}\ , \tag{2.16b}$$

$$Y^{bc}_{ij}Y^{ab}_{ik}Y^{bc}_{jk}=Y^{ab}_{jk}Y^{bc}_{ik}Y^{ab}_{ij}\ . \tag{2.16c}$$

To understand the significance of these identities, let us consider the cases $N=3$ and $N=4$.

For $N=3$, we may assign the six permutations of 123 to the vertices of a hexagon (see Fig. 6) in such a way that permutations which differ from one another by a transposition of neighboring symbols are situated on neighboring vertices. We see that we can reach any permutation P starting from 123 by going either clockwise or counterclockwise around the hexagon. Thus, for example, we may construct ξ_{321} by following either the route $123\to132\to312\to321$ or, alternatively, $123\to213\to231\to321$:

$$\xi_{321}=Y^{23}_{12}\xi_{312}=Y^{23}_{12}Y^{12}_{13}\xi_{132}=Y^{23}_{12}Y^{12}_{13}Y^{23}_{23}\xi_{123}\ , \tag{2.17}$$

$$\xi_{321}=Y^{12}_{23}\xi_{231}=Y^{12}_{23}Y^{23}_{13}\xi_{213}=Y^{12}_{23}Y^{23}_{13}Y^{12}_{12}\xi_{123}\ . \tag{2.18}$$

If the right-hand members of (2.17) and (2.18) are to coincide for arbitrary ξ_{123}, we must have

$$Y^{23}_{12}Y^{12}_{13}Y^{23}_{23}=Y^{12}_{23}Y^{23}_{13}Y^{12}_{12}\ ,$$

which is a special case of (2.16c). Of course, there are infinitely many other paths connecting 123 to 321, namely, those involving reversals of direction; that these yield the same result as the two paths without reversals is a consequence of (2.16a).

For N greater than 3, the connectivity properties of the permutations rapidly become much more complex. For $N=4$, we can place the permutations of 1234 on the vertices of a truncated octahedron, as in Fig. 7, in a "neighborliness"-preserving fashion. Path independence of ξ_P is assured by (2.16a)–(2.16c), which gives uniqueness, respectively, for

(a) going back and forth between two neighboring vertices.

(b) going around a square face; and

(c) going around a hexagonal face.

When one goes to arbitrarily large N, the conditions (2.16a)–(2.16c) remain sufficient for self-consistency (Zinn-Justin and Brezin, 1966).

Returning to the specific form (2.15) of the Y matrices in the Kondo model, we may ask whether they indeed satisfy the consistency conditions. Since the Hamiltonian does not restrict the Y matrix for equal purities (electrons do not interact with electrons), we may choose it at will, but in order to have a consistent ansatz, it must be chosen so as to satisfy the consistency conditions.

Let us assume the Y matrix for equal purity, $\alpha_i=\alpha_j$ is of the form

$$Y^{ab}_{ij}=c_i\mathbb{1}+d_i\mathscr{P}^{ab}\ ,\quad \alpha_i=\alpha_j\ , \tag{2.19}$$

with c_i and d_i depending only on α_i. Applying (2.16a) to (2.19), we find

$$Y^{ab}_{ij}Y^{ab}_{ji}=(c_i^2+d_i^2)\mathbb{1}+2c_id_i\mathscr{P}^{ab}=\mathbb{1}\ ,$$

which implies either

(I) $c_i=\pm1$, $d_i=0$, or

(II) $c_i=0$, $d_i=\pm1$.

Turning now to the remaining consistency relations, we see that (2.16b) is trivially satisfied for Y matrices of the form (2.15). Equation (2.16c) is also easily verified for the case $\alpha_i=\alpha_j=\alpha_k$. For the remaining case, $\alpha_i=\alpha_j\neq\alpha_k$,

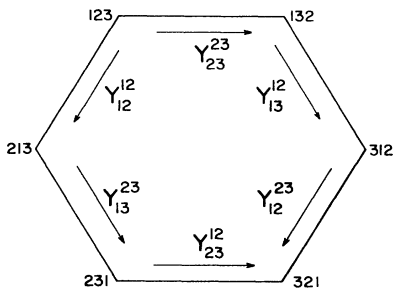

FIG. 6. Diagram for neighboring regions, i.e., regions that are related by a permutation of adjacent particles, for three particles. We indicated two paths to reach ξ_{321} starting from ξ_{123}. They must be equivalent if the Bethe ansatz is consistent.

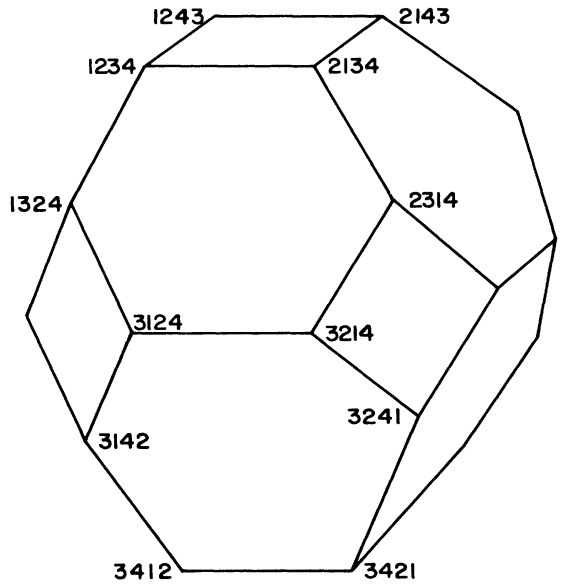

FIG. 7. Diagram for neighboring regions in the four-particle case.

we consider separately the possibilities (I) and (II):

(I)

$$\pm Y_{ik}^{ab}Y_{jk}^{bc}=\pm(-a_{ik}\mathscr{P}^{ab}+b_{ik}\mathbb{1})(-a_{jk}\mathscr{P}^{bc}+b_{jk}\mathbb{1})$$
$$=\pm(-a_{jk}\mathscr{P}^{ab}+b_{jk}\mathbb{1})(-a_{ik}\mathscr{P}^{bc}+b_{ik}\mathbb{1})$$
$$=\pm Y_{jk}^{ab}Y_{ik}^{bc}\ .$$

(II)

$$\pm\mathscr{P}^{bc}Y_{ik}^{ab}Y_{jk}^{bc}=\pm\mathscr{P}^{bc}(-a_{jk}\mathscr{P}^{ab}+b_{jk}\mathbb{1})$$
$$\times(-a_{ik}\mathscr{P}^{bc}+b_{ik}\mathbb{1})$$
$$\neq\pm Y_{jk}^{ab}Y_{ik}^{bc}\mathscr{P}^{ab}\ .$$

We see that (II) is incompatible with (2.16c), and hence

$$Y_{ij}^{ab}=\pm\mathbb{1}\quad\text{for }\alpha_i=\alpha_j\ ,$$

with the sign possibly depending on α_i. In what follows, we shall consistently set

$$Y_{ij}^{ab}=-1\quad\text{for }\alpha_i=\alpha_j\ . \tag{2.20}$$

It turns out that different assignments of sign correspond to different bases, and the choice is purely a matter of convenience. Although physical quantities cannot depend, in the infinite cutoff limit, on the particular choice of basis, the *description* of the basis states will depend on the convention adopted. In particular, the choice (2.20) gives rise to a fermionic bookkeeping: the wave functions (2.7) are antisymmetric in the momentum labels $k_1,\ldots,k_N$. This antisymmetry has nothing to do with the electron's Fermi statistics; in fact, without violating the overall antisymmetry of the wave function one can choose [as in Andrei (1980)] Y_{ij}^{ab} to be $+\mathbb{1}$ for $\alpha_i=\alpha_j$, in which case the wave function would be symmetric under permutations of $k_1,\ldots,k_N$.

To summarize, the set of Y matrices which solve the Kondo Hamiltonian and yield a consistent Bethe ansatz to the wave functions is given by

$$Y_{ij}^{ab}=\begin{cases}a_{ij}\mathbb{1}\mathscr{P}^{ab}-b_{ij}\mathbb{1}\ , & \alpha_i\neq\alpha_j\\ -\mathbb{1}\ , & \alpha_i=\alpha_j\ ,\end{cases} \tag{2.21}$$

where

$$-b_{ij}=\frac{2i(\alpha_i-\alpha_j)J}{1+J^2-\frac{1}{4}J''^2+i(\alpha_i-\alpha_j)J''}\ ,$$
$$a_{ij}=\frac{1-J^2+\frac{1}{4}J''^2}{1+J^2-\frac{1}{4}J''^2+i(\alpha_i-\alpha_j)J''}\ . \tag{2.22}$$

In the absence of interaction ($J=0,J'=0$), one finds $b_{ij}=0,a_{ij}=1$, so that

$$Y_{ij}^{ab}=\begin{cases}\mathscr{P}^{ab}\ , & \alpha_i\neq\alpha_j\\ -\mathbb{1}\ , & \alpha_i=\alpha_j\ ,\end{cases}$$

indicating discontinuities in the wave functions, even though the interaction has been turned off. The energy levels of

$$h=h_0=\sum_{i=1}^{N}-i\beta_i\partial_i=-i\sum_{i=1}^{N^e}\partial_i\ ,$$

however, are infinitely degenerate. Thus, for example, the energy level of two particles is given by

$$E=\frac{2\pi}{L}(n_1+n_2)\ ,$$

and all combinations of n_1 and n_2 whose sum is $n=n_1+n_2$ give the same energy. The fact that discontinuities persist when $J=0$ means that the zeroth-order approximation (in the sense of degenerate perturbation theory) must contain discontinuities.

While discontinuities may come as a surprise when $J=0$, they are certainly expected in the interacting theory. They are obviously required by the linear derivatives and δ-function potentials. If we insisted on expanding the wave functions in Fourier series and cut off the expansion (which would amount to the imposition of the conventional momentum cutoff $\mathscr{D}$), then the discontinuities would be smeared out and no exact solution could be constructed. Removing the cutoff ($\mathscr{D}\to\infty$) will again allow for the construction of exact wave functions. It is desirable, however, to find a scheme which allows the construction of exact eigenfunctions even in the presence of a finite cutoff. Our scheme (Andrei and Lowenstein, 1979; Andrei, 1980) accomplishes that without, apparently, taking us out of the universality class defined by the $\mathscr{D}$ scheme (Andrei and Lowenstein, 1980). The scheme will be discussed in Secs. III and VI.

Thus far, we have seen that a function F of the form (2.9) will satisfy the Hamiltonian (2.8), provided that the columns ξ_P of the coefficient matrix ξ are connected by the Y matrix (2.21). This means that, say, the first column ξ_0 is not constrained by (2.21) alone. Let us now turn our attention to the remaining properties which must be satisfied by the Bethe ansatz. This will determine ξ_0.

D. Permutation symmetry

The wave function $\mathscr{F}(x\beta a)$ was written in the form

$$\mathscr{F}(x\beta a)=F(x,\beta)t(a)\ ,$$

and we found a Bethe ansatz for $F(x,\beta)$. The spin-wave function $t(a)=t(a_1,\ldots,a_n)$ is specified by a Young tableau of up to two rows (because spin-$\frac{1}{2}$ has only two states available for antisymmetrization). For each of the $\binom{N}{M}-\binom{N}{M-1}$ standard Young tableaux with lower row of length M and upper row of length $N-M$ [see Fig. 8(a)] there will be $2S+1$ [$S=\frac{1}{2}(N-2M)$] mutually orthogonal wave functions t, each labeled by the total spin S and spin projection S_z in the range $-S\leq S_z\leq S$. We remind the reader that distinct Young tableaux correspond to independent ways of coupling N spin-$\frac{1}{2}$ to yield total spin S [see, for instance, Hamermesh (1962)].

If the wave function $\mathscr{F}(x\beta a)$ is to survive antisymmetrization, the function F must possess a permutation symmetry complementary to that of $t(a)$, namely, it must be described by a Young tableau of up to two columns [Fig. 8(b)]. More precisely (Hamermesh, 1962),

$$\sum_R a_R F(x_{R1}\dots x_{RN},\beta_{R1}\dots\beta_{RN})=F(x_1\dots x_N,\beta_1\dots\beta_N)\ , \tag{2.23}$$

where a_R are numerical coefficients determined by the specific Young tableau. Substituting the Bethe form of F the left-hand side of (2.23) becomes

$$\sum_R a_R \sum_{Q,P}\xi_{Q,P}\theta(x_{RQ})\exp\left[i\sum_j k_{Pj}x_{RQj}\right]\prod_l \delta_{\alpha_{Pl}}^{\beta_{RQl}}=\sum_{Q,P}\left[\sum_R a_R\xi_{R^{-1},Q,P}\right]\theta(x_Q)\exp\left[i\sum_j k_{Pj}x_{Qj}\right]\prod_l \delta_{\alpha_{Pl}}^{\beta_{Ql}}\ ,$$

so that we must have

$$\xi_{Q,P}=\sum_R a_R\xi_{R^{-1}Q,P}\ . \tag{2.24}$$

It is sufficient to impose the condition

$$\xi_{Q,0}=\sum_R a_R\xi_{R^{-1}Q,0}\ , \tag{2.25}$$

since multiple application of Y matrices then yields (2.24). [Note that $\mathscr{P}^{ab}$ in Y^{ab} acts on $\xi_{Q,P}$ by multiplication of Q from the right, an operation which commutes with the multiplication by R^{-1} from the left in (2.24)].

To understand the implications of (2.25), let us consider $\xi_{Q,0}$ as a function which assigns a complex number to every way of ordering N billiard balls, labeled 1,2, . . . , N, on a line. The relation (2.25) assigns a particular permutation symmetry to this function, corresponding, say, to a Young tableau with integers $g_1,g_2,\dots,g_{N-M}$ in the first column and $h_1,h_2,\dots,h_M$ in the second column. With this mixed permutation symmetry, the function will be antisymmetric with respect to permutations which move only balls $g_1,\dots,g_{N-M}$, and also antisymmetric for permutations of balls $h_1,\dots,h_M$ (see Appendix B for further discussion).

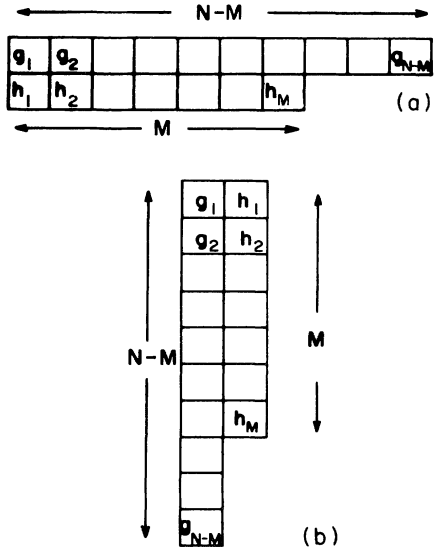

FIG. 8. (a) Young tableau $[N-M,M]$. The numbers $h_1,\dots,h_M$, $g_1,\dots,g_{N-M}$ form a permutation of $1,\dots,N$. It indicates a spin-S state with $+S=\frac{1}{2}(N-2M)$. (b) Young tableau $[2^M,1^{N-M}]$. It is conjugate to the tableau $[N-M,M]$.

E. Imposition of periodic boundary conditions

The final set of constraints on the Bethe-ansatz coefficients is provided by the periodic boundary conditions. To study these, we rewrite the wave function (2.9), collecting terms having the same exponential dependence on $x_1,\dots,x_N$:

$$F=\sum_{R\in S_N}F_R\ , \tag{2.26}$$

where

$$F_R=\exp\left[i\sum k_{Rj}x_j\right]\prod_i \delta_{\beta_i}^{\alpha_{Ri}}\sum_Q \xi_{Q,RQ}\theta(x_Q)\ .$$

Each F_R must separately satisfy the boundary conditions

$$F_R(\dots x_i=0\dots)=F_R(\dots x_i=L\dots)\ ,\quad i=1,\dots,N\ , \tag{2.27}$$

which imply

$$\xi_{Q,RQ}=\exp(ik_{RQ1}L)\xi_{\tilde{Q},R\tilde{Q}}\ , \tag{2.28}$$

where

$$Q=Q1\,Q2\dots QN\ ,$$
$$\tilde{Q}=Q2\,Q3\dots QN\,Q1\ .$$

Thus for all P and Q

$$\xi_{Q,P}=\exp(ik_{P1}L)\xi_{\tilde{Q},\tilde{P}}\ . \tag{2.29}$$

Expressing this as a relation between column vectors, we have

$$\xi_P=\exp(ik_jL)\mathscr{P}^{12}\mathscr{P}^{23}\cdots\mathscr{P}^{N-1,N}\xi_{\tilde{P}}\ , \tag{2.30}$$

where $j=P1$. If (2.30) is satisfied for any particular P, then it will be satisfied for any P with $j=P1$, since, if $P'=P1\,P2\dots P(a+1)\,Pa\dots PN$, then

$$\begin{aligned}\xi_{P'}&=Y^{a,a+1}_{Pa,P(a+1)}\exp(ik_jL)\mathscr{P}^{12}\mathscr{P}^{23}\cdots\mathscr{P}^{N-1,N}\xi_{\tilde{P}}\\&=\exp(ik_jL)\mathscr{P}^{12}\mathscr{P}^{23}\cdots\mathscr{P}^{N-1,N}Y^{a-1,a}_{Pa,P(a+1)}\xi_{\tilde{P}}\\&=\exp(ik_jL)\mathscr{P}^{12}\cdots\mathscr{P}^{N-1,N}\xi_{\tilde{P}}\ ,\end{aligned} \tag{2.31}$$

where we have used the commutation relation

$$\mathscr{P}^{ab}\mathscr{P}^{bc}=\mathscr{P}^{ac}\mathscr{P}^{ab} \tag{2.32}$$

to move the Y matrix to the right. It is convenient to choose as representative P with $P1=j$ in the following (where there is no confusion, we represent a permutation simply by the appropriate string of numbers in permuted order):

$$P=j\,1\,2\ldots j-1,j+1\ldots N\ ,$$
$$\tilde{P}=1\,2\ldots j-1,j+1\ldots Nj\ . \tag{2.33}$$

By using the Y matrices to write both ξ_P and $\xi_{\tilde{P}}$ in terms of ξ_0, we obtain, upon substitution into (2.30), a discrete eigenvalue equation for ξ_0. Thus

$$\xi_P=Y_{ij}^{12}\cdots Y_{j-2,j}^{j-2,j-1}Y_{j-i,j}^{j-i,j}\xi_0\ ,$$
$$\xi_{\tilde{P}}=Y_{jN}^{N-1,N}\cdots Y_{j,j+2}^{j+1,j+2}Y_{j+1}^{j,j+1}\xi_0\ ,$$

leads to

$$\begin{aligned}\exp(ik_jL)\xi_0=&\ Y_{j+1,j}^{j,j+1}Y_{j+2,j}^{j+1,j+2}\cdots\\ &\times Y_{N,j}^{N-1,N}\mathscr{P}^{N-1,N}\mathscr{P}^{N-2,N-1}\cdots\\ &\times\mathscr{P}^{12}Y_{ij}^{12}\cdots Y_{j-1,j}^{j-1,j}\xi_0\ .\end{aligned} \tag{2.34}$$

Note that we have used (2.16a) to throw all Y matrices onto the right-hand side of (2.34). We now commute $\mathscr{P}^{23}$, $\mathscr{P}^{34}$, $\ldots$, $\mathscr{P}^{j-1,j}$ to the right and $\mathscr{P}_{N-2,N-1}$, $\ldots$, $\mathscr{P}_{j,j+1}$ to the left to obtain

$$Z_j\xi_0=\exp(ik_jL)\xi_0\ , \tag{2.35}$$

where

$$Z_j=X_{j+1,j}X_{j+2,j}\cdots X_{Nj}X_{1j}\cdots X_{j-1,j} \tag{2.36a}$$

$$X_{ij}=\mathscr{P}^{ij}Y_{ij}^{ij}=\begin{cases}-\mathscr{P}_{ij}\ , & \alpha_i=\alpha_j\\ a_{ij}-b_{ij}\mathscr{P}^{ij}\ , & \alpha_i\neq\alpha_j\ .\end{cases} \tag{2.36b}$$

As a consequence of (2.16a)–(2.16c) the matrices X_{ij} satisfy

$$X_{ij}X_{ji}=1\ ,$$
$$X_{ij}X_{kl}=X_{kl}X_{ij}\ ,$$
$$X_{ij}X_{ik}X_{jk}=X_{jk}X_{ik}X_{ij}\ ,$$

for i,j,k,l all different. By a straightforward application of these identities, one can show that the unitary matrices commute with one another and hence can be simultaneously diagonalized. Actually, it is sufficient to diagonalize one of the Z_j (say, Z_N), as can be seen by substituting the explicit expressions for X_{ij} into (2.36b):

$$Z_j=\sum_c\left[\prod_{i\in c}a_{ij}\right]\left[\prod_{l\in c}b_{lj}\right](-1)^{|c|}\mathscr{P}_c\ ,$$

where the summation is over all subsets c of $\{1,2,\ldots,N\}$ with the property $i\in c\Rightarrow\alpha_i=\alpha_j$, and where $\mathscr{P}_c$ is the cyclic permutation of the elements $c_1,\ldots,c_{|c|}$ of c with $c_i>c_j$ for $i>j$, i.e., where $\mathscr{P}_c$ maps $c_n\to c_{n-1}\to\cdots c_1\to c_n$. Owing to the cyclic property of $\mathscr{P}_c$, we have

$$Z_i=Z_j\equiv Z_{\alpha_j}\ \text{ if }\alpha_i=\alpha_j\ .$$

But even the two matrices Z_1 and Z_0 are not independent: Another application of (2.32) yields

$$1=\prod_{j=1}^{N}Z_j=Z_1^{Ne}Z_0^{Ni}\ .$$

Consequently it will be sufficient to diagonalize any one of the matrices Z_j.

We now wish to convert (2.35) into a more convenient form by modifying slightly its permutation symmetry. If we define a column vector Φ with components

$$\Phi_Q=(-1)^{\delta(Q)}\xi_{Q,0}\ , \tag{2.37}$$

then (2.35) may be written

$$IZ_jI\Phi=\exp(ik_jL)\Phi\ ,$$

where I is the matrix with components

$$I_{QQ'}=(-1)^{\delta(Q)}\delta_{QQ'}\ .$$

Since

$$I\mathscr{P}^{ij}I=-\mathscr{P}^{ij}\ ,$$

Eq. (2.35) becomes

$$Z_j'\Phi=\exp(ik_jL)\Phi\ , \tag{2.38}$$

where

$$Z_j'=X_{j+1,j}'\cdots X_{Nj}'X_{1j}'\cdots X_{j-1,j}' \tag{2.39}$$

and

$$X_{ij}'=\begin{cases}\mathscr{P}^{ij}, & \alpha_i=\alpha_j\\ a_{ij}+b_{ij}\mathscr{P}^{ij}, & \alpha_i\neq\alpha_j\ ,\end{cases} \tag{2.40}$$

which we may write as follows:

$$X_{ij}'=e_{ij}\frac{i(\alpha_i-\alpha_j)+c\mathscr{P}^{ij}}{i(\alpha_i-\alpha_j)+c}\ , \tag{2.41}$$

where

$$c=\frac{2J}{1-J^2+\frac{1}{4}J''^2}\ ,\quad e_{ij}=e^{i(\alpha_i-\alpha_j)\phi}\ ,$$

$$e^{i\phi}=\frac{1-J^2+\frac{1}{4}J''^2-2iJ}{1+J^2-\frac{1}{4}J''^2+iJ''}\ . \tag{2.42}$$

We shall see that henceforth c will be the only effective coupling constant.

The advantage of the transformation to X' is that whereas ξ_0 satisfies (2.25) with coefficients a_R given by the tableau in Fig. 8(b), and hence is antisymmetric with respect to permutations of the numbers $g_1,\ldots,g_{N-M}$ appearing in the first column of a Young tableau, and also antisymmetric with respect to permutations of the numbers $h_1,\ldots,h_M$ in the second column, Φ is *symmetric* with respect to such permutations. Thus we may write

$$\Phi_Q=\phi(y_1,\ldots,y_M),\ y_1<y_2<\cdots<y_M\ , \tag{2.43}$$

where y_i is the position of the jth element of $\{h_1,\ldots,h_M\}$ which one encounters in running down the list of numbers $Q\,1,Q\,2,\ldots,Q_N$.

The matrix equation (2.38), as an eigenvalue equation for the discrete "wave function" $\phi(y_1,\ldots,y_M)$, resembles the eigenvalue problem originally solved by Bethe in 1931 for the one-dimensional Heisenberg magnet. Of course, in

Bethe's case, the role of our Z_j^i is played by the Hamiltonian with nearest-neighbor spin-spin interaction, seemingly a very different operator. Nevertheless, Bethe's strategy may be adapted to solve the eigenvalue problem (2.38), as Yang and Gaudin did for the nonrelativistic one-dimensional gas of fermions with δ-function interaction.

We now write down the modified Bethe-ansatz solution of the discrete eigenvalue problem, referring the interested reader to Appendixes A and B for a detailed proof of its validity.

F. Solution of the ansatz

The function $\phi(y_1,\ldots,y_M)$ is again expressible in a Bethe-ansatz form, and thus is given as a sum of products of "single-particle" wave functions $f(\Lambda,y)$. This wave function describes the situation when the lower row has only one box ($M=1$), and is of the form (see Appendix A)

$$f(\Lambda,y)=\prod_{j=1}^{y-1}\left[\frac{i(\alpha_j-\Lambda)+\frac{c}{2}}{i(\alpha_{j+1}-\Lambda)-\frac{c}{2}}\right] \tag{2.44}$$

where Λ plays the same role as the momentum k in the original Bethe ansatz (2.9).

The Bethe ansatz for $M>1$ is then given by

$$\phi(y_1,\ldots,y_M)=\sum_{P\in S_M} A_P \prod_{\gamma=1}^{M} f(\Lambda_{P\gamma},y_\gamma)\ , \tag{2.45}$$

where the coefficients A_P are defined uniquely, up to a common factor, by

$$\frac{A_{P'}}{A_P}=\frac{i(\Lambda_{P\gamma}-\Lambda_{P(\gamma+1)})+c}{i(\Lambda_{P\gamma}-\Lambda_{P(\gamma+1)}-c}\ ,$$
$$P'\gamma=P(\gamma+1),\quad P'(\gamma+1)=P\gamma\ , \tag{2.46}$$

and the parameters $\Lambda_i,\ldots,\Lambda_M$ must satisfy the coupled equation

$$\left[\frac{i(1-\Lambda_\gamma)+\frac{c}{2}}{i(1-\Lambda\gamma)-\frac{c}{2}}\right]^{N^e}\left[\frac{-i\Lambda_\gamma+\frac{c}{2}}{-i\Lambda_\gamma-\frac{c}{2}}\right]^{N^i}=-\prod_{\delta=1}^{M}\left[\frac{i(\Lambda_\delta-\Lambda_\gamma)+c}{i(\Lambda_\delta-\Lambda_\gamma)-c}\right]\ . \tag{2.47}$$

The function $\phi(y_1,\ldots,y_M)=\phi_Q$ thus defined, corresponds to the column $\xi_{Q,0}=\Phi_Q(-1)^{\delta(Q)}$, which in turn defines a wave function $F(x,\beta)$ with a permutation symmetry given by a Young tableau of two columns of length $N-M$ and M, respectively (see Appendix B). This in turn is conjugate to a spin-wave function $t(a)$ specified by a Young tableau of rows of lengths $N-M$ and M, respectively. For spin-$\frac{1}{2}$ wave functions this permutation symmetry determines the SU(2) spin properties (see, for example, Landau and Lifshitz, 1960a) which correspond in this case to total spin $S=\frac{1}{2}(N-2M)$. The wave function we constructed has maximum spin projection $S_z=S$. The other states, with $S_z<S$, are obtained by applications of the lowering operator S^-. This operator acts on the spin part of the wave function and has no effect on $F(x,\beta)$.

Finally, the eigenvalue $\lambda_j=e^{ik_jL}$ of Eq. (2.38) is given in terms of the parameters $\{\Lambda_1,\ldots,\Lambda_M\}$ determined from (2.47) by

$$\lambda_j=e^{ik_jL}=\prod_{i=1}^{N} e_{ij}\prod_{\gamma=1}^{M}\frac{i(\alpha_j-\Lambda_\gamma)+\frac{c}{2}}{i(\alpha_j-\Lambda_\gamma)-\frac{c}{2}}\ . \tag{2.48}$$

Our original operator diagonalization problem has now been reduced to a purely algebraic one: to solve Eq. (2.47) for $\Lambda_1,\ldots,\Lambda_M$. Once these have been determined, the corresponding wave function can be completely constructed, up to a normalization factor, using Eqs. (2.9), (2.21), (2.22), (2.36), (2.39), (2.40), (2.45), and (2.48). The energy momentum eigenvalues, on the other hand, will be given by (2.10) with each k_j obtained by taking the logarithm of (2.48).

Our task now is to find what solution of the $\{\Lambda_\gamma\}$ corresponds to the ground state, what the elementary excitations are, how the system responds when coupled to a magnetic field, and so on. This will be done in the following sections, to which the reader may turn at this point. Before concluding the present long section, we shall, however, discuss an alternate, though equivalent, formulation of the Bethe ansatz.

G. Alternate form of the Bethe ansatz—completeness

Thus far we have followed quite closely the approach of Yang. One may, however, use an equivalent form which simplifies the algebra somewhat and is convenient for establishing completeness of the basis. Let us start with the second quantized expression for the basis state; the expression

$$|\mathcal{F}\rangle=\int d^Nx\sum_{a,\beta}\sum_{Q,P}\xi_{Q,P}\theta(x_Q)\exp\left[i\sum_j k_{Pj}x_{Qj}\right]\times t_{a_1,\ldots,a_N}\prod_{l=1}^{N}\delta_{\beta_{Ql}}^{\alpha_{Pl}}\prod_i\psi_{a_i\beta_i}^\dagger(x_i)\,|0\rangle\ , \tag{2.49}$$

where

$$\theta(x_Q)=\theta(x_{Q1}<x_{Q2}<\cdots<x_{QN})$$

becomes, upon summing over $\beta_1,\ldots,\beta_N$,

$$|\mathcal{F}\rangle=\int d^Nx\sum_a\sum_{Q,P}\xi_{Q,P}\theta(x_Q)\exp\left[i\sum_i k_{Pj}x_{Qj}\right]t_{a_1,\ldots,a_N}\prod_i\psi^\dagger_{a_i\alpha_{PQ}-1_1}(x_i)\,|0\rangle$$
$$=\int d^Nx\sum_a\sum_{Q,P}\xi_{Q,PQ}\theta(x_Q)\exp\left[i\sum_i k_{Pj}x_j\right]t_{a_1,\ldots,a_N}\prod_i\psi^\dagger_{a_i\alpha_{Pi}}(x_i)\,|0\rangle\ . \tag{2.50}$$

But the $\psi^\dagger$'s may be reordered, using canonical anticommutation relations, to give

$$|\psi\rangle = \int d^N x \sum_a \sum_{Q,P} \xi_{Q,PQ}\theta(x_{PQ}) \exp\left[i\sum_i k_j x_j\right] t_{a_{P1},\ldots,a_{PN}}(-1)^{\delta(P)} \prod_i \psi^\dagger_{a_i\alpha_i}(x_i)|0\rangle$$
$$= \int d^N x \sum_a \sum_Q \phi^a_Q \theta(x_Q) \exp\left[i\sum_j k_j x_j\right] \prod_i \psi^\dagger_{a_i\alpha_i}(x_i)|0\rangle \ , \tag{2.51}$$

where

$$\phi^a_Q = \sum_P (-1)^{\delta(P)} \xi_{P^{-1}Q,Q} t_{a_{P1},\ldots,a_{PN}} \ . \tag{2.52}$$

We see that our original diagonalization problem can be rewritten as

$$hG = EG \ , \tag{2.53}$$

where h is given by (2.1) and

$$G_{a_1,\ldots,a_N}(x_1,\ldots,x_N) = \exp\left[i\sum_j k_j x_j\right] \sum_Q \phi^a_Q \theta(x_Q) \ , \tag{2.54}$$

with periodic boundary conditions with respect to each x_i. If we operate on G with the differential operator h, and demand the cancellation of terms proportional to $\delta(x_i - x_j)$, we obtain the analog of (2.14), namely,

$$\phi_{Q'} = S_{Qa,Q(a+1)}\phi_Q \ , \tag{2.55}$$

where S_{ij} has the same formal structure as X'_{ij}, except that S_{ij} acts in spin space,

$$S_{ij} = \begin{cases} e_{ij}\dfrac{i(\alpha_i-\alpha_j)+c\mathscr{P}^{\text{spin}}_{ij}}{i(\alpha_i-\alpha_j)+c}, & \alpha_i \neq \alpha_j \\ \mathscr{P}^{\text{spin}}_{ij}, & \alpha_i = \alpha_j \ , \end{cases} \tag{2.56}$$

where

$$(\mathscr{P}^{\text{spin}}_{ij})_{a_1,\ldots,a_N,a'_1,\ldots,a'_N} = \delta_{a_i a'_j}\delta_{a_j a'_i} \prod_{k\neq i,j} \delta_{a_k a'_k} \ .$$

The matrix S_{ij} has the interpretation of the two-particle S matrix for the collision of particle i (purity α_i, momentum k_i) with particle j (purity α_j, momentum k_j). The algebraic conditions which S_{ij} must satisfy in order for us to have a consistent Bethe-ansatz solution, namely [from (2.16)] (i,j,k,l all distinct),

$$S_{ij}S_{ji} = 1 \ ,$$
$$S_{ij}S_{kl} = S_{kl}S_{ij} \ , \tag{2.57}$$
$$S_{ij}S_{ik}S_{jk} = S_{jk}S_{ik}S_{ij} \ ,$$

are nothing but the factorizability conditions for the two-body S matrix.

Imposition of periodic boundary conditions leads to the discrete eigenvalue problem for ϕ_0,

$$Z^{\text{spin}}_j \phi_0 = \lambda_j \phi_0 = \exp(ik_j L)\phi_0, \quad j = 1,\ldots N \ , \tag{2.58}$$

where

$$Z^{\text{spin}}_j = S_{j+1\,j}\cdots S_{Nj}S_{1j}\cdots S_{j-1\,j} \ .$$

But the spinorial tensor ϕ_0 has components

$$\Phi_{a_1,\ldots,a_N} = \Phi(y_1,\ldots,y_M) \ ,$$

where y_j is the position of the jth down spin ($a_i = -\frac{1}{2}$), and so (2.58) gives us *precisely* the same eigenvalue problem as in the Yang approach.

Equation (2.52) gives a precise relation between solutions of the respective eigenvalue problems (2.34) and (2.58), with the same set of eigenvalues λ_j. In particular, it is a straightforward exercise to show that, given the satisfaction of (2.34) by ξ_0, then ϕ_0 defined by (2.52) is a solution of (2.58) with the same set of eigenvalues.

The form of the wave function (2.54) can be simplified even further if there is only one impurity. Then we may replace Q by a pair (Q,ν), where now Q refers only to the order of electrons on the line, and ν locates the impurity between electrons in positions ν and $\nu+1$. For convenience, we may shift the end points of our line segment to $\pm\frac{1}{2}L$, and fix the impurity at the origin. Exploiting the particularly simple action of S_{ij} for $\alpha_i = \alpha_j$, formula (2.54) becomes

$$G_{a_0,a_1,\ldots,a_{N_e}}(x_1,\ldots,x_{N_e}) = \exp\left[i\sum_{j=1}^{N_e} k_j x_j\right] \times \sum_Q \sum_{\nu=0}^{N_e} \phi^{(S_z)}_{\nu a_0 a_Q}\theta_\nu(x_Q) \ , \tag{2.59}$$

where $\phi^{(S_z)}_\nu = (S_-)^{S-S_z}\phi^{(S)}_\nu$,

$$\theta_\nu(x_Q) = \theta(x_{Q1} < x_{Q2} < \cdots < x_{Q\nu} < 0 < x_{Q(\nu+1)} \cdots < x_{QN_e}) \ ,$$

and a_0 is the impurity spin index.

The coefficients ϕ_ν for different values of ν may be expressed in terms of ϕ_0 by applying the periodic boundary condition,

$$\phi^{(S_z)}_{\nu a_0 a_Q} = \exp(ik_{Q1}L)\phi^{(S_z)}_{(\nu-1)a_0 a_{Q\tilde{P}}} \ , \tag{2.60}$$

where $\tilde{P}$ is the cyclic permutation $23\ldots N1$. Iterating (2.60), we get

$$\phi^{(S_z)}_{\nu a_0 a_Q} = \exp\left[i\sum_{j=1}^{\nu} k_{Qj}L\right]\phi^{(S_z)}_{0 a_0 a_{Q\tilde{P}^\nu}} \ , \tag{2.61}$$

where $Q\tilde{P}^\nu$ is the permutation $Q(\nu+1)Q(\nu+2)\ldots QN_eQ1\ldots Q\nu$. Moreover, from (2.58), we have

$$k_j = (iL)^{-1}\ln\lambda + 2\pi n_j/L \tag{2.62}$$

(where n_j is an integer) and thus (2.59) becomes

$$G_{a_0,a_1,\ldots,a_{N_e}}(x_1,\ldots,x_{N_e})=\exp\left[2\pi\sum_{j=1}^{N_e}n_jx_j/L\right]\sum_{Q,\nu}\theta_\nu(x_Q)\exp\left[(\ln\lambda)\left[\nu+\sum_j x_j/L\right]\right]\phi_{0a_0a_{Q\tilde{P}^\nu}}^{(S_z)}. \tag{2.63}$$

The completeness of the set of wave functions (2.63) in the Hilbert space $\mathscr{L}^2(\mathscr{R}^{N_e})C^{2^{(N_e+1)}}$ is relatively easy to establish. It is sufficient to show that any wave function of the product form

$$f(x_1,\ldots,x_{N_e})t_{a_0,a_1,\ldots,a_{N_e}} \tag{2.64}$$

can be expanded in the members (2.63) of our basis. Without loss of generality, we may assume $f(x_1,\ldots,x_{N_e})=0$ for $\theta_\nu(x_Q)=0$, some ν,Q.

That the tensors $\phi_0^{(S_z)}$ form a complete basis in spin space follows from the treatment of the analogous problem in the Heisenberg model (Takahashi, 1971). Permutating the indices by $Q\tilde{P}^\nu$ does not disturb the completeness property. Thus we may write

$$t_a=\sum_{\substack{\{\Lambda_1,\ldots,\Lambda_M\}\\ S_z}}c(\Lambda,S_z)\phi_{0a_0a_{Q\tilde{P}^\nu}}^{(S_z)}, \tag{2.65}$$

where the summation runs over $S=\frac{1}{2}N-M$ $(=0,1,2,\ldots)$, S_z $(=-S,-S+1,\ldots,+S)$ and the solution sets $\{\Lambda_1,\ldots,\Lambda_M\}$ of (2.39) with $N_i=1$. For fixed S,S_z and $\{\Lambda_1,\ldots,\Lambda_M\}$, we may expand

$$f(x_1,\ldots,x_{N_e})\exp\left[-(\ln)\lambda\right)\left[\nu+\sum_j x_j/L\right]\Big]$$

in a Fourier series,

$$\sum_{\{n_1,\ldots,n_{N_e}\}}d_{n_1,\ldots,n_{N_e}}\exp\left[2\pi\sum_{j=1}^{N_e}n_jx_j/L\right].$$

Thus, we have

$$f(x_1,\ldots,x_{N_e})t_{a_0,a_1,\ldots,a_{N_e}}=\sum_{\substack{\{\Lambda_1,\ldots,\Lambda_M\}\\ S_z}}c(\Lambda,S_z)\sum_{\{n_1,\ldots,n_{N_e}\}}d_{n_1,\ldots,n_{N_e}}G_{a_0,\ldots,a_{N_e}}^{(\Lambda,S_z,n)}(x_1,\ldots,x_{N_e}),$$

which is the desired expansion.

The completeness may be deduced also from S-matrix considerations. Thus the scattering matrix was calculated for the Gross-Neveu model (Andrei and Lowenstein, 1980b) and found to be complete and unitary, indicating that the Bethe states do, indeed, span the full Hilbert space.

III. THE SPECTRUM OF THE KONDO MODEL

In Sec. II we have derived the expressions determining the spectrum. Thus the energy is given as a sum of the electronic momenta,

$$E=\sum_{i=1}^{N^e}k_i^e, \tag{2.10'}$$

and the momenta are obtained from the eigenvalues λ

$$\lambda^e=e^{ik^eL}=\exp(-iN^i\phi)\prod_{\gamma=1}^{M}\frac{i(1-\Lambda_\gamma)+\frac{c}{2}}{i(1-\Lambda_\gamma)-\frac{c}{2}}, \tag{2.48'}$$

where we have set $\alpha=1$ in Eq. (2.48) to obtain electronic momenta k^e. The numbers $\Lambda_1,\ldots,\Lambda_M$, the Λ momenta or spin momenta, are in turn found from

$$\left[\frac{i(1-\Lambda_\gamma)+\frac{c}{2}}{i(1-\Lambda_\gamma)-\frac{c}{2}}\right]^{N^e}\left[\frac{-i\Lambda_\gamma+\frac{c}{2}}{-i\Lambda_\gamma-\frac{c}{2}}\right]^{N^i}=-\prod_{\delta=1}^{M}\frac{i(\Lambda_\delta-\Lambda_\gamma)+c}{i(\Lambda_\delta-\Lambda_\gamma)-c},\quad \gamma=1,\ldots,M. \tag{2.47}$$

A. The quantum numbers of the Kondo eigenstates

Casting the above expression into a more convenient form, we find (dropping inessential terms)

$$E=\sum_{j=1}^{N^e}\frac{2\pi}{L}n_j+D\sum_{\gamma=1}^{M}[\Theta(2\Lambda_\gamma-2)-\pi], \tag{3.1}$$

where $D=N^e/L$ is the electronic density and $\Theta(x)=-2\tan^{-1}x/c$ and where the spin momenta $\{\Lambda_\gamma\}$ satisfy the following set of coupled equations:

$$N^e\Theta(2\Lambda_\gamma-2)+N^i\Theta(2\Lambda_\gamma)=\sum_{\delta=1}^{M}\Theta(\Lambda_\gamma-\Lambda_\delta)-2\pi I_\gamma,\quad \gamma=1,\ldots,M, \tag{3.2}$$

where the n_j are integers and I_γ are integers (half integers) depending on $N-M$'s being odd (even) and arising when we take the logarithm of Eqs. (2.44) and (2.43).

Note that all electrons are equally shifted from their free value by $\phi(M,\{I_\gamma\})=\sum_{\gamma=1}^{M}[\Theta(2\Lambda_\gamma-2)-\pi]$. In other words, the phase shift of the electrons due to their interaction is independent of their motion. This fortunate circumstance is due to the fact that the coupling constant J is dimensionless and that the generalized phase shift matrix Y therefore cannot depend on the momentum. This is in contradistinction to the model treated by Yang and Gaudin, where the coupling constant has dimensions and the phase shift matrix is of the form $Y=Y(c/K_i-K_j)$. This, in turn, leads to a coupling of the equations that are analogous to (3.1) and (3.8), such that they cannot be solved analytically.

Each allowed choice (see below) of the integers n_j and I_γ uniquely determines an eigenstate of the Hamiltonian.

We shall refer to the $\{n_j, I_\gamma\}$ configurations as the quantum numbers of the state they determine. These quantum numbers replace, for example, the $\{n_j^+, n_j^-\}$ quantum numbers of the free fermion gas, where each level could be populated by a spin-up or a spin-down electron. In our case the quantum numbers n_j have no relevance to spin. Thus in the fermionic bookkeeping they can be only singly occupied, while in the bosonic all n_j may have the same value (see discussion related to the choice of $Y_{(\text{equal purity})}$). Indeed, we shall see that n_j are density fluctuation and I_γ spin fluctuation quantum numbers.

From the structure of the wave functions (Sec. II) we learn that if two of the Λ_γ's coincide, the wave function vanishes identically. Furthermore, as $|\Theta(x)| \leq \pi$, it is obvious that if an I_γ is too large there will be no solution to Eq. (3.2). The set of $\{I_\gamma\}$ such that there exists a solution with all Λ_γ momenta distinct is called an allowed configuration. Counting all allowed configurations (Takahashi, 1971), one can show that there are $\binom{N}{M} - \binom{N}{M-1}$ of them, as required by the dimension of the representation specified by the Young tableau $(N-M, M)$, namely, having an upper and lower row of lengths $N-M$ and M, respectively. The questions of what the ground-state configuration $\{I_\gamma^0\}$ is and what the associated spin $S = \frac{1}{2}(N-2M)$ is are dynamic ones and will be discussed soon.

Obviously, the spectrum is not bounded from below, as the integers n_j can take arbitrarily large and negative values. This is expected, since we linearized the spectrum.

To define the model we introduce a cutoff K, which we impose as follows:

$$\left| \frac{2\pi}{L} n_j \right| < K \tag{3.3}$$

(only the electronic n_j quantum numbers are cut off).

The cutoff K is imposed on the eigenstates of the fully interacting Hamiltonian and thus differs substantially from the conventional momentum cutoff $\mathscr{D}$ which is imposed on the eigenfunctions of the free Hamiltonian (Andrei and Lowenstein, 1979 and 1981; Andrei, 1980).

B. Discussion of the cutoff scheme

The choice of the K scheme is quite necessary if we want to preserve solubility in the presence of a finite cutoff. It respects the discontinuities that are present in the wave function (due to the δ-functions' interaction and the first-order derivatives) while ensuring finite energy. Thus the space of functions allowed by the two schemes is different, since the adoption of the conventional scheme allows only wave functions that are expressible in terms of a large but finite number of Fourier modes (free field eigenstates). This precludes discontinuities and an exact solution. In the scaling regime [$\mathscr{D} \to \infty, g(\mathscr{D}) \to 0$, where T_0 is fixed] the discontinuous functions may be reconstructed out of plane waves, and one may expect the two schemes to coincide. This expectation is formalized by the statement that the model, being a renormalizable field theory, could be regularized in various schemes, but if the cutoffs are removed and the model is renormalized the same way (namely, with the same physical quantities held fixed while the cutoff is removed), then one expects the various constructions to yield the same results. This is a statement of universality. Thus as we want to understand the low-temperature properties of the model, $T \ll \mathscr{D}$ (keeping the temperature low is equivalent to keeping the cutoff large or removing it altogether), we may use our favorite definition, which should yield the same results in the universal region.

This is obviously a strong statement which up to now has been rigorously proven only in perturbation theory. It was shown that the same Green's functions are obtained irrespective of whether one used Pauli-Villars regulators, Bogoliubov-Parasiuk-Hepp-Zimmermann subtractions, dimensional regularization, or any other of a slew of schemes available. We know of no proof applicable to nonperturbative constructions. We shall, however, provide ample evidence below that indeed this is the case (see Sec. VI). The same scheme has previously been used in the analysis of the back scattering model (Andrei and Lowenstein, 1979, 1980a, 1980b), and, again, the results obtained there coincide, in the universal regime, with results of other schemes.

One of the interesting points in the K scheme is that it leads to a coupling constant $J(K)$ which is nonanalytically related to the conventionally defined coupling constant $g(\mathscr{D})$ (a relation to be discussed in Sec. VI). This way we have shown that the universality class is actually larger than expected and that analyticity is not a necessary criterion for deciding the question.

We would like to emphasize, however, that once we are out of the universal regime the results of different constructions may differ substantially. Thus much more care needs to be taken in the choice of an appropriate scheme if one wishes to explore temperatures of the order of the cutoff.

C. The eigenstates

Returning now to the construction of the spectrum, we want to calculate the energy eigenvalues induced by each choice of the quantum numbers $\{n_j, I_\gamma\}$, and the transformation properties of the induced state under the action of the rotation group. We shall study the model in the thermodynamic limit $N^e \to \infty, D = N^e/L$ fixed.

1. The ground state

The state with the lowest energy will be a spin singlet, induced by a consecutive $\{I_\gamma^0\}$ configuration

$$I_{\gamma+1}^0 = I_\gamma^0 + 1 \ . \tag{3.4}$$

The n_j quantum numbers will be taken at the minimum allowed by the cutoff. Since we chose the fermionic bookkeeping, namely, $Y_{(\text{equal purity})} = -1$, all the n_j must be different; they form a consecutive sequence from

$n_0=-KL/2\pi$ and upwards. Had we chosen the bosonic bookkeeping $Y_{(\text{equal purity})}=+1$, as we did before (Andrei and Lowenstein, 1979; Andrei, 1980), then we could set $n_j=-KL/2\pi$ for all j. The physical results are identical, though the two choices correspond to different choices of bases of states in the Hilbert space. Another point to notice is that we have taken the Fermi surface to be at $p_F=0$. This corresponds to a relation between the depth of the Fermi sphere (namely, the cutoff K) and the density of electrons $D=N^e/L$. The relation is trivially given by $K=\pi D$, and we shall alternately refer to K or D to denote the cutoff scheme.

We turn now to the solution of Eq. (3.2) with the consecutive configuration (3.4). As we are interested only in the thermodynamic limit, $N,M\to\infty$, it is sufficient to determine the density of solutions $\sigma(\Lambda)$ induced by each configuration $\{I_\gamma\}$. Thus we define

$$\sigma(\Lambda_\gamma)=\frac{1}{\Lambda_{\gamma+1}-\Lambda_\gamma}\ . \tag{3.5}$$

In the case where all Λ solutions are real (which is the case of the ground state and also of maximum-spin excitations) we may rewrite Eqs. (3.1) and (3.2) as

$$E=\sum_{j=1}^{N^e}\frac{2\pi}{L}n_j+D\int d\Lambda\,\sigma(\Lambda)[\Theta(2\Lambda-2)-\pi] \tag{3.6}$$

$$N^e\Theta(2\Lambda_\gamma-2)+N^i\Theta(2\Lambda_\gamma)=\int d\Lambda'\sigma(\Lambda')\Theta(\Lambda_\gamma-\Lambda')-2\pi I_\gamma\ . \tag{3.7}$$

One may obtain an equation for the density of the ground state $\sigma_{\rm gs}$ by subtracting Eq. (3.7) written for Λ_γ from that written for $\Lambda_{\gamma+1}$ and expanding in the difference $\Delta\Lambda=\Lambda_{\gamma+1}-\Lambda_\gamma$ assumed to be of order $1/N$. One then finds

$$\sigma_{\rm gs}(\Lambda)=f(\Lambda)-\int K(\Lambda-\Lambda')\sigma_{\rm gs}(\Lambda')d\Lambda'\ , \tag{3.8}$$

where

$$f(\Lambda)=\frac{2c}{\pi}\left[\frac{N^e}{c^2+4(\Lambda-1)^2}+\frac{N^i}{c^2+4\Lambda^2}\right],$$
$$K(\Lambda)=\frac{c}{\pi}\frac{1}{c^2+\Lambda^2}\ . \tag{3.9}$$

Here we used $\Theta'(\Lambda)=-(2c/c^2+\Lambda^2)$ and the fact that, for the ground state, $I_{\gamma+1}-I_\gamma=1$ for all γ. The solution is easy to find,

$$\sigma_{\rm gs}(\Lambda)=\frac{1}{2c}\left[\frac{N^e}{\cosh\frac{\pi}{c}(\Lambda-1)}+\frac{N^i}{\cosh\frac{\pi}{c}\Lambda}\right]. \tag{3.10}$$

We can now determine the transformation properties of the ground state by calculating

$$M=\int\sigma_{\rm gs}(\Lambda)d\Lambda=\tfrac{1}{2}N\ . \tag{3.11}$$

Therefore, the state has a Young tableau of two equal-length rows. Regarding it now as determining the SU(2) representation, we recognize it as a singlet (Mattis, 1967).

The ground-state energy, on the other hand, is given by

$$E_{\rm gs}=D\int d\Lambda\,\sigma_{\rm gs}(\Lambda)[\Theta(2\Lambda-2)-\pi]+\sum\frac{2\pi}{L}n_j$$
$$=-\frac{\pi}{2L}(N^e)^2-N^iDi\ln\frac{\Gamma(1+ic)\Gamma(\frac{1}{2}-ic)}{\Gamma(1-ic)\Gamma(\frac{1}{2}+ic)}\ , \tag{3.12}$$

which is the ground-state energy of a free gas of fermions if $N^i=0$ or $c=0$.

To show that, indeed, this is the lowest energy state we shall study variations from the ground-state configuration $\{n_j^0,I_\gamma^0\}$. These correspond to elementary excitations.

2. Elementary excitations

a. Density excitations (particle hole). These are obtained by exciting an electron from the Fermi sphere. Thus we change a given n_j where $-K\le(2\pi/L)n_j<0$ to $n_j'=n_j+\Delta n\ge 0$. The change in energy involved is

$$\Delta E=\frac{2\pi}{L}\Delta n>0\ .$$

Obviously M, which depends only on the I_γ quantum numbers, did not change and neither did the spin. These are therefore massless density fluctuations which decouple from the rest of the spectrum. For a neat discussion of their origin see Witten (1978) [see also Andrei and Lowenstein (1979)].

b. Spin excitations. These are obtained by varying the $\{I_0\}$ sequence from its ground-state configuration. One way to modify it is to put "holes" into it, where by a "hole" we mean an integer omitted from the consecutive sequence. For example, we choose γ_0 such that $I_{\gamma_0+1}=I_{\gamma_0}+2$, and $I_{\gamma+1}=I_\gamma+1,\gamma\neq\gamma_0$. Thus Λ^h, the spin momentum corresponding to the omitted integer $I_{\gamma_0}+1$, constitutes a "hole." This means that we have to solve Eq. (3.1) or (3.7) in the presence of a (bare) hole density $\sigma^h(\Lambda)=\delta(\Lambda-\Lambda^h)$. [This hole density will be "dressed" by the "back flow of the Fermi sea" to give $\Delta\sigma^h(\Lambda)$.] The equation for the density $\sigma(\Lambda)$ now becomes

$$\sigma(\Lambda)+\sigma^h(\Lambda)=f(\Lambda)-\int K(\Lambda-\Lambda')\sigma(\Lambda')d\Lambda'\ , \tag{3.13}$$

where for the case of a few holes at Λ_i^h

$$\sigma^h(\Lambda)=\sum_{i=1}^{N^h}\delta(\Lambda-\Lambda_i^h)\ .$$

[A more precise derivation of Eq. (3.13) will be given below.] The solution to Eq. (3.13) is (in Fourier space)

$$\tilde\sigma(p)=\tilde\sigma_{\rm gs}(p)-\sum_{j=1}^{N^h}\frac{e^{-i\Lambda_j^hp}\exp\left[\frac{c}{2}|p|\right]}{2\cosh\frac{c}{2}p}\ . \tag{3.14}$$

The form of the change in the density (the dressed hole density)

$$\Delta\tilde{\sigma}(p)=-e^{-i\Lambda^h p}\frac{\exp\frac{c}{2}|p|}{2\cosh\frac{c}{2}p} \tag{3.15}$$

includes the effect of the back flow of the sea of spin momenta Λ. Since all Λ momenta are coupled through Eq. (3.2), removing one of them affects them all and leads to the redistribution given by $\Delta\tilde{\sigma}(p)$.

Now, M, the number of "down spins," or the length of the lower row in the Young tableau, is given by

$$M=\int\sigma(\Lambda)d\Lambda=\tilde{\sigma}(p=0)=\tfrac{1}{2}N-\tfrac{1}{2}N^h ,$$

so that each hole contributes $(\Delta M)_h=-\frac{1}{2}$. The simplest excitation, then, is made of two holes at Λ_1^h and Λ_2^h, say. It is a triplet, since $\Delta M=-1$, so that we move one box from the lower row to the upper row in the Young tableau (see Fig. 9). In the language of spin representations we have now a symmetrized product of two spin-half objects which yields a spin-one state—a triplet. The excitation energy ΔE^t is

$$\Delta E^t=D\int\Delta\sigma(\Lambda)[\Theta(2\Lambda-2)-\pi]d\Lambda$$
$$=2D(\tan^{-1}e^{(\pi/c)(\Lambda_1^h-1)}+\tan^{-1}e^{(\pi/c)(\Lambda_2^h-1)}) . \tag{3.16}$$

Thus ΔE^t is a sum of two terms carrying *spin-half* each and coupled to yield a spin-one state. We call the corresponding objects *dressed electrons.* These one-hole states cannot be obtained if we hold the number of electrons N^e fixed. To generate a one-hole state it is necessary to add an electron to the system whose total spin will then change to be spin-half (Fig. 10). The corresponding change in energy is, not too surprisingly, given by (see Fig. 11)

$$\Delta E^d=2D\tan^{-1}e^{(\pi/c)(\Lambda^h-1)}+\frac{2\pi}{L}n . \tag{3.17a}$$

In other words, for N odd, the ground state contains a spin-half excitation (n is the level into which the electron was inserted). This electronlike excitation (or chiral excitation in the language of the chiral Gross-Neveu model) is responsible for zero-temperature conduction and will be further discussed in Sec. IX. For the rest of the discussion N^e will be held fixed.

Returning to Eq. (3.17a), we learn, then, that the Kondo system has no mass gap, as we can have arbitrarily low-lying spin-flip excitation by choosing Λ^h arbitrarily large and negative, in which case the doublet excitation takes the form

$$\Delta E^d=2T_0e^{(\pi/c)\Lambda^h} . \tag{3.17b}$$

The scale $T_0=De^{-\pi/c}$, which we call the *fluctuation scale,* will play a fundamental role throughout this work.

We would like to point out that Λ^h is not a continuous, independent variable. It is determined by the $\{I_\gamma\}$ configuration and can occupy slots in the Λ plans which are determined by the choice of the omitted $I_{\gamma_0}+1$. This point will be discussed more below.

It was shown, thus far, that inserting two "holes" into the ground-state configuration $\{I_\gamma\}$ leads to a triplet excitation composed of two spin-half objects coupled symmetrically. Can we construct a state where they are coupled antisymmetrically to yield a singlet state? This question leads us to consider configurations of $\{I_\gamma\}$ that lead to complex Λ solutions. Obviously, these complex solutions occur in conjugate pairs. We shall show now that a sing-

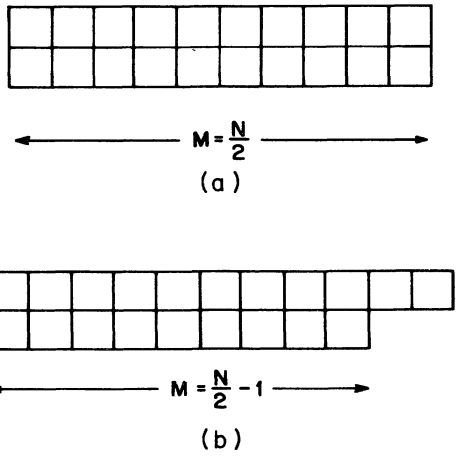

FIG. 9. (a) The ground-state tableau. It corresponds to a spin singlet. (b) An excited-state tableau. Here $\Delta M=-1$, so that a box has been moved from the lower to the upper row. The two boxes now represent a symmetrized product of the two spin-$\frac{1}{2}$ functions, thus a triplet.

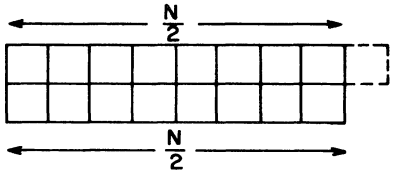

FIG. 10. A "dressed electron" excitation. An electron has been added to the system and the ground state has total spin-$\frac{1}{2}$. This is a one-hole excitation (spin-$\frac{1}{2}$ kink) and is present even in the ground state if N is odd.

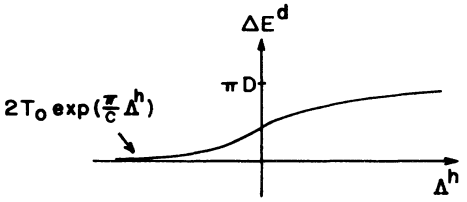

FIG. 11. The energy dependence on the hole Λ momentum for the fundamental spin-$\frac{1}{2}$ excitation. All elementary excitation energies are given as linear combinations of $\Delta E^d(\Lambda)$.

let excitation can be constructed and is composed of a "sea" of real spin momenta Λ with two holes, at Λ_1^h and Λ_2^h, and a two-string, namely, a pair of complex Λ located at $\Lambda^{\pm}=\bar{\Lambda}\pm ic/2$, where $\bar{\Lambda}=\frac{1}{2}(\Lambda_1^h+\Lambda_2^h)$ (see Fig. 12). That this indeed satisfies Eq. (2.43) in the limit $N\to\infty$ will be discussed below (see Appendix D).

The density of real Λ momenta $\sigma(\Lambda)$ now satisfies

$$\sigma_{\rm sing}(\Lambda)+\sigma^h(\Lambda)+\sigma^{\rm string}(\Lambda)=f(\Lambda)-\int K(\Lambda-\Lambda')\sigma_{\rm sing}(\Lambda')d\Lambda' ,$$

where

$$\sigma^h(\Lambda)=\delta(\Lambda-\Lambda_1^h)+\delta(\Lambda-\Lambda_2^h) ,$$

$$\sigma^{\rm string}(\Lambda)=+\frac{\frac{3}{2}c}{(\frac{3}{2}c)^2+(\Lambda-\bar{\Lambda})^2}+\frac{c}{(\frac{1}{2}c)^2+(\Lambda-\bar{\Lambda})^2} , \tag{3.18}$$

where σ^h arises, as before, from generating holes in the ground-state (consecutive) sequence. The two-string contribution $\sigma^{\rm string}(\Lambda)$ arises from the fact that in the singlet configuration we added a two-string at $\Lambda^{\pm}=\bar{\Lambda}\pm ic/2$, and thus the sum in Eq. (3.2) includes two more terms $\Theta(\Lambda_\gamma-\Lambda^+)+\Theta(\Lambda_\gamma-\Lambda^-)$. This contribution can be rewritten as $\Theta[2(\Lambda-\bar{\Lambda})]+\Theta[\frac{2}{3}(\Lambda-\bar{\Lambda})]$ with $\bar{\Lambda}$ real. When we convert this set of algebraic equations to an integral equation, by the method described before, we obtain Eq. (3.18), whose solution is (in Fourier space)

$$\tilde{\sigma}_{\rm sing}(p)=\sigma_{\rm gs}(p)+\Delta\sigma^{\rm string}(p)+\Delta\sigma^h(p) , \tag{3.19}$$

with

$$\Delta\tilde{\sigma}^h(p)=-\frac{\exp\frac{c}{2}|p|}{2\cosh\frac{c}{2}p}(e^{-i\Lambda_1^h p}+e^{-i\Lambda_2^h p}) ,$$

$$\Delta\tilde{\sigma}^{\rm string}(p)=-e^{-(c/2)|p|}e^{-i\bar{\Lambda}p} .$$

This state is indeed a singlet, as can be deduced from calculating

$$M=2+\int d\Lambda\,\sigma_{\rm sing}(\Lambda)=2+\tilde{\sigma}_{\rm sing}(0)$$
$$=2+\tilde{\sigma}_{\rm gs}(0)+\Delta\tilde{\sigma}^h(0)+\Delta\tilde{\sigma}^{\rm string}(0)=\tfrac{1}{2}N \tag{3.20}$$

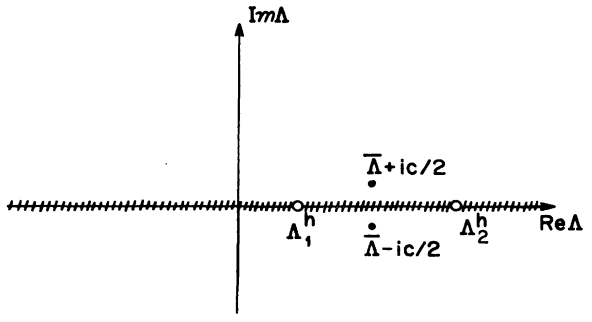

FIG. 12. The singlet configuration depicted in the Λ plane. It consists of a distribution (3.19) of Λ momenta on the real line (with two holes, at Λ_1^h and Λ_2^h) as well as a two-string at $\Lambda^{\pm}=\bar{\Lambda}\pm ic/2$, where $\bar{\Lambda}=\frac{1}{2}(\Lambda_1^h+\Lambda_2^h)$.

[where the term $(+2)$ is contributed by the string which consists of 2Λ-momenta $\Lambda^{\pm}=\bar{\Lambda}\pm ic/2$] so that the Young tableau has two rows of equal length.

The change in the energy, $\Delta E^{\rm sing}$, is also easy to calculate

$$\Delta E^{\rm sing}=D\int[\Delta\sigma^h(\Lambda)+\Delta\sigma^{\rm string}(\Lambda)][\Theta(2\Lambda-2)-\pi]d\Lambda$$
$$+D[\Theta(2\Lambda^+-2)+\Theta(2\Lambda^--2)-2\pi]$$
$$=2D(\tan^{-1}e^{(\pi/c)(\Lambda_1^h-1)}+\tan^{-1}e^{(\pi/c)(\Lambda_2^h-1)}) , \tag{3.21}$$

where we used

$$\int d\Lambda\,\Delta\sigma^{\rm string}(\Lambda)[\Theta(2\Lambda-2)-\pi]+\Theta(2\Lambda^+-2)+\Theta(2\Lambda^--2)=0 .$$

The string induces a change in the sea of real Λ momenta which exactly cancels the direct contribution [the same phenomenon occurs in the backscattering model (Andrei and Lowenstein, 1979) or in the massive Thirring model (Bergknoff and Thacker, 1978)].

Although the excitation energy has the same form for a singlet and triplet excitation, it does not follow that the interaction between the dressed electrons is spin independent. Indeed, one can calculate the phase shift when these collective excitations cross each other and find that they are different in the singlet case and the triplet case (Andrei and Lowenstein, 1980).

These features are general. Elementary excitations are made of holes and complex pairs. If we have only real Λ momenta with n holes in the Λ sea, then the state will have spin $S=\frac{1}{2}n$. (For N^e fixed n must be even.) These are maximum-spin excitations. The energy associated with each hole is given in Eq. (3.17), so for a state with holes at $\Lambda_1^h,\ldots,\Lambda_n^h$ we have

$$(\Delta E)^{n\ \rm holes}=\sum_{j=1}^{M}D\tan^{-1}(e^{(\pi/c)(\Lambda_j^h-1)}) .$$

When complex pairs are added, they couple spins to lower total spin. The various complex structures allowed will be discussed below, but they all have the feature that their contribution to the energy cancels, so that it is determined only by the holes in the sea of real Λ momenta. The contribution of these complexes shows up, however, not only in the counting of states and their total spin but also in the S matrix determining the interaction of the various excitations. All these features appear also in other models.

As remarked in the Introduction, the Kondo model is intimately related to the Gross-Neveu and Heisenberg models, all of them describing particles interacting via spin exchange. Using the Bethe ansatz, one arrives at equations of the type (3.2),

$$N^\alpha\Theta(2\Lambda_\gamma-2\alpha)+N^\beta\Theta(2\Lambda_\gamma-2\beta)=\sum_\delta\Theta(\Lambda_\gamma-\Lambda_\delta)-2\pi I_\gamma .$$

In the backscattering model $\alpha=1$ and $\beta=-1$ to indicate

left and right movers, in the Kondo problem $\alpha=1$ and $\beta=0$ to indicate a right mover (electron) and a stationary particle (impurity), and in the Heisenberg model (see, for example, Takahashi, 1971) $\alpha=\beta=0$, as all spins are stationary.

It is therefore not surprising that these three models have the identical types of excitation. Thus the Gross-Neveu (GN) model (Andrei and Lowenstein, 1979) has a singlet ground state induced by the consecutive configuration $I_{\gamma+1}=I_\gamma+1$. The triplet excitation is composed of a two-hole state, namely, a spin-$\frac{1}{2}$ object (dressed electrons) which may also be coupled to give an excited singlet by adding a two-string.

The energy of each dressed electron is given by

$$\Delta E^d_{\rm GN}=D\int d\Lambda\,\Delta\sigma(\Lambda)[\Theta(2\Lambda-2)-\Theta(2\Lambda+2)-2\pi]$$
$$=D\left[\tan^{-1}\left[\exp\frac{\pi}{c}(\Lambda^h-1)+\tan^{-1}\exp\frac{\pi}{c}(\Lambda^h-1)\right]\right],$$

with $\Delta\sigma(\Lambda)$ given by Eq. (3.15) (the same $\Delta\sigma$ appears also in the Heisenberg model) and with Λ^h the position of the hole. The Kondo form for the energy carried by each Λ momentum, namely, $g^K(\Lambda)=[\Theta(2\Lambda-2)-\pi]$, is replaced here by $g^{\rm GN}=[\Theta(2\Lambda-2)-\Theta(2\Lambda+2)-2\pi]$, as the energy is carried by both left and right movers. In the scaling regime the excitation energy assumes a relativistic form,

$$\Delta E^d_{\rm GN}=m\cosh\left[\frac{\pi}{c}\Lambda^h\right],$$

with $m=De^{-\pi/c}$.

These massive spin-$\frac{1}{2}$ particles interact via the following S matrix (Andrei and Lowenstein, 1980):

$$S^{\rm (triplet)}_{\rm GN}=\frac{\Gamma\left[\frac{1}{2}+i\frac{\chi}{2\pi}\right]\Gamma\left[-i\frac{\chi}{2\pi}\right]}{\Gamma\left[\frac{1}{2}-i\frac{\chi}{2\pi}\right]\Gamma\left[\frac{i\chi}{2\pi}\right]},$$
$$S^{\rm (singlet)}_{\rm GN}=\frac{1+i\frac{\chi}{\pi}}{1-i\frac{\chi}{\pi}}S^{\rm (triplet)},$$

where $\chi=(\pi/c)(\Lambda_1-\Lambda_2)$ is the relative rapidity.

In the Heisenberg case matters are identical. The ground state (for the antiferromagnetic case) is given by a consecutive configuration, and the excitations consist of holes and strings (Takahashi, 1971). The triplet excitation is formed out of two holes in the sequence. These, again, are spin-$\frac{1}{2}$ objects, usually referred to as *spin waves* in this context, and they may be coupled to give a singlet by adding a two-string between them.

The energy of each hole is

$$\Delta E^d_{\rm Heisenberg}=\int\Delta\sigma(\Lambda)\left[\frac{-2J}{1+\Lambda^2}\right]d\Lambda=\frac{2J}{\cosh\frac{\pi}{2}\Lambda^h},$$

with $\Delta\sigma$ given as before by Eq. (3.15), and $g_H=-2J(1/1+\Lambda^2)$ is the corresponding energy function for the Heisenberg model. The scattering matrix for two spin waves is obtained from the corresponding expression in the backscattering model replacing c by 2 in the expression for χ. Thus

$$S^{\rm (trip)}_H=\frac{\Gamma\left[\frac{1}{2}+i\frac{\Lambda}{2\pi}\right]}{\Gamma\left[\frac{1}{2}-i\frac{\Lambda}{2\pi}\right]}\frac{\Gamma\left[-i\frac{\Lambda}{2\pi}\right]}{\Gamma\left[i\frac{\Lambda}{2\pi}\right]},$$
$$S^{\rm (sing)}_H=\frac{1+i\frac{\Lambda}{\pi}}{1-i\Lambda\pi}S^{\rm (triplet)}_H.$$

[The Heisenberg model spectrum was recently also considered by Faddeev and Takhatajan (1981) with the same conclusions.]

The S-matrix concept in the Kondo model is more delicate and will be discussed in Sec. IX where it will be used to find the magnetoresistance of the model.

In what follows we shall develop the Kondo formalism and occasionally remark on correspondences in the Heisenberg model (usually well known) and in the backscattering model (usually new).

D. Strings and holes

In this segment we shall discuss holes and strings more carefully following Yang and Yang (1969), Takahashi (1971), and Gaudin (1971). Consider again Eq. (3.2), which determines the Λ momenta for a given sequence $\{I_\gamma\}$. At the moment we choose $\{I_\gamma\}$ to induce only real Λ momenta. Having determined the allowed momenta by solving the equations we can form the function

$$v(\Lambda)=-\frac{1}{2\pi}\left[N^e\Theta(2\Lambda-2)+N^i\Theta(2\Lambda)-\sum_{\delta=1}^{M}\Theta(\Lambda-\Lambda_\delta)\right], \quad (3.22)$$

where the Λ_δ's occurring in the sum are those that have been determined before. Those values of the variable Λ that satisfy

$$v(\Lambda_\gamma)=I_\gamma,$$

where I_γ is an integer belonging to the sequence, are just the allowed momenta, while those values of Λ satisfying

$$v(\Lambda^h_\varepsilon)=J_\varepsilon,$$

where J_ε are the integers omitted from the $\{I_\gamma\}$ sequence, are the holes. Introducing the distribution functions $\sigma(\Lambda)$ and $\sigma^h(\Lambda)$ of the Λ momenta and Λ-momenta holes, respectively, we can rewrite Eq. (3.2) as follows:

$$\frac{1}{2\pi}[N^e\Theta(2\Lambda-2)+N^i\Theta(2\Lambda)]=v(\Lambda)+\frac{1}{2\pi}\int d\Lambda'\Theta(\Lambda-\Lambda')\sigma(\Lambda'), \quad (3.23a)$$

$$\frac{dv}{d\Lambda}=\sigma(\Lambda)+\sigma^h(\Lambda)\,, \tag{3.23b}$$

where the last equation follows from the definitions of the function v and of the density functions σ and σ^h. Indeed, the number of holes and Λ's in the interval $d\Lambda$ is given by $[\sigma(\Lambda)+\sigma^h(\Lambda)]d\Lambda$ and by the number of values of J_δ and I_γ which $v(\Lambda)$ takes as it ranges over the interval $d\Lambda$. We thus have

$$f(\Lambda)-\int K(\Lambda-\Lambda')\sigma(\Lambda')d\Lambda'=\sigma(\Lambda)+\sigma^h(\Lambda)\,, \tag{3.24}$$

which we used before.

To discuss complex Λ momenta we make the following "string hypothesis": the solutions of Eq. (3.2) in the limit $N\to\infty$ always occur in the form of n-strings, where an n-string is a complex of n Λ momenta given by

$$\Lambda_j^{(n)}=\Lambda^{(n)}+i\frac{c}{2}(n+1-2j),\quad j+1,2,\ldots,n\,. \tag{3.25}$$

This hypothesis has a long history (Bethe, 1931; Katsura, 1965; Ovchinnikov, 1967; Takahashi, 1971; Lai, 1971) and has been successful in all its applications. Recently it has been put to scrutiny (Destri and Lowenstein, 1981; Woynarovich, 1982) and it was shown that while the hypothesis holds in the presence of a macroscopic number of holes, it is not necessarily true in case we consider excitations containing only a small number of them. Two-string solutions always exist (when two holes or more are present), but the conjugate pairs organize themselves into n-strings, $n>2$, only if driven macroscopically. Thus more care need be taken when analyzing scattering events of elementary excitations than in thermodynamic applications, where, in studying the response of the system to external probes, we excite a sufficiently large number of holes and the string hypothesis is valid.

Let us develop then the form that Eq. (2.43) or Eq. (3.2) takes in this case. Consider the case where we have M_n n-strings $\Lambda_{\gamma,j}^{(n)}=\Lambda_\gamma^{(n)}+i(c/2)(n+1-2j)$, $\gamma=1,\ldots,M_n$ with $\Lambda_\gamma^{(n)}$ being the real part of the γth n-string. Equation (3.2) then takes the following form:

$$N^e\Theta\left[\frac{2\Lambda_\gamma^{(n)}-2}{n}\right]+N^i\Theta\left[\frac{2\Lambda_\gamma^{(n)}}{n}\right]=\sum_{m,\delta}{}'\Theta_{n,m}(\Lambda_\gamma^{(n)}-\Lambda_\delta^{(m)}),-2\pi I_\gamma^{(n)}\,, \tag{3.26}$$

where the summation is over all strings different from the particular $\Lambda_\gamma^{(n)}$ string. We consider real Λ momenta as one-strings. The function $\Theta_{mn}(x)$ is defined as

$$\Theta_{nm}(x)=\begin{cases}\Theta\left[\dfrac{2x}{|n-m|}\right]+2\Theta\left[\dfrac{2x}{|n-m|+2}\right]+\cdots+2\Theta\left[\dfrac{2x}{n+m-2}\right]+\Theta\left[\dfrac{2x}{n+m}\right], & n\neq m\\ 2\Theta\left[\dfrac{x}{2}\right]+\cdots+2\Theta\left[\dfrac{x}{2n-2}\right]+\Theta\left[\dfrac{x}{2n}\right], & n=m\end{cases} \tag{3.27}$$

and as a reminder $\Theta(x)=-2\tan^{-1}x/c=\Theta_{11}(x)$. Equation (3.26) is obtained by using Eq. (3.2) and summing over all members of a string. Thus, for example,

$$\sum_{j=1}^{n}\Theta(\Lambda-\Lambda_j^{(n)})=\frac{1}{i}\sum_{j=1}^{n}\ln\frac{c-i(\Lambda-\Lambda_j^{(n)})}{c+i(\Lambda-\Lambda_j^{(n)})}=\Theta\left[\frac{2}{n+1}(\Lambda-\Lambda^n)\right]+\Theta\left[\frac{2}{n-1}(\Lambda-\Lambda^n)\right],$$

$$\sum_{j=1}^{n}\Theta(2\Lambda_j^{(n)}-2)=\Theta\left[\frac{2\Lambda^{(n)}-2}{n}\right].$$

The integers $I_\gamma^{(n)}$ determine the allowed string solutions $\Lambda_\gamma^{(n)}$, and are the spin quantum numbers of the system.

The rest is just as before. For a chosen set of configurations $\{I_\gamma^{(n)}\}$ one determines the corresponding $\Lambda_{\gamma,j}^{(n)}$ complex spin momenta and then forms the function

$$v_n(\Lambda)=\frac{1}{2\pi}\left[N^e\Theta\left[\frac{2\Lambda-2}{n}\right]+N^i\Theta\left[\frac{2\Lambda}{n}\right]-\sum_{(m,\delta)}\Theta_{nm}(\Lambda-\Lambda_\delta^{(m)})\right], \tag{3.28}$$

with the $\Lambda_\delta^{(m)}$ determined earlier, so that solutions, $\Lambda_\gamma^{(n)}$, of $v_n(\Lambda_\gamma^{(n)})=I_\gamma^{(n)}$ are the allowed strings, while solutions $\Lambda_\gamma^{h(n)}$ of

$$v_n(\Lambda_\gamma^{h(n)})=J_\gamma^{(n)}\,,$$

where $J_\gamma^{(n)}$ are the integers omitted in the sequence, correspond to n-string holes. In the limit of $N\to\infty$ we may introduce n-string density $\sigma_n(\Lambda)$ and n-string hole density $\sigma_n^h(\Lambda)$, which obviously satisfy

$$\frac{dv_n(\Lambda)}{d\Lambda}=-[\sigma_n^h(\Lambda)+\sigma_n(\Lambda)] \tag{3.29}$$

and together with Eq. (3.28) [by taking a derivative with respect to Λ of Eq. (3.28) and combining with (3.29)] yield

$$f_n(\Lambda)=\sigma_n^h(\Lambda)+\sum_{m=1}^{\infty}A_{nm}\sigma_m(\Lambda)\,, \tag{3.30}$$

where

$$f_n(\Lambda)=\frac{nc}{2\pi}\left[\frac{N^e}{\left[\frac{nc}{2}\right]^2+(\Lambda-1)^2}+\frac{N^i}{\left[\frac{nc}{2}\right]^2+\Lambda^2}\right], \tag{3.31a}$$

and A_{nm} is an operator defined by

$$A_{nm}=[\,|n-m|\,]+2[\,|n-m|+2]+\cdots+2[n+m-2]+[n+m]\,, \tag{3.31b}$$

where $[n]$ is a linear integral operator given by

$$[n]f(\Lambda)=\frac{1}{\pi}\int\frac{\frac{nc}{2}}{\left[\frac{nc}{2}\right]^2+(\Lambda-\Lambda')^2}f(\Lambda')d\Lambda'\,,$$

or where in Fourier space $[n]$ is multiplicative and given by $[n](p)=e^{-nc|p|/2}$. Using this notation, we may rewrite the one-string equation (3.13) as

$$f_1(\Lambda)=\sigma_1^h(\Lambda)+A_{11}\sigma_1(\Lambda)\,. \tag{3.13'}$$

In terms of the string variables $\Lambda_\gamma^{(n)}$ the energy can be expressed as

$$E=\sum_n D\int d\Lambda\,\sigma_n(\Lambda)\left[\Theta\left[\frac{2\Lambda-2}{n}\right]-\pi\right]+\sum_j\frac{2\pi}{L}n_j\,.$$

Here we have performed the sum over the individual members of a string and are left with integration over the string locations only.

In the following sections we shall study the response of the system to macroscopic probes such as the external magnetic field, finite temperature, and the applied electric field.

IV. THE ZERO-TEMPERATURE MAGNETIZATION OF THE SPIN-$\frac{1}{2}$ KONDO MODEL

In this section we discuss, in the framework of the exact solution, the zero-temperature magnetization for all values of the magnetic field H in the scaling regime, $H\ll D$. (Here we adopt the convention that H is expressed as multiple of μ, the magnetic moment of the electron, or of the impurity, which we take to be the same.) We shall find that the magnetization $\mathcal{M}$ scales and becomes a universal function $\mathcal{M}=\mathcal{M}(H/T_0)$ in the scaling regime. Here $T_0=T_0(D,J)$ is a scale held fixed when we remove the cutoff (Anderson *et al.*, 1970). The universality of the function means here that various constructions which lead to the same value of T_0—$T_0=0.0001$ eV, say—will have the same magnetization curve even if they have different functional dependence on the coupling constant.

We shall also see that the impurity part of the magnetization $\mathcal{M}^i$ exhibits a crossover behavior. For high magnetic field, $T_0\ll H\ll D$, the magnetization $\mathcal{M}^i$ approaches that of a free spin with logarithmic corrections accessible by perturbative theory. Lowering the magnetic field, we cross over to a strong coupling regime with a screened impurity.

The methods that we use were developed in the context of the Heisenberg model (Griffith, 1964, Yang and Yang, 1966). They were applied to the Kondo problem in Andrei (1980) for small magnetic fields and in Wiegmann (1980) and Andrei and Lowenstein (1981) for arbitrary magnetic fields. In the last reference the universal number W' (see Introduction), characterizing the crossover, was also deduced.

A. The magnetization equation

In the presence of a magnetic field we have to add to the Hamiltonian a magnetic term

$$\mathcal{H}_{\rm mag}=-2SH\,,$$

where S is the total spin component of the system in the direction of H. Since this term commutes with the Kondo Hamiltonian, the eigenstates constructed in Sec. III will also diagonalize $\mathcal{H}_{\rm total}=\mathcal{H}_{\rm Kondo}+\mathcal{H}_{\rm mag}$. However, the ground state will be different. As a result of the magnetic term, the system will gain energy by flipping spins to align with the magnetic field. Each flipped spin corresponds to two holes in the Λ sea, two spin-$\frac{1}{2}$ objects (dressed electrons) indicative of two broken bonds. The excitation energy of each hole is given by Eq. (3.17),

$$\Delta E^{\rm hole}=2D\tan^{-1}\left[\exp\frac{\pi}{c}(\Lambda^h-1)\right]$$

(see Fig. 11). This expression can be made arbitrarily close to zero by choosing Λ^h sufficiently large and negative. But as all Λ momenta must be distinct, we shall be led to a depletion region where no Λ solution exists from $\Lambda=-\infty$ to $\Lambda=B$. B is determined by an equilibrium argument equating the magnetic energy gained by flipping spins to align with the magnetic field and the energy cost of these holes. Using this argument, we shall later determine $B=B(H)$, the dependence of the Λ-sea Fermi level, B, on the magnetic field H.

It is obvious that no complex strings are excited, as these reduce the total spin of the system. For example, consider the fundamental singlet and triplet excitations discussed in Sec. II. While their cost in interaction energy is the same, the singlet does not gain magnetic energy.

We thus have to consider the following magnetization equation:

$$\sigma_B(\Lambda)+\int_B^\infty K(\Lambda-\Lambda')\sigma_B(\Lambda')d\Lambda'=f(\Lambda)\,, \tag{4.1}$$

where, as before

$$K(x)=\frac{1}{\pi}\frac{c}{c^2+x^2}\,,$$
$$f(x)=\frac{N^e}{\pi}\frac{(c/2)}{(c/2)^2+(x-1)^2}+\frac{N^i}{\pi}\frac{(c/2)}{(c/2)^2+x^2}\,. \tag{4.2}$$

The energy and total spin of the system are given by

$$E_B=D\int_B^\infty\sigma_B(\Lambda)[\Theta(2\Lambda-2)-\pi]+\sum_j\frac{2\pi}{L}n_j-2HS \tag{4.3}$$

and

$$S=\tfrac{1}{2}N-M=\tfrac{1}{2}N-\int_B^\infty\sigma_B(\Lambda)d\Lambda\,, \tag{4.4}$$

where M, as before, is the total number of down spins.

The quantum numbers $\{n_j\}$ have no spin content and are not excited by the magnetic field.

Our task then is to solve for the Λ density $\sigma_B(\Lambda)$. By minimizing the energy we shall determine the parameter B in terms of the magnetic field H and thus finally find the magnetization curve $\mathcal{M}=\mathcal{M}(H)$, where $\mathcal{M}=2\mu S$.

B. The Wiener-Hopf technique

Equation (4.1) is a Wiener-Hopf integral equation and has been discussed using this technique by Yang and Yang in connection with a similar problem in the Heisenberg model. It is amusing to point out that while Eq. (4.1) is exact in the case of the Kondo model, it is only approximate in the Heisenberg case. The reason is simple. The excitation spectrum is asymmetric in Λ momenta in our case (see Fig. 11) and is symmetric in the Heisenberg (or backscattering model). Thus, while we have only one depletion region, $[-\infty,B]$, there are two in the symmetric cases, namely, holes will be excited in the regions $[-\infty,-B]$ and $[B,+\infty]$ for the Heisenberg model (or in the region $[-B,B]$ in the backscattering model). The resulting equation is of the Wiener-Hopf form only if the second depletion region is ignored, which is valid for small magnetic fields (Yang and Yang, 1966).

We proceed to discuss the application of the Wiener-Hopf technique [see, for example, Morse and Feshbach (1953)] to the Kondo model.

While $\sigma_B(\Lambda)$ has no direct physical meaning for $\Lambda<B$, we may use Eq. (4.1) to define it there. We shift the origin, by introducing $\rho(\Lambda)=\sigma_B(\Lambda+B)$, which leads to

$$\rho(\Lambda)+\int_0^\infty K(\Lambda-\Lambda')\rho(\Lambda')d\Lambda'=f(\Lambda+B)\ . \tag{4.5}$$

Let us define

$$\rho_\pm(\Lambda)=\theta(\pm\Lambda)\rho(\Lambda)\ , \tag{4.6}$$

with $\theta(\Lambda)$ being a step function. The magnetization equation, Eq. (4.5), becomes, in Fourier space,

$$[1+\tilde{K}(p)]\tilde{\rho}_+(p)+\tilde{\rho}_-(p)=\tilde{f}(p)e^{ipB}\ , \tag{4.7}$$

where

$$\tilde{f}(p)=\int_{-\infty}^{\infty}dx\,e^{-ipx}f(x)$$
$$=N^e e^{-(c/2)|p|}e^{-ip}+N^i e^{(c/2)|p|}\ , \tag{4.8a}$$

$$1+\tilde{K}(p)=2e^{-(c/2)|p|}\cosh\left[\frac{c}{2}p\right]\ . \tag{4.8b}$$

We now wish to factorize the kernel,

$$1+\tilde{K}(p)=\frac{K_+\left[\frac{c}{2\pi}p\right]}{K_-\left[\frac{c}{2\pi}p\right]}\ , \tag{4.9}$$

in such a way that K_+ (K_-) has singularities only in the upper (lower) half plane of the complex p plane (see Morse and Feshbach, 1953). The factorization is given by

$$K_+(q)=K_-^{-1}(-q)$$
$$=\frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2}+iq)}\exp\left[-iq\left[1+\frac{i\pi}{2}-\ln(-q+i\varepsilon)\right]\right]\ . \tag{4.10}$$

Equation (4.7) can then be rewritten as

$$K_-\left[\frac{cp}{2\pi}\right]\tilde{\rho}_-(p)+K_+\left[\frac{cp}{2\pi}\right]\tilde{\rho}_+(p)=K_-\left[\frac{cp}{2\pi}\right]\tilde{f}(p)e^{ipB}$$
$$=K_+\left[\frac{cp}{2\pi}\right]\tilde{g}(p)p^{ipB}\ , \tag{4.11}$$

where

$$\tilde{g}(p)=\frac{\tilde{f}(p)}{1+\tilde{K}(p)}=\frac{N^e e^{-ip}+N^i}{2\cosh\left[\frac{c}{2}p\right]}\ . \tag{4.12}$$

We shall find occasion to use the two versions of the right-hand side (rhs) of Eq. (4.11).

The next step is to express Eq. (4.11) as a sum of two terms $q_\pm(p)$ such that $q_+(p)$ [$q_-(p)$] has singularities only in the upper (lower) half p plane. In that case

$$K_+(p)\tilde{\rho}_+(p)-q_+(p)=K_-(p)\tilde{\rho}_-(p)-q_-(p)=0\ , \tag{4.13}$$

and we may solve for $\tilde{\rho}_\pm(p)$,

$$\tilde{\rho}_\pm(p)=\frac{q_\pm(p)}{K_\pm(p)}\ . \tag{4.14}$$

We proceed to find $q_\pm(p)$. Our strategy is to Laplace-transform $f(\Lambda)$ and $g(\Lambda)$ and consider each amplitude separately. We have

$$f(\Lambda)=\frac{1}{\pi}\int_0^\infty dt\sin\frac{ct}{2}(N^e e^{-|\Lambda-1|t}+N^i e^{-|\Lambda|t})\ , \tag{4.15a}$$

$$g(\Lambda)=\frac{1}{2c}\left[\frac{N^e}{\cosh\frac{\pi}{c}(\Lambda-1)}+\frac{N^i}{\cosh\frac{\pi}{c}\Lambda}\right]$$
$$=\frac{N^e}{c}\sum_{k=0}^{\infty}(-1)^k e^{-(\pi/c)(2k+1)|\Lambda-1|}$$
$$+\frac{N^i}{c}\sum_{k=0}^{\infty}(-1)^k e^{-(\pi/c)(2k+1)|\Lambda|}\ . \tag{4.15b}$$

Replacing $f(\Lambda)$ in the rhs of Eq. (4.11) by a single amplitude $f_t(\Lambda)=e^{-|\Lambda|t}$ whose Fourier transform is

$$\tilde{f}_t(p)=\frac{2t}{p^2+t^2}=i\left[\frac{1}{p+it}-\frac{1}{p-it}\right]\ , \tag{4.16}$$

we discover (4.11) takes the form

$$K_-\left[\frac{cp}{2\pi}\right]\tilde{f}_t(p)e^{ipB}=K_-\left[\frac{cp}{2\pi}\right]e^{ipB}\left[\frac{1}{p+it}-\frac{1}{p-it}\right]$$
$$=q'_-(p,t;B)+q'_+(p,t;B)\ . \tag{4.17}$$

Similarly, using the function $g(\Lambda)$ instead, we have for the rhs of Eq. (4.11)

$$K_+\left[\frac{cp}{2\pi}\right]\tilde{g}_t(p)e^{ipB}=iK_+\left[\frac{cp}{2\pi}\right]e^{ipB}\left[\frac{1}{p+it}-\frac{1}{p-it}\right]$$
$$=q''_-(p,t;B)+q''_+(p,t;B)\ , \tag{4.18}$$

where

$$q'_+(p,t;B)=-i\frac{K_-\left[\frac{ict}{2\pi}\right]e^{-Bt}}{p-it}\ , \tag{4.19a}$$

$$q'_-(p,t;B)=iK_-\left[\frac{cp}{2\pi}\right]\frac{e^{ipB}}{p+it}$$
$$-i\left[K_-\left[\frac{cp}{2\pi}\right]e^{ipB}-K_-\left[\frac{ict}{2\pi}\right]e^{-Bt}\right]\frac{1}{p-it}\ . \tag{4.19b}$$

$$q''_+(p,t;B)=(-i)K_+\left[\frac{cp}{2\pi}\right]\frac{e^{ipB}}{p-it}$$
$$+i\left[K_+\left[\frac{cp}{2\pi}\right]e^{ipB}-K_+\left[\frac{-ict}{2\pi}\right]e^{Bt}\right]\frac{1}{p+it}\ . \tag{4.20a}$$

$$q''_-(p,t;B)=iK_+\left[\frac{-ict}{2\pi}\right]\frac{e^{Bt}}{p+it}\ . \tag{4.20b}$$

For the amplitude $f_t(\Lambda-1)=e^{-|\Lambda-1|t}$ one has to replace B by $B-1$ in the above formulas.

We now distinguish between two cases, $B\le 0$ and $B\ge 0$ (but still $B\ll 1$). In the first case we use the first version of the rhs of Eq. (4.11) and in the second case the second version. We thus have

$$\tilde{\rho}_\pm(p,t;B)=\begin{cases}\dfrac{q''_\pm(p,t;B)}{K_\pm(p)}, & B\le 0\\ \dfrac{q'_\pm(p,t;B)}{K_\pm(p)}, & 0\le B\ ,\end{cases} \tag{4.21}$$

with $q''_\pm(p,t;B)$ and $q'_\pm(p,t;B)$ given in (4.19) and (4.20).

When the full amplitudes $f(\Lambda)$ and $g(\Lambda)$ are considered, we finally have

$$\tilde{\rho}_\pm(p)=\begin{cases}\displaystyle\sum_{k=0}^{\infty}(-1)^k\left[\frac{N^e}{c}\tilde{\rho}_\pm\left[p,t=(2k+1)\frac{\pi}{c},B-1\right]+\frac{N^i}{c}\tilde{\rho}_\pm\left[p,t=(2k+1)\frac{\pi}{c},B\right]\right], & B\le 0\\ \dfrac{1}{\pi}\displaystyle\int_0^\infty dt\sin\left[\frac{ct}{2}\right][N^e\tilde{\rho}_\pm(p,t;B-1)+N^i\tilde{\rho}_\pm(p,t;B)], & 0\le B\ll 1\ .\end{cases} \tag{4.22}$$

C. The magnetization curve

The z component of the total spin of the system is given by

$$S=\tfrac{1}{2}N-M=\tfrac{1}{2}N-\int_B^\infty\sigma_B(\Lambda)d\Lambda$$
$$=\tfrac{1}{2}N-\int_0^\infty\rho_+(\Lambda)d\Lambda$$
$$=\int\rho_-(\Lambda)d\Lambda=\tilde{\rho}_-(0)\ , \tag{4.23}$$

so that the magnetization for small magnetic field, $B\le 0$, is given by

$$\mathcal{M}=2\mu S$$
$$=\frac{\mu}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{K_+\left[\frac{1}{i}(k+\frac{1}{2})\right]}{(2k+1)K_-(0)}$$
$$\times\left[N^e\exp\left[(B-1)(2k+1)\frac{\pi}{c}\right]+N^i\exp\left[B(2k+1)\frac{\pi}{c}\right]\right]\ . \tag{4.24}$$

Using $K_-(0)=1/\sqrt{2}$,

$$K_+\left[\frac{1}{i}(k+\tfrac{1}{2})\right]=(k+\tfrac{1}{2})^{k+(1/2)}e^{-[k+(1/2)]}\frac{\sqrt{2\pi}}{\Gamma(k+1)}$$

we have

$$\mathcal{M}=\frac{\mu}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(k+\tfrac{1}{2})^{k-(1/2)}e^{-[k+(1/2)]}$$
$$\times\left[LT_0\exp\left[\frac{\pi}{c}B(2k+1)\right]\exp\left[-\frac{\pi}{c}2k\right]+N^i\exp\frac{\pi}{c}B(2k+1)\right]\ , \tag{4.25}$$

where we introduced the scale $T_0=(N^e/L)e^{-\pi/c}=De^{-\pi/c}$ [see (3.17)].

We choose the scale T_0 to be held fixed while removing D, $D\to\infty$. In other words, we concentrate on magnetic fields H that are small compared to D. Obviously, the effective coupling constant c becomes a function of the cutoff, $c=c(D)=\pi/(\ln D/T_0)$, and tends to zero, with increasing D, the scale being set by our choice of T_0. This is the phenomenon of asymptotic freedom, as discussed in

the Introduction. The limit we are considering ($D\to\infty, c\to 0, T_0$ fixed) leads, in renormalizable models, to universal answers.

We proceed now to show that the above limit exists, and that in the scaling regime, $H \ll D$, the magnetization becomes, a function of the ratio H/T_0.

As $c\to 0$, we may neglect powers of $e^{-\pi/c}$, and thus the magnetization becomes

$$\mathcal{M}=\mu\left[\frac{2}{\pi e}\right]^{1/2} LT_0 e^{B\pi/c} + \mu\frac{N^i}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(k+\tfrac{1}{2})^{k-(1/2)}e^{-[k+(1/2)]} \times e^{(B\pi/c)(2k+1)}, \quad B\le 0 . \tag{4.26}$$

We wish to identify the first term as the contribution of the electrons in the absence of interaction, namely, the Pauli magnetization. Indeed, setting $N^i=0$ (thus no impurities), one finds

$$\mathcal{M}^e=\mu\left[\frac{2}{\pi e}\right]^{1/2} LT_0 e^{B\pi/c} = \mu\frac{HL}{\pi}=\mathcal{M}_{\text{Pauli}} . \tag{4.27}$$

This allows us to relate H, the magnetic field, to the parameter B, namely, the Λ-sea Fermi level.

$$e^{\pi B/c}=\left[\frac{e}{2\pi}\right]^{1/2}\frac{H}{T_0}\equiv\frac{H}{T_1} . \tag{4.28}$$

We shall give a more careful derivation of this relation below.

We turn now to the region $B\ge 0$. The expression for the electronic contribution holds for $B<1$, while for the impurity part we use the second version of (4.23), which yields an analytic continuation past $B=0$. The result thus is,

$$\mathcal{M}=\mathcal{M}^e+\mathcal{M}^i , \tag{4.29a}$$

$$\mathcal{M}^e=\mu\left[\frac{2}{\pi e}\right]^{1/2} LT_0 e^{\pi B/c}=\frac{HL}{\pi} , \quad \text{where } B\ll 1 \text{ (i.e., } H\ll D) \tag{4.29b}$$

$$\mathcal{M}^i=\begin{cases} N^i\dfrac{\mu}{\sqrt{\pi}}\displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!}(k+\tfrac{1}{2})^{k-(1/2)}e^{-[k+(1/2)]}\left[\dfrac{H}{T_1}\right]^{2k+1}, & H\le T_1 \\ N^i\mu\left[1-\pi^{-3/2}\displaystyle\int_0^{\infty}\frac{dt}{t}\sin(\pi t)e^{-t(\ln t-1)}\left[\dfrac{T_1}{H}\right]^{2t}\Gamma(t+\tfrac{1}{2})\right], & T_1\le H\ll D . \end{cases} \tag{4.29c}$$

The universal impurity magnetization curve thus obtained is displayed in Fig. 13.

D. Asymptotics and scales

Consider first the small magnetic field behavior. Obviously, as $H\to 0$ we have (Andrei, 1980)

$$\mathcal{M}^i\to N^i\frac{\mu^2}{\pi T_0}H . \tag{4.30}$$

This simple result exhibits the Kondo effect in the magnetization. Indeed, due to its coupling to the conduction band, the impurity spin is screened (quenched) and thus has finite susceptibility at zero temperature $\chi^i=\mu^2/\pi T_0$, in contrast to the divergence occurring for a free spin $\chi^{\text{Curie}}=\mu^2/T$.

We shall use the scale $T_0=De^{-\pi/c}$ to characterize the small magnetic field, low-temperature region in the H-T plane. It is therefore the strong coupling scale.

Consider now the high magnetic field region $H\gg T_0$ (still $H\ll D$ to ensure scaling). The asymptotics is controlled by the behavior of the integrand in Eq. (4.29c) near $t=0$. Hence, expanding the integrand for small t, we have

$$\mathcal{M}^i\sim\left[1-\int_0^{\infty}dt\exp\left[-t\ln\frac{H}{T_1}\right]\times[1-t\ln t+(1-C-2\ln 2)t+O(t^2)]\right], \tag{4.31}$$

where $T_1=(2\pi/e)^{1/2}T_0$, and C is Euler's constant. Performing the integration with the help of the formula

$$\int_0^{\infty}x^{p-1}e^{-\lambda x}\ln x\,dx=\frac{\Gamma^{(p)}}{p}[\psi(p)-\ln\lambda] , \quad p,\lambda>0 , \tag{4.31a}$$

where $\psi(p)=-C+\sum_{k=1}^{p-1}1/k$, we obtain

$$\mathcal{M}^i\underset{D\gg\tilde{H}\gg T_0}{\sim}\mu\left[1-\frac{1}{2\ln\frac{H}{T_1}}+\frac{\ln 2}{\left[\ln\frac{H}{T_1}\right]^2}-\frac{\ln\frac{H}{T_1}}{\left[2\ln\frac{H}{T_1}\right]^2}+O\left[\left[\frac{H}{T_0}\right]^2\right]^{-3}\right] . \tag{4.32}$$

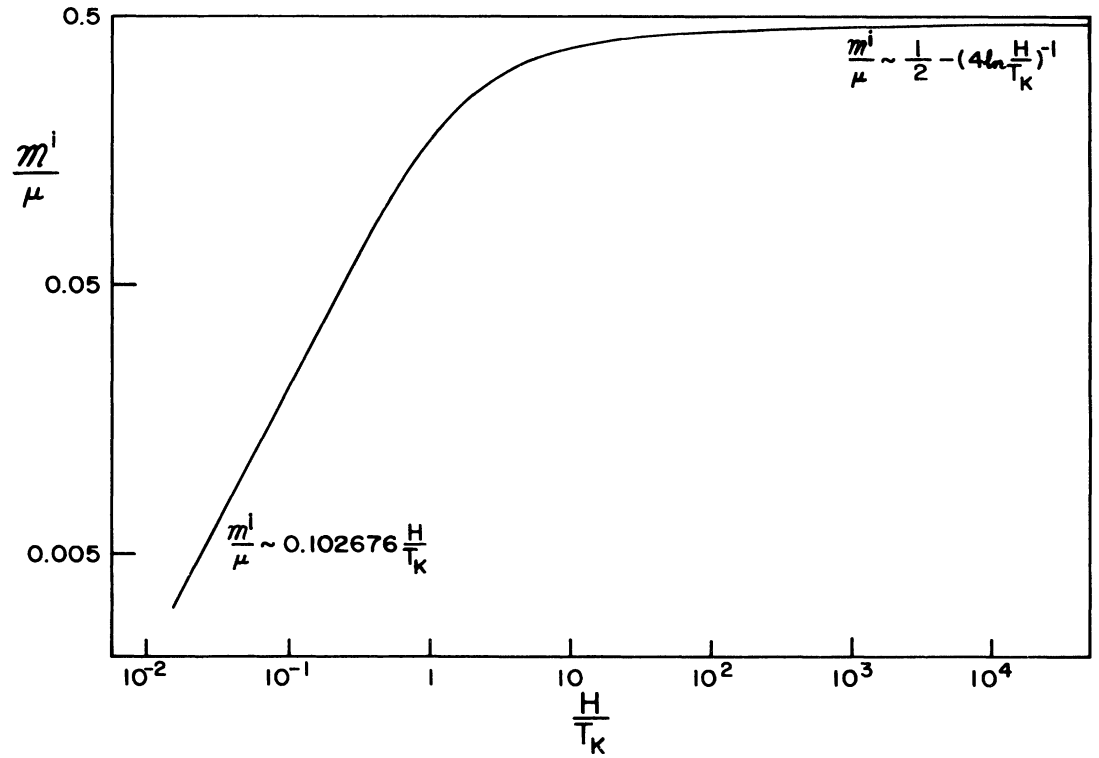

FIG. 13. The magnetization curve [Eq. (4.29)]. The convention here is that the moment is $\frac{1}{2}\mu$. The scale T_k is determined in Sec. VI.

We now introduce the high magnetic field scale T_H, parametrizing a weak coupling region in the H-T plane, by requiring that no term of $O[\ln^{-2}(H/T_H)]$ should appear in the expansion. One finds

$$\mathcal{M}^i \underset{D\gg H\gg T_0}{\sim} \mu\left[1-\frac{1}{2}\frac{1}{\ln\left[\frac{H}{T_H}\right]}+\frac{1}{4}\frac{\ln\ln\left[\frac{H}{T_H}\right]}{\left[\ln\left[\frac{H}{T_H}\right]\right]^2}+0\left[\frac{1}{\ln\frac{H}{T_H}}\right]^3+\cdots\right], \tag{4.33}$$

with

$$T_H=\left[\frac{\pi}{e}\right]^{1/2}T_0=W'T_0\ . \tag{4.34}$$

The ratio $W'=T_H/T_0=\sqrt{\pi/e}$ (Andrei and Lowenstein, 1981) (expressed in terms of the two most important numbers in mathematics!), characterizes the crossover from the perturbative, weak coupling regime of high magnetic field, where the impurity spin behaves essentially (up to logarithmic corrections) like a free spin to a strong coupling regime, where it is completely screened.

E. The energy in the magnetic field

We wish to rederive the relation between B and H from energy consideration. The change in energy in the presence of a magnetic field is given by

$$\Delta E(B)=\Delta E_0(B)-HS\ ,$$

where $\Delta E_0(B)$ includes the contribution of the holes (which fill the negative end of the Λ axis from $-\infty$ to B),

$$\begin{aligned}\Delta E_0(B)&=\int_B^\infty d\Lambda\,\sigma_B(\Lambda)[\Theta(2\Lambda-2)-\pi]\\&=\int_{-\infty}^B d\Lambda\,\sigma_B(\Lambda)2D\tan^{-1}(e^{(\pi/c)(\Lambda-1)})\ ,\end{aligned} \tag{4.35}$$

where $\sigma_B(\Lambda)$ for $-\infty<\Lambda<B$ is defined by Eq. (4.1) and is the hole density in the macroscopically excited Λ sea. In the scaling region ($D\to\infty, c\to 0, T_0$ held fixed), this becomes

$$\begin{aligned}\Delta E_0(B)&=2T_0\int_{-\infty}^B d\Lambda\,\sigma_B(\Lambda)e^{\pi\Lambda/c}\\&=2T_0\int d\Lambda\,\rho_-(\Lambda)e^{\pi/c}e^{\pi B/c}\\&=2T_0e^{\pi B/c}\tilde{\rho}_-\left[\frac{i\pi}{c}\right].\end{aligned} \tag{4.36}$$

Also from (4.23)

$$S=\int\rho_-(\Lambda)d\Lambda=\tilde{\rho}_-(0)\ , \tag{4.37}$$

so that

$$\Delta E(B)=2T_0 e^{\pi B/c}\tilde{\rho}_-\left[\frac{i\pi}{c}\right]-H\tilde{\rho}_-(0)\ . \tag{4.38}$$

Using the solution of the Wiener-Hopf equation for $\tilde{\rho}_-(i\pi/c)$ and $\tilde{\rho}_-(0)$, we can write

$$\Delta E(B)=-\frac{i}{2}\left[\frac{1}{\pi}\right]^2\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}dt\left[\left[\frac{2T_0 e^{\pi B/c}K_+\left[-\frac{i}{2}\right]}{2t+1}-\frac{HK_+(0)}{2t}\right]\right]\eta(B,t)\ , \tag{4.39}$$

where

$$\eta(B,t)=\Gamma(\tfrac{1}{2}-t)\Gamma(\tfrac{1}{2}+t)K_+(-ict/2\pi)\times[N^e\exp(B-1)t+N^i\exp Bt]\ .$$

Varying now Eq. (4.39) with respect to B, we find

$$0=\frac{\partial\Delta E(B)}{\partial\left[\exp\frac{\pi B}{c}\right]}=\left[2T_0 e^{\pi B/c}\left[\frac{\pi}{e}\right]^{1/2}-H\sqrt{2}\right]\int dt\,\eta(B,t)dt \tag{4.40}$$

and hence

$$e^{\pi B/c}=\left[\frac{e}{2\pi}\right]^{1/2}\frac{H}{T_0}\ , \tag{4.41}$$

as before.

V. THE THERMODYNAMICS OF THE KONDO MODEL

Having diagonalized the Kondo Hamiltonian, we are now in a position to write down a formal expression of the partition function Z of the system at nonzero temperature T and external magnetic field H. We shall deduce a set of coupled integral equations determining the free energy F, and demonstrate scaling behavior and crossover properties in the full H-T plane. After deriving many of their features we shall rewrite the equations in a way that provides a basis for an iterative solution. Computer results displaying the full thermodynamics will be displayed.

The first to formulate the thermodynamics of a Bethe-ansatz system were Yang and Yang (1969). Their results were extended to the Heisenberg model in Gaudin (1971) and Takahashi (1971) and to many other soluble systems by the latter author. The same ideas were adopted to the Kondo model in Filyov *et al.* (1981) and Andrei and Lowenstein (1981).

A. The partition function

The formal expression for the partition function is

$$Z=\mathrm{Tr}\exp\left[-\frac{1}{T}(\mathscr{H}-2HS_z)\right]=\sum_{S=0}^{N/2}\sum_{S_z=S}^{S}\mathrm{Tr}_{SS_z}\exp\left[-\frac{1}{T}(\mathscr{H}-2HS_z\right]\ , \tag{5.1}$$

where $\mathscr{H}$ is the zero-field Hamiltonian and Tr_{SS_z} is the trace over all basis states with values S and S_z of total spin and z component of the spin. Since $\mathscr{H}$ is invariant under simultaneous rotations of all spins, we may split off the sum over S_z to obtain

$$Z=\sum_{S=0}^{N/2}\frac{\sinh\left[(2S+1)\frac{H}{T}\right]}{\sinh\left[\frac{H}{T}\right]}\mathrm{Tr}_{SS}\exp\left[-\frac{\mathscr{H}}{T}\right]$$

$$\approx\sum_{S=0}^{N/2}\mathrm{Tr}_{SS}\exp\left[-\frac{1}{T}(\mathscr{H}-2SH)\right]\ . \tag{5.2}$$

In the last approximation we have dropped terms proportional to $\exp(-SH/T)$, as well as an overall factor $[2\sinh(H/T)]^{-1}$, since these terms contribute negligibly to the calculation of thermodynamic quantities in the limit $L\to\infty$ (note that $S\sim L$).

Now let us exploit the specific form of the energy for our basis states. Recall that each such state is labeled by a set of quantum numbers $\{n_j,I_\gamma\}$ with $n_j\geq-\frac{1}{2}N^e$. The corresponding energy is

$$E=\frac{2\pi}{L}\sum_{i=1}^{N^e}n_j+E(\Lambda) \tag{5.3}$$

with

$$E(\Lambda)=D\sum_{\gamma=1}^{M}[\Theta(2\Lambda_\gamma-2)-\pi] \tag{5.4}$$

and the complex Λ momenta satisfying Eq. (3.2). The partition function factorizes accordingly,

$$Z=Z_0Z_\Lambda\ , \tag{5.5}$$

where

$$Z_0=\sum_{\{n_j\}}\exp\left[-\frac{1}{T}\sum_{j=1}^{N^e}\frac{2\pi}{L}n_j\right]$$

is just the partition function for N^e noninteracting spinless fermions with linear kinetic energy, and

$$Z_\Lambda = \exp\left[\frac{NH}{T}\right] \sum_M \sum_{\{\Lambda_1,\cdots,\Lambda_M\}} \exp\left[-\frac{1}{T}[E(\Lambda) + 2MH]\right] . \quad (5.6)$$

The summation is facilitated by employing the n-string and n-string hole density $\sigma_n(\Lambda)$ and $\sigma_n^h(\Lambda)$ discussed in Sec. III. The densities satisfy (3.30)

$$\sigma_n^h(\Lambda) + \sum_{m=1}^{\infty} A_{nm}\sigma_m(\Lambda) = f_n(\Lambda) ,$$

where A_{nm} are integral operators given by

$$A_{nm} = [\,|n-m|\,] + 2[\,|n-m|+2] + \cdots + 2[n+m-2] + [n+m] ,$$

$$[k]a(\Lambda) = \frac{1}{\pi}\int \frac{\frac{kc}{2}}{\left[\frac{kc}{2}\right]^2 + (\Lambda-\Lambda')^2} a(\Lambda')d\Lambda' ,$$

and the function f_n by

$$f_n(\Lambda) = \frac{nc}{2\pi}\left[\frac{N^e}{\left[\frac{nc}{2}\right]^2 + (\Lambda-1)^2} + \frac{N^i}{\left[\frac{nc}{2}\right]^2 + \Lambda^2}\right]$$

$$= [n][N^e\delta(\Lambda-1) + N^i\delta(\Lambda)] .$$

In terms of the string densities the partition function Z_Λ becomes

$$Z_\Lambda = \exp\left[\frac{NH}{T}\right] \int \prod D\sigma_n D\sigma_n^h \exp\mathcal{S} \times \exp\left[-\frac{1}{T}[E(\Lambda) + 2HM]\right] , \quad (5.7)$$

with

$$E(\Lambda) + 2HM = \sum_n \int d\Lambda\, \sigma_n(\Lambda) g_n(\Lambda) , \quad (5.8)$$

where

$$g_n(\Lambda) = D\left[\Theta\left[\frac{2\Lambda-2}{n}\right] - \pi\right] + 2Hn \quad (5.9)$$

and where $\mathcal{S}(\{\sigma_n,\sigma_n^h\})$ is the entropy associated with the densities $\{\sigma_n,\sigma_n^h\}$—in other words, where $\exp\mathcal{S} \times(\{\sigma_n,\sigma_n^h\})$ is a functional, counting how many configurations $\{I_\gamma\}$ lead to solutions $\{\Lambda_\gamma\}$ that are consistent with a given set of densities $\{\sigma_n,\sigma_n^h\}$. To determine $\mathcal{S}$ divide the Λ axis into intervals $d\Lambda$, chosen sufficiently small that the densities are approximately constant over each interval, yet sufficiently large that $(\sigma_n+\sigma_n^h)d\Lambda \gg 1$. As discussed in Sec. III, the number of possible slots for Λ-strings in the interval $d\Lambda$ is $dv_n = (\sigma_n^h+\sigma_n^h)d\Lambda$. However, $\sigma_n d\Lambda$ of these are occupied, while $\sigma_n^h d\Lambda$ of these are empty; thus the number of ways of distributing the n-strings among the slots is

$$\frac{[(\sigma_n^{(\Lambda)}+\sigma_n^h(\Lambda))d\Lambda]!}{[\sigma_n(\Lambda)d\Lambda]![\sigma_n^h(\Lambda)d\Lambda]!} .$$

Using Stirling's formula, we can simplify this to give

$$d\mathcal{S}_n = \ln\frac{[(\sigma_n+\sigma_n^h)d\Lambda]!}{[\sigma_n d\Lambda]![\sigma_n^h d\Lambda]!}$$

$$= [(\sigma_n+\sigma_n^h)\ln(\sigma_n+\sigma_n^h) - \sigma_n^h\ln\sigma_n^h - \sigma_n\ln\sigma_n]d\Lambda ,$$

so that the entropy, $\mathcal{S}$, becomes

$$\mathcal{S} = \sum_n \int d\Lambda[(\sigma_n+\sigma_n^h)\ln(\sigma_n+\sigma_n^h) - \sigma_n^h\ln\sigma_n^h - \sigma_n\ln\sigma_n] . \quad (5.10)$$

B. Derivation of the thermodynamic equations

In thermodynamic limit, $N\to\infty$, we may evaluate Z_Λ by the method of stationary phase. Varying the functional

$$F_\Lambda\{\sigma_n,\sigma_n^h\} = E(\Lambda) + 2HM - T\mathcal{S}$$

$$= \sum_n \int d\Lambda\left[\sigma_n g_n - T\sigma_n\ln\left[1+\frac{\sigma_n^h}{\sigma_n}\right] - T\sigma_n^h\ln\left[1+\frac{\sigma_n}{\sigma_n^h}\right]\right] , \quad (5.11)$$

subject to the constraint,

$$\delta\sigma_n^h = -\sum_m A_{nm}\delta\sigma_m , \quad (5.12)$$

one has

$$0 = \delta F_\Lambda = T\sum_n \delta\sigma_n\left[\frac{g_n}{T} - \ln\left[1+\frac{\sigma_n^h}{\sigma_n}\right] - \sum_m A_{nm}\ln\left[1+\frac{\sigma_n}{\sigma_n^h}\right]\right] , \quad (5.13)$$

from which follows

$$\ln[1+\eta_n(\Lambda)] = \frac{g_n(\Lambda)}{T} + \sum_{m=1}^{\infty} A_{nm}\ln[1+\eta_m^{-1}(\Lambda)] , \quad (5.14)$$

where

$$\eta_n(\Lambda)=\frac{\sigma_n^h(\Lambda)}{\sigma_n(\Lambda)} .$$

This set of equations may be rewritten by the use of the following identities [see Takahashi (1971), noting erratum]:

$$\begin{aligned} &A_{nm}-G(A_{n-1,m}+A_{n+1,m})=\delta_{nm} , \quad n>1 \\ &A_{1m}-GA_{2m}=\delta_{1m} , \end{aligned} \tag{5.15}$$

where G is defined by

$$\begin{aligned} Gf(\Lambda)&\equiv\frac{[1]}{[0]+[2]}f(\Lambda) \\ &=\frac{1}{2c}\int d\Lambda'\frac{1}{\cosh\frac{\pi}{c}(\Lambda-\Lambda')}f(\Lambda') . \end{aligned}$$

By multiplying Eq. (5.15) by $\ln(1+\eta_m^{-1})$ and summing over m, one obtains, using Eq. (5.14),

$$\ln\eta_n=G[\ln(1+\eta_{n+1})+\ln(1+\eta_{n-1})] , \tag{5.16a}$$

$$\ln\eta_1=-\frac{2D}{T}\tan^{-1}(e^{(\pi/c)(\Lambda-1)})+G\ln(1+\eta_2) , \tag{5.16b}$$

where the driving term,

$$\Delta E^d=2D\tan^{-1}(e^{(\pi/c)(\Lambda-1)}) ,$$

is the result of

$$\Delta E^d=\frac{1}{[0]+[2]}(g_1-2H)=2D\tan^{-1}(e^{(\pi/c)(\Lambda-1)}) \tag{5.17}$$

and should be familiar as the one-hole excitation energy.

To close the set of Eq. (5.16a) one has to supply a boundary condition for $n\to\infty$. We use another identity,

$$[n+1]A_{n,m}-[n]A_{n+1,m}=\begin{cases}0 , & m<n \\ -([m+1]+[m-1]) , & m>n\end{cases} \tag{5.18}$$

which, when multiplied by $\ln(1+\eta_m^{-1})$ and summed over m, gives

$$\lim_{n\to\infty}([n+1]\ln(1+\eta_n)-[n]\ln(1+\eta_{n+1}))=-\frac{2H}{T} . \tag{5.19}$$

We shall refer to the set of Eq. (5.14) or to the equivalent set (5.16) and (5.19) as the GT equations, as they are similar to those derived by Gaudin and Takahashi for the Heisenberg model (note, however, the difference in the boundary condition). The main difference is that the driving term there takes the form

$$\Delta E^d_{\text{Heisenberg}}=\frac{1}{[0]+[2]}\frac{2J}{1+\Lambda^2}=J\frac{1}{ch\frac{\pi}{2}\Lambda} . \tag{5.20}$$

Similarly, the thermodynamic equations for the chiral Gross-Neveu model (the backscattering model) are obtained by inserting

$$\Delta E^d_{\text{GN}}=D\tan^{-1}\left[\frac{\cosh\frac{\pi}{c}\Lambda}{\sinh\frac{\pi}{c}}\right] , \tag{5.21}$$

the one-hole excitation energy in that model (Andrei and Lowenstein, 1979), into the GT equation. This similarity is obviously the result of the identical structure of the spin excitation of the three models, as discussed in Sec. III.

C. The free energy

Once a set of η_n solutions satisfying the GT equations has been found, the spin free energy may be obtained from Eq. (5.11) and simplified by the use of Eqs. (3.14) and (5.14):

$$\begin{aligned} F_\Lambda&=\sum_n\int d\Lambda\Big[\sigma_n g_n-T\sigma_n\ln(1+\eta_n) \\ &\qquad -T\Big[f_n-\sum_m A_{nm}\sigma_m\Big]\ln(1+\eta_n^{-1})\Big] \\ &=-\sum_n T\int d\Lambda\, f_n\ln(1+\eta_n^{-1}) \\ &=-\sum_n T\int d\Lambda[N^e\delta(1-\Lambda) \\ &\qquad +N^i\delta(\Lambda)][n]\ln(1+\eta_n^{-1}) . \end{aligned} \tag{5.22}$$

This may be simplified using Eq. (5.14) for $n=1$, which may be rewritten as

$$[1]\left[\ln(1+\eta_1)-\frac{g_1}{T}\right]=([0]+[2])\sum_n[n]\ln(1+\eta_n^{-1}) ,$$

where we used the identity $[n][m]=[n+m]$. The expression for F_Λ thus becomes

$$\begin{aligned} F_\Lambda&=-T[N^e\delta(1-\Lambda)+N^i\delta(\Lambda)]G\left[\ln(1+\eta_1)-\frac{g_1}{T}\right] \\ &=\int d\Lambda\,\sigma_0(\Lambda)\{g_1(\Lambda)-T\ln[1+\eta_1(\Lambda)]\} \\ &=E_{\text{gs}}-T\int d\Lambda\,\sigma_0\ln[1+\eta_1(\Lambda)] , \end{aligned} \tag{5.23}$$

where

$$\begin{aligned} \sigma_0(\Lambda)&=G[N^e\delta(1-\Lambda)+N^i\delta(\Lambda)] \\ &=\frac{1}{2c}\left[\frac{N^e}{\cosh\frac{\pi}{c}(\Lambda-1)}+\frac{N^i}{\cosh\frac{\pi}{c}\Lambda}\right] \end{aligned}$$

is the expression for the ground-state density and E_{gs} is the ground-state energy, independent of H and T.

We also have to add the density fluctuation free energy F_0, which described massless, noninteracting spinless fermions. It is given, in the limit N^e and L tending to infinity with $D=N^e/L \gg T$, by

$$F_0=-\frac{LT}{2\pi}\int_{-\pi D}^{\infty} dK \ln\left[1+\exp\left[-\frac{K}{T}\right]\right]$$
$$=-\frac{LT^2}{2\pi}\left[\frac{\pi^2}{6}+\left[\frac{2\pi D}{T}\right]^2+O(e^{-2\pi D/T})\right]. \tag{5.24}$$

We thus have for the free energy

$$F=F_0+F_\Lambda=E_0-\frac{\pi LT^2}{12}$$
$$-T\int d\Lambda\,\sigma_0(\Lambda)\ln[1+\eta_1(\Lambda)]\,, \tag{5.25}$$

with E_0 now containing also the temperature-independent contribution of the density fluctuations.

In the case of the Gross-Neveu model the free energy takes the same form, with σ_0 now given by

$$\sigma_0^{\mathrm{GN}}=\frac{1}{2c}\left[\frac{N^+}{\cosh\frac{\pi}{c}(\Lambda-1)}+\frac{N^-}{\cosh\frac{\pi}{c}(\Lambda+1)}\right],$$

where $N^\pm$ are the numbers of the left- and right-moving electrons.

D. Scaling

We shall now show the scaling properties of the thermodynamic equations for the Kondo (or backscattering) model.

We are interested in Eq. (5.16) in the regime where $T \ll D$. This implies that the function η_1 has a very sharp decrease, proportional to $\exp[-(2D/T)\tan^{-1}z]$, where $z=\exp[(\pi/c)(\Lambda-1)]$, for Λ tending to infinity. Thus if z is not kept much less than unity, the η_1 is of order $\exp(-2D/T)$ and contributes negligibly to the integrals in Eqs. (5.16) and (5.25). Hence for $D \gg T$, we may replace $\tan^{-1}z$ by z in these integrals and compute η_n from a modified version of the GT equations

$$\ln\eta_n=G[\ln(1+\eta_{n+1})+\ln(1+\eta_{n-1})]\,, \tag{5.26a}$$
$$\ln\eta_1=-2e^{\zeta}+G\ln(1+\eta_2)\,, \tag{5.26b}$$

where we now regard η_n as a function of the new variable ζ,

$$\zeta=\frac{\pi}{c}\Lambda=\ln\frac{T_0}{T}\,, \tag{5.27}$$

with

$$T_0=De^{-\pi/c}$$

and

$$G(\zeta-\zeta')=\frac{1}{2\pi}\frac{1}{\cosh(\zeta-\zeta')}$$

[for the backscattering model replace the driving term by $-(T_0/T)\cosh\zeta$]. In terms of the new variables the function η_n depends on H and T only through the ratio H/T. The free energy becomes, moreover,

$$F=E_0-\frac{\pi LT^2}{12}-\frac{T}{2\pi}\int d\zeta\left[\frac{N^e}{\cosh\left[\zeta-\ln\frac{T_0}{T}-\frac{\pi}{c}\right]}+\frac{N^i}{\cosh\left[\zeta-\ln\frac{T_0}{T}\right]}\right]\ln\left[1+\eta_1\left[\zeta,\frac{H}{T}\right]\right], \tag{5.28}$$

which we rewrite as

$$F=E_0+F^e+F^i+O\left[\exp\left[-\frac{\pi D}{T}\right]\right], \tag{5.29}$$

with the impurity contribution

$$F^i=-TN^i\frac{1}{2\pi}\int d\zeta\frac{\ln(1+\eta_1)}{\cosh\left[\zeta-\ln\frac{T}{T_0}\right]}$$
$$\equiv -TN^i f\left[\frac{T}{T_0},\frac{H}{T}\right] \tag{5.30}$$

and electron contribution

$$F^e=-\frac{\pi LT^2}{12}-TN^e f\left[\frac{T}{D},\frac{H}{T}\right]. \tag{5.31}$$

We thus find that the impurity free energy F^i, has a nontrivial limit in the scaling regime and depends on the coupling constant and the cutoff only through the contribution defining $T_0=De^{-\pi/c}$. It is universal to all materials or constructions with the same T_0.

We have lumped together the density contribution and the spin contribution of the electrons to give F^e. Only that part of $N^e f(T/D,H/T)=LDf(T/D,H/T)$ which does not vanish for $D/T\to\infty$ has a universal character. Note also that the same function $f(t,h)$ appears in both expressions.

The scaling form of the free energy for the backscattering model is given by

$$F^{\mathrm{GN}}=E_0-\frac{LT^2}{\pi 12}-\frac{LT}{2\pi}m\int d\zeta\cosh\zeta\ln(1+\eta_1)\,,$$

where m is the mass of the spin-$\frac{1}{2}$ excitation, $m=D\exp(-\pi/c)$.

While the thermodynamic equation for the Heisenberg and Kondo models is isomorphic, the difference in the driving term is very significant at high temperatures. In the case of the Heisenberg chain,

$$\frac{\Delta E_H^d}{T}=\frac{J}{T}\frac{1}{\cosh\frac{\pi}{2}\Lambda}\,,$$

and as $1/[\cosh(\pi/c)\Lambda]$ is a bounded function, one may obtain a high-temperature expansion in powers of J/T. This is not the case in the Kondo density driving term, which is of the form

$$\frac{\Delta E_K^d}{T}=\frac{2D}{T}\tan^{-1}e^{(\pi/c)(\Lambda-1)},$$

and where we always have $D\gg T$. In the scaling form regime we can write

$$\frac{\Delta E_K^d}{T}=\frac{2T_0}{T}e^{\zeta},$$

with T_0/T small in the high-temperature region (yet $T\ll D$). However, e^{ζ} is not bounded and therefore no conventional high-temperature expansion exists. This is expected, since in the Kondo model, which is renormalizable and asymptotically free, the high-temperature expansion is in powers of $(1/\ln T)$. The same discussion holds in the case of the backscattering model. At low temperatures, however, we expect similar behavior.

E. Various properties

We wish to establish that there exist solutions $\eta_n(\zeta,H/T)$ for the thermodynamic equations with the following properties:

(1) Limits for $\zeta\to\pm\infty$

$$\eta_n^{\pm}\equiv\eta_n(\pm\infty)=\frac{\sinh^2(nx_0+y_0^{\pm})}{\sinh^2 x_0}-1,\quad n=1,2,\ldots,$$
$$x_0=\frac{H}{T},\quad y_0^-=x_0,\quad y_0^+=0. \tag{5.32}$$

(2) Monotonic decrease in ζ (fixed n)

$$\eta_n(\zeta)>\eta_n(\zeta')\ \text{ if }\ \zeta<\zeta'.$$

(3) Monotonic increase in n (fixed ζ)

$$\eta_n(\zeta)>\eta_{n'}(\zeta)\ \text{ if }\ n>n'.$$

(4) Analyticity

$\eta_n(\zeta)$ is an analytic function, regular and free of zeros in the strip $|\operatorname{Im}\zeta|<\pi/2$.

We now proceed to establish properties (1)–(4). That the η_n, assuming for the moment that they exist, approach constant values at $\zeta=\pm\infty$ is a simple consequence of the fact that the exponential of the driving term in (5.26), $e^{-2e^{\zeta}}$, has this property, and the kernel $G(\zeta-\zeta')$ has exponential falloff at infinity.

The extreme values of (5.32) may be found by replacing η_n in the GT equations by constants, the kernel by $\frac{1}{2}\delta(\zeta-\zeta')$, and the inhomogeneous term by 0 or $-\infty$. The resulting equations,

$$\ln\eta_n^{\pm}=\tfrac{1}{2}[\ln(1+\eta_{n+1}^{\pm})+\ln(1+\eta_{n-1}^{\pm})],\quad n>1 \tag{5.33a}$$

$$\ln\eta_1^-=\tfrac{1}{2}\ln(1+\eta_2^-),\quad \eta_1^+=0, \tag{5.33b}$$

together with the large-n boundary condition [consequence of (5.19)],

$$\lim\frac{\eta_{n+1}^{\pm}}{\eta_n^{\pm}}=e^{2H/T}, \tag{5.34}$$

can be solved as in Takahashi (1971) to give (5.32).

To show the existence of a solution of (5.26) with each η_n decreasing from η_n^- to η_n^+ as ζ runs from $-\infty$ to $+\infty$, consider the following recursive scheme [reminiscent of an algorithm employed by Johnson (1974) to solve numerically a similar set of equations in the anisotropic Heisenberg model]:

$$\eta_n^{(0)}=\eta_n^-,$$
$$\ln\eta_n^{(\nu)}=G\ln(1+\eta_{n-1}^{(\nu)})+G\ln(1+\eta_{n+1}^{(\nu-1)}), \tag{5.35}$$
$$\ln\eta_1^{(\nu)}=G\ln(1+\eta_n^{(\nu-1)})-2e^{\zeta},$$

where ν is the iteration number, $1,2,\ldots$.

It is easy to see that for $\nu\geq 1$, $\eta_n^{(\nu)}$ is a strictly decreasing function of ζ, with $0<\eta_n^{(\nu)}<\eta_n^{(\nu-1)}$. Thus $\eta_n=\lim_{\nu\to\infty}\eta_n^{(\nu)}$ exists and, by the continuity of the functions $\ln x$, $\ln(1+x)$ and of the integral operator G, the η_n satisfy Eqs. (5.26). As such, they are infinitely differentiable functions, and one readily verifies that $d\eta_n/d\zeta$ is strictly negative.

To verify the analyticity of $\eta_n(\zeta)$, we take the Fourier transform of the right-hand side of (5.26a). It is of the form $T_n(p)/\cosh(\pi p/2)$, where the tempered distribution $T_n(p)$ falls off faster than any power at infinity. It follows that for $n>1$, $\ln\eta_n$ is regular in the strip $|\operatorname{Im}\zeta|<\pi/2$. This also is true of $\ln\eta_1$ as a consequence of (5.26b). Thus all η_n are regular and free of zeros in the strip $|\operatorname{Im}\zeta|<\pi/2$.

To complete our discussion of the existence and properties of η_n satisfying (5.26), we must show that the unique solution obtained (in principle) by the iterative procedure described above satisfies the large-n boundary condition (5.19). To do this, we introduce functions ξ_n by

$$\eta_n=e^{2\xi_n}\frac{\sinh^2 nx_0}{\sinh^2 x_0}-1. \tag{5.36}$$

The GT equations (5.26), written in terms of ξ_n, become

$$\xi_n=G(\xi_{n-1}+\xi_{n+1})+b_n,\quad n>1 \tag{5.37a}$$
$$\xi_1=\tfrac{1}{2}\ln[1+2\cosh x_0\exp(-2\exp\zeta)\exp(2G\xi_2)], \tag{5.37b}$$

where

$$b_n=-\tfrac{1}{2}\ln[1+\varepsilon_n(1-e^{-2\xi_n})], \tag{5.38a}$$
$$\varepsilon_n=\left[\frac{\sinh^2 nx_0}{\sinh^2 x_0}-1\right]^{-1}. \tag{5.38b}$$

We see that ε_n, and hence b_n, in (5.37) are of order $\exp(-2nx_0)$ for $n\gg x_0^{-1}$. If the "remainder terms" b_n were not present, Eqs. (5.37) and (5.38), together with (5.34), would have a solution of the form $\xi_n=[n-1]\xi_1$. For nonzero b_n, we have instead

$$\xi_n=[n-1]\xi_1+\sum_{k=2}^{\infty}H_{nk}b_k\ , \tag{5.39}$$

where

$$H_{nk}(\zeta)=\int\frac{dp}{2\pi}e^{ip\zeta}\tilde{H}_{nk}(p) \tag{5.40a}$$

$$\tilde{H}_{nk}(p)=\begin{cases}2\cosh(\pi p/2)\exp(-n\pi\,|p\,|/2)\left[\left[\dfrac{\sinh(\pi kp/2)}{\sinh(\pi p/2)}\right]-e^{-(k-1)\pi\,|p\,|/2}\right]\ , & k\le n\\ 2\cosh(\pi p/2)\exp(-n\pi\,|p\,|/2)\left[\left[\dfrac{\sinh(n\pi p/2)}{\sinh(\pi p/2)}\right]-e^{-(n-1)\pi\,|p\,|/2}\right]\ , & k\ge n\ .\end{cases} \tag{5.40b}$$

The operator H_{nk} can be written as a finite linear combination of $[l]$ operators with positive integer coefficients. Hence $H_{nk}(\zeta)$ is a positive definite function with ζ^{-2} behavior at infinity. The bound

$$|H_{nk}b_k(\zeta)|<(n-1)\ln(1+\varepsilon_k)\ ,\quad k>n$$

assures the uniform convergence of the representation (5.39) and we are thus permitted to write

$$\begin{aligned}[n+1]\xi_n-[n]\xi_{n+1}&=\sum_{k=2}^{\infty}([n+1]H_{nk}-[n]H_{n+1,k})b_k\\&=-\sum_{k=n+2}^{\infty}([k+1]+[k-1])b_k\ .\end{aligned} \tag{5.41}$$

For $n\to\infty$, the right-hand side of (5.41) vanishes, giving

$$\lim_{n\to\infty}([n+1]\xi_n-[n]\xi_{n+1})=0\ , \tag{5.42}$$

but this is nothing but (5.19) expressed in terms of the ξ_n.

F. Low-temperature behavior

Having satisfied ourselves that well-defined solutions η_n exist, we turn now to study the low-temperature, $T\ll T_0$, properties of the model. As remarked before, we expect many to be similar to those of the corresponding antiferromagnetic Heisenberg model (and, obviously, to those of the Gross-Neveu model).

We saw that the impurity contribution to the free energy F^i is given by the universal function

$$F^i=-TN^if\left[\frac{T}{T_0},\frac{H}{T}\right]$$

where

$$f(t,h)=\frac{1}{2\pi}\int d\zeta\frac{1}{\cosh(\zeta+\ln t)}\ln[1+\eta_1(\zeta,h)]\ . \tag{5.43}$$

This allows us to study asymptotic behavior for $\ln T/T_0\to\infty$ $(-\infty)$, as T is very large (small) compared to T_0.

The electron contribution F^e is expressed by the same function $f(t,h)$

$$\begin{aligned}F^e&=-\frac{\pi LT^2}{12}-TN^ef\left[\frac{T}{D},\frac{H}{T}\right]\Bigg|_{D\to\infty}\\&=-\frac{\pi LT^2}{12}-LTDf\left[\frac{T}{D},\frac{H}{T}\right]\Bigg|_{D\to\infty}\ .\end{aligned}$$

For sufficiently small t we may expand $1/\cosh(\zeta+\ln t)$ in (5.43),

$$\frac{1}{\cosh(\zeta+\ln t)}=2t\exp\zeta(1-t^2\exp2\zeta+t^4\exp4\zeta+\cdots)\ ,$$

to obtain (we may perform the expansion within the integral as η_1 vanishes for large ζ as $e^{-2e^{\zeta}}$)

$$f(t,h)=\frac{t}{\pi}\int d\zeta\exp\zeta\ln[1+\eta_1(\zeta,h)]+O(t^2)\ , \tag{5.44}$$

so that

$$\frac{F^e}{L}=-\frac{\pi}{12}T^2-\frac{T^2}{\pi}\int d\zeta\exp\zeta\ln\left[1+\eta_1\left[\zeta,\frac{H}{T}\right]\right]+O\left[\frac{T^4}{D^2}\right]\ . \tag{5.45}$$

Equation (5.45) is valid even when $N^i=0$, i.e., when the system consists of $N^e=DL$ noninteracting, spin-$\frac{1}{2}$ electrons in the presence of a field H. Thus we may compare (5.45) with F^e/L calculated in the conventional manner

$$\begin{aligned}\frac{F^e}{L}&=-\frac{T}{2\pi}\left[\int_{-(\pi D+H)}^{\infty}dk\ln(1+e^{-k/T})+\int_{-(\pi D-H)}^{\infty}dk\ln(1+e^{-k/T})\right]\\&=-\frac{T^2}{2\pi}\left[\frac{1}{3}\pi^2+\left[\frac{\pi D}{T}\right]^2+\left[\frac{H}{T}\right]^2\right]+O(e^{-\pi D/T})\end{aligned} \tag{5.46}$$

to obtain

$$\int d\zeta\,e^{\zeta}\ln\left[1+\eta_1\left[\zeta,\frac{H}{T}\right]\right]=\frac{\pi^2}{12}+\frac{H^2}{2T^2}\ . \tag{5.47}$$

As the same integral appears in the impurity low-temperature expression,

$$\frac{F^i}{N^i}=-\frac{T^2}{\pi T_0}\int d\zeta\, e^{\zeta}\ln\left[1+\eta_1\left[\zeta,\frac{H}{T}\right]\right]+O\left[\frac{T^4}{T_0^3},\frac{H^4}{T_0^3}\right]$$

$$=-\frac{1}{\pi T_0}\left[\frac{\pi^2}{12}T^2+\frac{1}{2}H^2\right]+O\left[\frac{T^4}{T_0^3},\frac{H^4}{T_0^3}\right]. \quad (5.48)$$

We find that the impurity contribution to specific heat at low temperature is

$$C_v^i=N^i\frac{\pi}{6T_0}T \quad (5.49)$$

and the magnetic susceptibility [reproducing Eq. (4.30)]

$$\chi^i=N^i\frac{\mu^2}{\pi T_0}=N^i\frac{\mu_W^2}{4\pi T_0}, \quad (5.50)$$

where μ_W is the magnetic moment using Wilson's convention.

This gives the low-temperature ratio U (Wilson, 1975)

$$U=\mu_W^2\frac{C_v^i}{T\chi^i}=\tfrac{2}{3}\pi^2. \quad (5.51)$$

Being defined within the low-temperature region, U does not explore crossover properties and can be determined without solving the model, using Fermi liquid theory (Nozières, 1976) or perturbation theory (Yamada, 1975) or Ward's identities (Zawadowski, 1981) or, on the other hand, using low-temperature properties of the solution by a method similar to the above (Filyov *et al.*, 1981) or phase shift analysis similar to the one to be discussed in Sec. IX. That the same ratio appears in other models like the Heisenberg or Gross-Neveu models is expected in the light of the remarks in Sec. V.D.

G. High-temperature behavior

The high-temperature behavior, while accessible to perturbation expansion in the convention approach, is annoyingly hard to obtain from the equations. Still, most of the physics can be extracted.

For $T\gg T_0$ only the behavior of the smooth function η_1 for $\zeta\to-\infty$ is relevant in an asymptotic expansion for F^i. Using now

$$1+\eta_1=\frac{\sinh^2 2H/T}{\sinh^2 H/T}$$

[Eq. (5.32)], we find

$$F^i=-\frac{T}{2\pi}N^i\int d\zeta\frac{1}{\cosh\left[\zeta+\ln\frac{T}{T_0}\right]}\ln\left[1+\eta_1\left[\zeta,\frac{H}{T}\right]\right]$$

$$\to -TN^i\ln\left[2\cosh\frac{H}{T}\right], \quad (5.52)$$

which is just the free energy of an isolated spin (if $N^i=1$) in the presence of a magnetic field H. The screening of the impurities magnetic moment, evident for low temperatures from $\chi^i=\mu^2/\pi T_0$, thus completely disappears for asymptotically large T. How rapidly is this "asymptotic freedom" approached? We expect a high-temperature expansion to be obtained in the form

$$-\frac{F^i}{T}=A_0+\frac{A_1}{\ln\frac{T}{T_0}}+\frac{A_2}{\ln^2\frac{T}{T_0}}+\frac{A_2'\ln\ln\frac{T}{T_0}}{\ln^2\frac{T}{T_0}}+O\left[\frac{\ln^2\ln\frac{T}{T_0}}{\ln^3\frac{T}{T_0}}\right] \quad (5.53)$$

in analogy to the high magnetic field expansion (4.33).

To what extent can the coefficients be determined? Motivated by the form of (5.52) we make the ansatz, to be inserted into

$$F^i=-\frac{T}{\pi}N^i\int d\zeta\frac{1}{\cosh\left[\zeta+\ln\frac{T}{T_0}\right]}\xi_1\left[\zeta,\frac{H}{T}\right],$$

that ξ_n is of the form

$$\xi_n(\zeta)=\xi_{no}^{\pm}+\xi_{n1}^{\pm}/\zeta+\xi_{n2}^{\pm}/\zeta^2+\xi_{n2}'^{\pm}\frac{\ln|\zeta|}{\zeta^2}+\cdots \quad (5.54)$$

for $\zeta\to\pm\infty$.

The corresponding tails of $\eta_n=e^{2\xi_n}(\sinh^2 nx_0/\sinh^2 x_0)-1$, again expanded as in (5.54), will satisfy (5.33), if they assume the form

$$\eta_n^{\pm}=\frac{\sinh^2(nx^{\pm}+y^{\pm})}{\sinh^2 x^{\pm}}-1, \quad (5.55a)$$

$$y^-=x^-,\quad y^+=0, \quad (5.55b)$$

where

$$x^{\pm}=x_0+\frac{x_1^{\pm}}{\zeta}+\frac{x_2^{\pm}}{\zeta^2}+x_2'^{\pm}\frac{\ln|\zeta|}{\zeta^2}+\cdots \quad (5.55c)$$

and the coefficients $x_i^{\pm}$ are to be determined by the large-n relation (5.42).

Rewriting Eq. (5.54) in terms of the $x_i^{\pm}$, we find for asymptotically large n

$$\xi_n^{\pm}\sim\ln\frac{\sinh(nx^{\pm}+y^{\pm})\sinh x_0}{\sinh x^{\pm}\sinh x_0}\sim n(x^{\pm}-x_0)+y^{\pm}+\ln\frac{\sinh x_0}{\sinh x^{\pm}}$$

$$\sim(n-\coth x_0)\left[\frac{x_1^{\pm}}{\zeta}+\frac{x_2^{\pm}}{\zeta^2}+\frac{x_2'^{\pm}}{\zeta^2}\ln|\zeta|\right]+\left[y_0^{\pm}+\frac{y_1^{\pm}}{\zeta}+\frac{y_2^{\pm}}{\zeta^2}+\frac{y_2'^{\pm}}{\zeta^2}\ln|\zeta|\right]+\frac{1}{2\sinh^2 x_0}\frac{x_1^{\pm 2}}{\zeta^2}. \quad (5.56)$$

Inserting (5.56) into Eq. (5.42) (note the form of the large-n boundary condition), we obtain

$$-\left[\frac{x_1^{\pm}}{\zeta}+\frac{x_2^{\pm}}{\zeta^2}+\frac{x_2'^{\pm}}{\zeta^2}\ln|\zeta|\right]+\frac{1}{2}\left[\frac{y_0^- - y_0^+}{\zeta}+\frac{y_1^+ - y_1^-}{\zeta^2}\ln|\zeta|\right]=O(\zeta^{-2}), \tag{5.57}$$

where the right-hand side contains terms (not directly calculable) of the order ζ^{-2} but not $\zeta^{-2}\ln|\zeta|$. This is due to the structure of the operator $[n]$ appearing in the large-n relation [Eq. (5.42)], which has a tail falling off with this power.

We thus may read off

$$x_1^+ = x_1^- = \frac{y_0^- - y_0^+}{2} = \frac{x_0}{2}, \tag{5.58a}$$

$$x_2'^+ = x_2'^- = \frac{y_1^+ - y_1^-}{\pi} = -\frac{x_0}{4}, \tag{5.58b}$$

with $x_2^+ = x_2^-$ undetermined. The constants $x_2^{\pm}$, which determine the Wilson number W, discussed in the Introduction, are thus unobtainable by this method. We shall, however, be able to calculate this important number, using ideas of universality, in Sec. VI.

Substituting (5.58) into (5.55), we obtain

$$\xi_1^- = \tfrac{1}{2}\ln(1+\eta_1^-) = \ln(2\cosh x_0)+\tfrac{1}{2}\tanh x_0\left[\frac{x_0}{\zeta}-\frac{x_0}{2\zeta^2}\ln(\zeta)\right] + O(\zeta^{-2}), \tag{5.59}$$

which yields Eq. (5.53), with

$$A_0 = \ln(2\cosh x_0)$$

as before

$$A_1 = 2A_1' = -\tfrac{1}{2}x_0\tanh x_0 . \tag{5.60}$$

The terms proportional to $\ln^{-2}(T/T_0)$ can be deleted, if at the same time T_0 is replaced by an appropriate new scale T_k. This indeed is the definition we adopt, following Wilson, for the Kondo temperature T_k. The yet undetermined coefficients $x_2^{\pm}$ will determine the numerical relation between T_0 and T_k. Thus finally we find for the free energy

$$F_i = -TN^i\left[\ln(2\cosh x_0)-\tfrac{1}{2}x_0\tanh x_0\left[\frac{1}{\ln T/T_k}+\frac{1}{2}\frac{\ln\ln(T/T_k)}{\ln^2 T/T_k}\right]+O\left[\frac{\ln^2\ln T/T_k}{\ln^3 T/T_k}\right]\right] \tag{5.61a}$$

with $x_0 = H/T$, and the resulting susceptibility

$$\chi^i = \frac{\mu^2}{T}\left[1-\left[\frac{1}{\ln T/T_k}+\frac{1}{2}\frac{\ln\ln(T/T_k)}{\ln^2 T/T_k}\right]+O\left[\frac{\ln^2\ln T/T_k}{\ln^3 T/T_k}\right]\right]. \tag{5.61b}$$

Note the relative factor of $\frac{1}{2}$ between the $\ln^{-1}(T/T_k)$ term and the $\ln\ln(T/T_k)/\ln^2(T/T_k)$ term. The same factor appears in the high-field expression (4.33) and, in the conventional perturbation theory, accounts for the $\frac{1}{2}\ln g$ term in the expression for T_k.

H. The region $H \gg T$

We now turn to analyze the asymptotic region $x_0 = H/T \gg 1$. Among other things this will enable us to describe the passage to the zero-temperature limit.

For large x_0 Eq. (5.39) reduces to

$$\xi_n = [n-1]\xi_1 + O(e^{-x_0}) \tag{5.62}$$

and so ξ_1 can be determined from a single equation $\ln\eta_1 = -2e^{\zeta}+G\ln(1+\eta_2)$, which now takes the form

$$\ln(\exp 2\xi_1 - 1) = -2e^{\zeta}+2G\xi_2+2\ln(2\cosh x_0) \approx -2e^{\zeta}+2G[1]\xi_1+x_0, \tag{5.63}$$

where we dropped corrections exponentially small in x.

We rewrite the equation introducing $\eta_1(\zeta,x_0) = \exp[x_0\psi(\zeta,x_0)-2e^{\zeta}]$, as follows:

$$\psi(\zeta,x_0) = 1+x_0^{-1}\int d\zeta' L(\zeta-\zeta')\times\ln[1+\exp(x_0\psi_0-2e^{\zeta'})], \tag{5.64}$$

with

$$L = G[1] = \frac{[2]}{[0]+[2]}.$$

Now we define $\bar{\zeta}$ by

$$2\exp\bar{\zeta} = x_0\psi(\bar{\zeta},x_0). \tag{5.65}$$

For $x_0 \gg 1$, $\bar{\zeta} \gg \zeta$, one can replace the logarithm in (5.64) by $x_0\psi - 2\exp\zeta$; for $\zeta \gg \bar{\zeta}$ one can replace it by zero. In fact, the only interval on the ζ axis for which one or the other approximation is not valid is of order x_0^{-1}. Hence

$$\psi(\zeta,x_0) = 1+x_0^{-1}\int_{-\infty}^{\bar{\zeta}} d\zeta' L(\zeta-\zeta')[x_0\psi(\zeta,x_0)-2e^{\zeta'}] + O(x_0^{-2}), \tag{5.66}$$

which becomes by a change of variables

$$\psi(y) = 1+\int_{-\infty}^{0} dy' L(y-y')[\psi(y')-Ce^{y'}], \tag{5.67}$$

where

$$y = \zeta-\bar{\zeta} = \frac{\pi}{c}\Lambda - \ln\left[\frac{HC}{2T_0}\right],$$

$$C=\psi(0)\ .$$

Equation (5.67) could be handled by the same Wiener-Hopf methods used in Sec. IV. This is not, however, necessary, as we shall soon see. The impurity contribution to the free energy becomes

$$F^i=-N^iH\int_{-\infty}^{0}dy\frac{1}{\cosh\left[y+\ln\frac{HC}{2T_0}\right]}[\psi(y)-Ce^y]$$

$$+O(x_0^{-1})\ . \tag{5.68}$$

This expression must coincide if the $T\to 0$ limit is not singular, with the corresponding expression for the energy, which is the same as the free energy at $T=0$, obtained in Sec. IV. To see that this indeed is the case we use (5.57) to isolate the asymptotic T dependence of $\eta_1(\Lambda)$ in the limit $T\to 0$:

$$\eta_1(\Lambda)\sim\exp\left\{H\psi\left[\frac{\pi}{c}\Lambda-\ln\left[\frac{HC}{2T_0}\right]\right]\right.$$

$$\left.-2T_0\exp\frac{\pi}{c}\Lambda\right\}\Big/T\ . \tag{5.69}$$

For $\Lambda<B\equiv(c/\pi)\ln(HC/2T_0)$ the exponent is positive and $\eta_1(\Lambda)\to\infty$; for $\Lambda>B$ the exponent is negative and $\eta_1(\Lambda)\to 0$. Recalling the definition $\eta_n=\sigma_n^h/\sigma_n$, we see that the limiting situation is precisely that described in Sec. IV: nothing but holes for $\Lambda<B$, nothing but one-strings for $\Lambda>B$. Comparison with (4.28) allows us to evaluate the constant C:

$$C=\left[\frac{2e}{\pi}\right]^{1/2}\ . \tag{5.70}$$

Without explicitly determining ψ, we have enlarged the validity of our expression from the $T=0$ axis to the entire region where $x_0=H/T\gg 1$.

I. Numerical solution

Having probed and analyzed our equations from various points of view, we proceed to present a numerical solution obtained by Rajan, employing the iteration scheme given in (5.39) (Rajan *et al.*, 1982).

The GT equations have been earlier analyzed numerically by Johnson, who investigated the anisotropic Heisenberg model. In his scheme, however, the isotropic limit is inaccessible, since the anisotropy is used to cut off the equations. The method used here, however, is applicable also in this limit. In a forthcoming work we shall present the isotropic Heisenberg model and the backscattering model thermodynamics.

Here we present the magnetization, susceptibility, and specific-heat curves as functions of the temperature for various values of the magnetic field (Figs. 14–16).

The curves illustrate the features we have discussed. In Fig. 14 we have plotted $T\chi/\mu_W^2$ as a function of the temperature. This is a measure of the effective spin squared

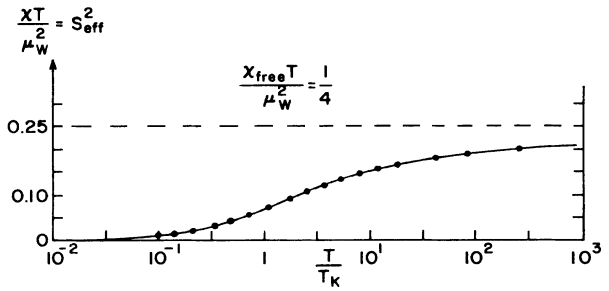

FIG. 14. The effective spin χT is plotted vs the temperature. It vanishes as $T\to 0$ indicating complete screening. The dots represent the renormalization-group calculation of Krishna-murthy *et al.* (1980).

of the impurity. At low temperature the spin is screened by the electron, while at high temperature, $T\gg T_k$, we cross over to the asymptotic freedom regime, and the value $S_{\rm eff}^2=\frac{1}{4}$ is logarithmically approached, the scale being set by T_k ($T_k/T_0=W=1.2902$, as will be calculated in Sec. VI). We also compare with a renormalization-group calculation (Krishna-murthy *et al.*, 1978). Although the model was constructed very differently, the agreement is excellent, providing support for the universality of the results in the scaling regime (see Sec. VI).

In Fig. 15 we have plotted the specific heat C_v^i as a function of the temperature, for various values of the magnetic field. Asymptotic freedom manifests itself by the tendency of the curves to approach with increasing field the free-spin specific heat (Schottky anomaly) $C_v^f=x_0^2\sec^2x_0$, $x_0=H/T$. The approach is logarithmically slow, and even at $H/T_k=9$ the curves still differ

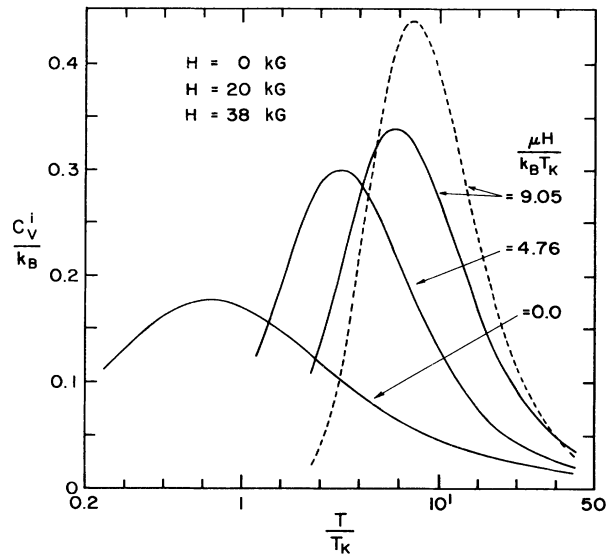

FIG. 15. The specific heat vs the temperature for various values of the magnetic field. The broken line is the corresponding free-spin specific heat. For comparison with experiment see Rajan *et al.* (1982).

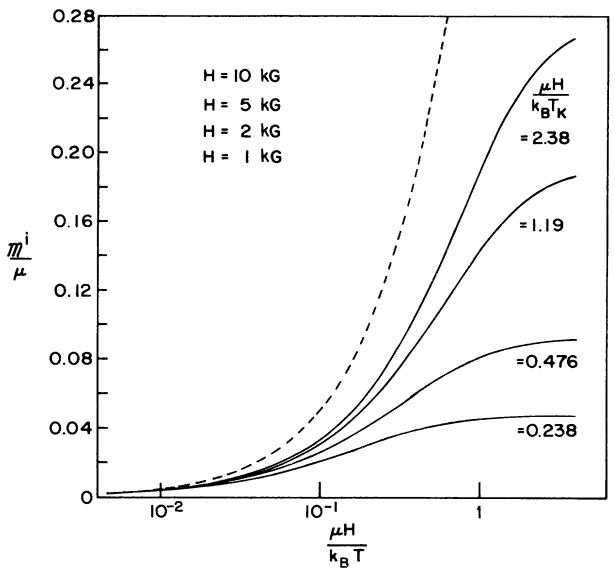

FIG. 16. The magnetization as a function of H/T for different values of the magnetic field. As it is raised, the free-spin curve (broken line) is approached. For comparison with experiment see Rajan *et al.* (1982).

significantly. The crossover also manifests itself in the temperature dependence, and every curve tends at high temperature to the appropriate free-spin curve plotted here for $H/T_k=9.05$.

In Fig. 16 we have plotted the impurity magnetization as a function of H/T for various values of the magnetic field. Again, the free-spin magnetization $\mathcal{M}_f^i=\mu\tanh(H/T)$ is logarithmically approached both with increasing magnetic field and/or increasing temperature. The scales determining this approach to asymptotic are T_H and T_k, respectively.

The graphs were confronted with experiment in Rajan *et al.* (1982) (after all, the Kondo model is not only a theorist's toy...). It was found that the agreement with experiment is much better than might be expected from such a simple model.

VI. UNIVERSALITY

In Sec. V we have determined the impurity free energy, $F^i=F^i(T,H:D,J)$, and studied it in the scaling limit ($D\to\infty, c\to 0$ with T_0 fixed). In this regime the function is expressible in the form

$$F^i=-Tf\left[\frac{T}{T_0},\frac{H}{T}\right],$$

with T_0 being the only scale in the problem. This limit describes the system at temperatures that are low compared with the bandwidth, and where, indeed, one expects the model to have physical relevance.

It is important to study the model in this limit, since in this case the results are independent of the particular way the model has been defined. The physical situation one tries to model is such that the details of the cutoff structure (i.e., bandwidth and density of states away from the Fermi surface) are irrelevant. This permits the construction of the model using various schemes which are expected to yield the same results in the scaling limit. When this is the case, these schemes are said to be in the same universality class. There is no obvious criterion which determines the extent of the class, but our construction violates some cherished notions about it. Still, we wish to show in this section that our definition of the model, though unusual, has not changed its universality class. The question as to what, indeed, does determine universality properties should therefore be further investigated.

A. Conventional lore

At first sight this hope may seem unfounded. Indeed, our expression for the dynamically generated scale, $T_0=De^{-\pi/c}$ differs in an essential way from the expression obtained in the more conventional momentum cutoff scheme ($\mathscr{D}$ scheme), where it is given by (Fowler and Zawadowski, 1971)

$$T_0=\mathscr{D}\sqrt{g}\,e^{[-(\pi/2g)+\alpha(g)]}, \tag{6.1}$$

with $\alpha(g)$ being a power series in the coupling constant g as defined in the $\mathscr{D}$ scheme. The word "essential" refers to the absence of the square root of the coupling constant in our expression. While it is well known that a coupling constant depends on the scheme defining the Hamiltonian, it has generally been felt that any other scheme which does not modify the universality class of the problem would lead to an expression for T_0 which could differ from (6.1) only in the analytic part $\alpha(g)$. In particular, the square root of the coupling has been expected always to be present.

This view was based on experience with scaling (Yuval and Anderson, 1970; Anderson, 1970) or renormalization-group (Wilson, 1975) techniques. The basic idea is to reduce the bandwidth $\mathscr{D}$ by integrating out high-energy intermediate states. For a small coupling g this merely renormalizes $\mathscr{D}$ thus leading to an equivalent description in terms of a new cutoff $\mathscr{D}'$ and a new coupling constant g'. As one is always integrating over a finite amount of the momentum range, no singularities may result and the couplings are analytically related,

$$g'=U(g)=g+U_1g^2+U_2g^3+\cdots. \tag{6.2}$$

The two couplings are equal to first order, since in the tree approximation there are no divergences and therefore no ambiguities as to the definition of the coupling [note that g is dimensionless, i.e., that it multiplies a marginal (or a renormalizable) operator].

If the change is infinitesimal, one can describe the dependence of the coupling constant on the cutoff by means of the Gell-Mann and Low (Callan-Symanzik) β function,

$$\mathscr{D}\frac{\partial}{\partial\mathscr{D}}g=\beta(g), \tag{6.3}$$

and for the Kondo model in the $\mathscr{D}$ scheme (Yosida and Okiji, 1964; Armytage, 1973),

$$\beta(g)=(2\rho g)^2-\tfrac{1}{2}(2\rho g)^3+O(\rho g)^4 , \tag{6.4}$$

where ρ is the density of states given in our convention by $\rho=1/\pi$.

The dynamic scale T_0 is invariant under the scaling procedure, so that it must satisfy

$$\left[\mathscr{D}\frac{\partial}{\partial\mathscr{D}}-\beta(g)\frac{\partial}{\partial g}\right]T_0(\mathscr{D},g)=0 \tag{6.5}$$

and is therefore given by

$$T_0=\mathscr{D}\exp\left[\int^{2\rho g}\frac{dx}{\beta(x)}\right], \tag{6.6}$$

where the lower limit is arbitrary and determines the normalization of the scale.

For small g we have

$$\int^{2\rho g}\frac{dx}{\beta(x)}=\int^{\rho g}\frac{dx}{x^2[1-\frac{1}{2}x+O(x^2)]}$$

$$\approx-\frac{1}{2\rho g}+\tfrac{1}{2}\ln 2\rho g+\cdots \tag{6.7}$$

(where the ellipsis represents terms regular in g), so that the scale is of the form (6.1):

$$T_0=\text{const}\exp\left[-\frac{\pi}{2g}+\tfrac{1}{2}\ln g+\alpha(g)\right].$$

The singular part comes from the first two terms in the β function and the regular part from the rest.

It turns out, however, that under transformation $g\to U(g)$ of the form (6.2) (which may relate any two schemes, and not necessarily just two related by scaling) the first two terms in the β function are unchanged. To see this, consider the function in the $\mathscr{D}'$ scheme,

$$\beta'(g')=\mathscr{D}\frac{\partial}{\partial\mathscr{D}}g'=\frac{\partial U}{\partial g}\beta(g)=(1+2U_1g+3U_2g^2+\cdots)[\beta_0(2\rho g)^2+\beta_1(2\rho g)^3+\beta_2(2\rho g)^4+\cdots]+\cdots , \tag{6.8}$$

which up to terms of order g^4 is

$$\beta_0(2\rho)^2(g+U_1g^2)^2+\beta_1(2\rho)^3(g+U_1g^2)^3+\cdots ,$$

so that we find, in terms of the new coupling g',

$$\beta'(g')=\beta_0(2\rho g')^2+\beta_1(2\rho g)^3+O(g')^4 \tag{6.9}$$

with the first two terms unchanged.

We find then, that as long as we consider schemes that can be analytically related, the β function will have the same first two terms. In the Kondo model, in particular, where $\beta_1=\frac{1}{2}$, the dynamic scale will contain the square root of the coupling constant.

While these considerations are valid, it does not follow that a scheme not analytically related to $\mathscr{D}$ is necessarily in a different universality class.

B. The *D* scheme

As already discussed in Sec. III, the D scheme differs substantially from the more conventional lattice or momentum cutoff schemes. In the latter, Eq. (1.1), one wishes to express the eigenstates in terms of a finite, though large, number of Fourier modes (noninteracting eigenstates). But as the wave functions necessarily contain discontinuities, the construction can only be approximate so long as the cutoff is finite. In our scheme, in contrast, the cutoff is imposed on the fully interacting wave function and is therefore operative in a different space of functions. No finite changes in the range of momentum integration can relate the schemes, as the one (D) always involves an infinite number of modes, while the other ($\mathscr{D}$) only a finite number. It is therefore conceivable that the connection between the coupling constant g (defined in the $\mathscr{D}$ scheme) and J (defined in the D scheme) is nonanalytic. When the cutoff is removed, however, the spaces of functions coincide and one expects the two constructions to yield the same answers in that limit (scaling limit). This was demonstrated in the framework of the Gross-Neveu model (Andrei and Lowenstein, 1979, 1980a, 1980b), where the D scheme (or K scheme) was first employed. The spectrum and the S matrix were calculated and found to be in accord with other approaches.

In a sense this is not too surprising. The Gross-Neveu model (and the Kondo model) are renormalizable in the field-theoretic sense. That means, indeed, that cutoff effects are irrelevant when the cutoff is removed. Still, the proofs of renormalizability are perturbative and it is not obvious that they apply to nonperturbative constructions.

Furthermore, the calculations of the spectrum and S matrix explore only the strong coupling regime of the theory, while here we are interested in analyzing the theory over the whole range.

Thus, though we may expect the D scheme to be in the $\mathscr{D}$ universality class, the "essential" difference in the form of the scale indicates that the schemes cannot be analytically related. To determine the connection let us consider the universal susceptibility function $\chi^i=\chi^i(T/T_k)$ in the high-temperature region, where a perturbative expansion in the coupling is reliable.

In the D scheme [Eq. (5.61b)] we find

$$\chi^i\to\frac{\mu^2}{T}\left[1-\frac{2J}{T}+\frac{1}{2}\left[\frac{2J}{\pi}\right]^2\ln\left[\frac{2J}{\pi}\right]+O(J^2)\right], \tag{6.10}$$

which is a double expansion in J and $J\ln J$.

The $\mathscr{D}$ scheme (see Appendix C), on the other hand, yields a power series in the coupling g,

$$\chi^i \to \frac{\mu^2}{T}\left[1-\frac{2g}{\pi}+O(g^2)\right] . \tag{6.11}$$

Hence we deduce that the coupling constants are related by

$$\frac{2g}{\pi}=\frac{2J}{\pi}-\frac{1}{2}\left[\frac{2J}{\pi}\right]^2 \ln\frac{2J}{\pi}+O(J^2) . \tag{6.12}$$

Owing to the scaling property of the theory the identification holds over the whole range of the temperature and in particular accounts for the "missing" square root in T_0. The β function in our scheme can now be deduced from (6.8) or from the form of T_0, and is given by

$$\beta(J)=\left[\frac{c}{\pi}\right]^2 , \tag{6.13}$$

where

$$c=\frac{2J}{1-(\frac{3}{4})J^2} ,$$

where the "second term," namely, $\frac{1}{2}(2J/\pi)^3$, is absent. If we look back at Eq. (5.61b) or (4.33), we find that the factor $\beta_1=\frac{1}{2}$ occurs in the combination of $\ln^{-1}(x)$ term with the $\ln^{-2}(x)\ln\ln(x)$ term in the universal functions. It is this combination that makes up the running coupling constant. It is when we insist on a construction where the weak coupling expansion is given by a power series that this factor makes its appearance in the β function.

We have established the connection between the schemes assuming universality, namely, that the same function $\chi^i(T/T_k)$ is obtained in the two schemes (once it is expressed in terms of the scale T_k, whose structure, in turn, need not be universal).

Evidence that χ^i is indeed universal is provided by comparing our results (Fig. 16) with those obtained earlier by renormalization-group calculation (Krishna-murthy *et al.*, 1978) plotted on the same graph. The agreement is excellent.

A more stringent test, though, is the calculation of $W=T_k/T_0$, the temperature crossover number, which forms our main result.

C. Calculation of *W*

The calculation of this number belongs, strictly speaking, to Sec. V.G. It is determined, however, by the coefficient $x_2^+=x_2^-$, which, due to the form of the boundary conditions at $n\to\infty$, we could not calculate using a high-temperature expansion. Still, the idea of universality suggests another way. Thus we express W as a product of $W'=T_H/T_0$ and $U=T_k/T_H$,

$$W=\frac{T_k}{T_0}=\frac{T_k}{T_H}\frac{T_H}{T_0} . \tag{6.14}$$

W' is the magnetic crossover number whose determination involved the complete solution of the model, yet which is much easier to find than W, as only maximum spin excitations for a given energy are involved. One finds (Andrei and Lowenstein, 1981)

$$W'=\frac{T_H}{T_0}=\left[\frac{\pi}{e}\right]^{1/2} .$$

The number $U=T_k/T_H$, on the other hand, can be calculated without having to solve the model at all (see Fig. 5). The asymptotic regions parametrized by T_k and T_H are the high-temperature and high-field regions, respectively, both of which are in the weak coupling regime and thus susceptible to perturbative expansions.

To determine U we construct the model in the $\mathscr{D}$ scheme (momentum cutoff), in which the weak coupling expansions are given as power series in the coupling constant. Being universal, U could be determined in *any* scheme.

In Appendix C we have calculated the impurity free energy to second order in g. For the case $S=\frac{1}{2}$,

$$\frac{F^i}{T}=-\ln\left[2\cosh\frac{H}{T}\right]+\frac{g}{\pi}\frac{H}{T}\tanh\frac{H}{T}$$
$$-\left[\frac{g}{\pi}\right]^2\left[\frac{1}{2}\frac{H^2}{T^2}\operatorname{sech}\frac{2H}{T}+\left[\frac{\exp(H/T)}{2\cosh(H/T)}\left\{\phi\left[\frac{H}{T}\right]-\frac{2H}{T}\left[1+\ln\frac{\mathscr{D}}{2H}+Ei\left[-\frac{2H}{T}\right]\right]\right\}+(H\leftrightarrow -H)\right]\right]+O(g^2) , \tag{6.15}$$

where we have dropped terms independent of the field. Here

$$\phi(t)=\int_0^1 dx\,(1-x)\exp(-2xt)[\pi^2\csc^2(\pi x)-x^{-2}-(1-x)^{-2}] , \tag{6.16}$$

and Ei is the exponential-integral function. Asymptotically,

$$F^i\to\begin{cases}-H+\dfrac{gH}{\pi}+2H\left[\dfrac{g}{\pi}\right]^2\ln\dfrac{e\mathscr{D}}{2H}+O(g^3), & \dfrac{T}{H}\to 0\\[2ex] -\dfrac{H^2}{2T}+\dfrac{gH^2}{\pi T}+2\dfrac{H^2}{T}\left[\dfrac{g}{\pi}\right]^2\ln\left[\beta\gamma e^{-7/4}\dfrac{\mathscr{D}}{T}\right]+O(g^3), & \dfrac{T}{H}\to\infty ,\end{cases} \tag{6.17}$$

where

$$\ln\beta = \int dx\,(1-x)^2 x\,(\pi^2 \csc^2 \pi x - x^{-2}) \tag{6.18}$$

and $\ln\gamma$ is Euler's constant. The magnetization therefore is given by

$$\mathcal{M}^i \to \begin{cases} \mu\left[1-\frac{1}{2}\left[\frac{2g}{\pi}\right]+\frac{1}{2}\left[\frac{2g}{\pi}\right]^2\left[\ln\frac{H}{\mathscr{D}}+\ln 2\right]\right]+O(g^3) \\ \mu\frac{H}{T}\left[1-\left[\frac{2g}{\pi}\right]+\left[\frac{2g}{\pi}\right]^2\left[\ln\frac{T}{\mathscr{D}}-\ln\beta\gamma e^{-7/4}\right]\right]+O(g^3)\,. \end{cases} \tag{6.19}$$

The constants in the second-order terms are used to normalize the scales $\mathscr{T}_k$ and $\mathscr{T}_H$ (the scales as defined in the $\mathscr{D}$ scheme)

$$\mathcal{M}^i \to \begin{cases} \mu[1-\frac{1}{2}(\ln H/\mathscr{T}_H)^{-1}-\frac{1}{4}(\ln H/\mathscr{T}_H)^{-2}\ln\ln H/\mathscr{T}_H+O(\ln H/\mathscr{T}_H)^{-3}] \\ \mu\frac{H}{T}[1-(\ln T/\mathscr{T}_k)^{-1}-\frac{1}{2}(\ln T/\mathscr{T}_k)^{-2}\ln\ln T/\mathscr{T}_k+O(\ln T/\mathscr{T}_k)^{-3}]\,, \end{cases} \tag{6.20}$$

so that no $(\ln H/\mathscr{T}_H)^{-2}$ or $(\ln T/\mathscr{T}_k)^{-2}$ terms appear in the expressions. Hence

$$U=\frac{\mathscr{T}_k}{\mathscr{T}_H}=2\beta\gamma e^{-7/4}\,. \tag{6.21}$$

But the ratio is universal, so

$$U=\frac{\mathscr{T}_k}{\mathscr{T}_H}=\frac{T_k}{T_H}\,.$$

We therefore deduce that the temperature crossover number is

$$W=2\beta\gamma\pi^{1/2}e^{-9/4}\,, \tag{6.22}$$

and substituting the values

$$\ln\gamma=0.577\,216\,,$$

$$\ln\beta=0.662\,122\,,$$

we have (Andrei and Lowenstein, 1981)

$$\frac{W}{4\pi}=0.102\,676\,. \tag{6.23}$$

This number was first calculated by Wilson using numerical renormalization-group techniques. His result (Wilson, 1975) is

$$\frac{W}{4\pi}=0.1032\pm 0.0005\,, \tag{6.24}$$

in good agreement with ours.

We thus find that our scheme, although not analytically connected to the conventional schemes, is most probably in their universality class.

To summarize: the concepts of universality and renormalizability have been shown to have a wider range of applicability than suspected earlier. This important question should be further studied.

VII. THE KONDO MODEL WITH ARBITRARY IMPURITY SPIN

We turn now to the consideration of a system of electrons interacting with a localized spin S of arbitrary magnitude. To realize this situation one needs a conduction band with only s-wave character or else the $(2S+1)$ spin states of the impurity would couple to the higher angular momentum states of the electrons leading to an SU($2S+1$) Kondo model (Coqblin and Schrieffer, 1973). Here we shall discuss the spin-S representation of SU(2), leaving the more realistic case of SU(n) symmetry to the next section.

We consider, then, the following model:

$$\mathscr{H}=-i\int \phi_a^*(x)\partial_x\phi_a(x)dx+2J\mathbf{S}\phi_a^*(0)\sigma_{ab}\phi_b(0)\,, \tag{7.1}$$

where now $\mathbf{S}$, the impurity spin operator, can be in any representation of SU(2).

The model is again Bethe soluble, and the construction of the eigenstates is similar to the corresponding construction in the fundamental, spin-$\frac{1}{2}$ representation, with the modifications occurring only at the second Bethe-ansatz level. Now, to use the billiard ball picture, the impurity site can hold up to $2S$ balls, whereas the electron sites hold up to one ball only!

This leads to a diagonalization procedure that is very similar to the one discussed thus far, which yields, however, a ground state characterized by a total spin $(S-\frac{1}{2})$; this partially screened impurity spin (Mattis, 1967), of course, manifests itself in the thermodynamic properties, as we shall discuss soon.

The spectrum of excitation, however, built on that ground state, has the same structure as the spin-$\frac{1}{2}$ spectrum. It consists of "hole states," namely, spin-$\frac{1}{2}$ dressed electrons, whose energy is given by

$$\Delta E=\sum_{i=1}^{N^h} 2D(\tan^{-1}c^{(\pi/c)(\Lambda_i^h-1)})\,.$$

The energy depends only on the positions of the holes, $\{\Lambda_i^h, i=1,\ldots,N^h\}$, and not on the total spin of the state, which may vary from maximal spin-$\frac{1}{2}N^h$ to zero, depending on whether we have added the appropriate n-strings to couple the spins of the dressed electrons.

The thermodynamics is developed along familiar lines:

it is again expressible by a set of functions $\{\eta_n\}$ satisfying the identical GT equation, with the difference that the impurity free energy is now determined by the (2S)th member of the set:

$$F^i=-\frac{T}{2\pi}\int d\zeta \frac{1}{\cosh\left[\zeta+\ln\frac{T}{T_0}\right]}\ln\left[1+\eta_{2S}\left[\zeta,\frac{H}{T}\right]\right].$$

The discussion of scaling, cutoff procedure, and universality may be taken over entirely and applied to the present case. Yet still in this framework the few changes we indicated lead to new and interesting features.

Thus consider the zero-temperature impurity magnetization. Its free value is given by $\mathcal{M}^F=\mu\mathcal{S}\,\mathrm{sgn}(H)$, and it is approached asymptotically at high field

$$\mathcal{M}^i \underset{H\gg T_H^h}{\to} \mu 2S\left[1-\frac{1}{2\ln(H/T_H^h)}-\frac{\ln\ln(H/T_H^h)}{4\ln^2(H/T_H^h)}+O\ln^{-3}(H/T_H^h)\right], \tag{7.2}$$

where we introduced the characteristic high-field scale T_H^h by absorbing the $\ln^{-2}(H/T_H^h)$ term.

For low fields, on the other hand, a free spin $(S-\frac{1}{2})$ behavior is logarithmically approached:

$$\mathcal{M}^i \underset{H\ll T_H^l}{\to} \mu(2S-1)\left[1+\frac{1}{2\ln(T_H^l/H)}-\frac{\ln\ln(T_H^l/H)}{4\ln^2(T_H^l/H)}+O\ln^{-3}(T_H^l/H)\right], \tag{7.3}$$

where again a low-field characteristic scale T_H^l has been analogously defined.

We thus observe a crossover from the asymptotically free high-field region where the system exhibits a free-spin behavior with the typical perturbation correction to the low-field region, where now the spin is only partially screened and approaches the values $(S-\frac{1}{2})$.

The ratio of the scales

$$\tilde{W}'=\frac{T_H^h}{T_H^l} \tag{7.4}$$

is a universal number which turns out to be very simple. We shall find, indeed,

$$T_H^l+T_H^h\equiv T_H=\left[\frac{\pi}{e}\right]^{1/2}T_0, \tag{7.5}$$

where $T_0=De^{-\pi/c}$ is the basic scale determined by the fluctuations, and which in the case $S=\frac{1}{2}$ uniquely parametrized the low-field, low-temperature region.

The same pattern, of course, occurs when we consider the temperature dependence. One finds

$$\mathcal{M}^i=\frac{4}{3}\mu\frac{H}{T}\begin{cases}S(S+1)\{1-\ln^{-1}(T/T_k^h)-\frac{1}{2}\ln\ln(T/T_k^h)\ln^{-2}(T/T_k^h)+O[\ln^{-3}(T/T_k^h)]\}, & T\gg T_k^h\\ (S-\frac{1}{2})(S+\frac{1}{2})\{1+\ln^{-1}(T_k^l/T)-\frac{1}{2}\ln\ln(T_k^l/T)\ln^{-2}(T_k^l/T_k)+O[\ln^{-3}(T_k^l/T)]\}, & T\ll T_k^l.\end{cases} \tag{7.6}$$

We observe the crossover to a partially screened spin $(S-\frac{1}{2})$ ground state. The scales T_k^h, T_k^l, the high and low Kondo temperatures, parametrize the respective regions in the temperature. Their ratio is again universal, and we define

$$\tilde{W}=\frac{T_k^h}{T_k^l}. \tag{7.7}$$

We shall find again

$$T_k^h=T_k^l=T_k.$$

Thus the numbers characterizing the crossover properties in the H-T plane are extremely simple, indicative of the fact that the physics in strong coupling regions is of the same nature as in the weak coupling limit.

To complete our account of the universal numbers in this case we also calculate U,

$$U=\frac{T_k}{T_H}. \tag{7.8}$$

This universal number is not *a priori* a crossover characteristic, being defined primarily in the high-field, high-temperature region. It can, however, be considered, in our case, as a generalization of Wilson's number, if we view it as

$$\frac{T_k^h}{T_H^l}=\frac{T_k}{T_H}=U. \tag{7.9}$$

A third way to generalize Wilson's number is to construct

$$W=\frac{T_k}{T_0}, \tag{7.10}$$

the ratio of the temperature scale T_k to the fluctuation scale T_0, which in the former case also played the role of the strong coupling scale. This proliferation of crossover characteristics is due to the fact that now there is no complete screening and thus no unique scale associated with it.

The diagonalization of the higher-spin model and the construction of the thermodynamics were carried out by Fateev and Wiegmann (1981a,1981b) using Baxter-type transfer matrix methods and also by Furuya and Lowenstein (1981) by means of Yang's method. The calculation of the universal numbers was given in the last reference, though it is presented here slightly differently.

A. Diagonalization of the Hamiltonian and the classification of states

The system we study consists of N^e electrons on an interval $-L/2 \le x \le L/2$ interacting with a spin-S impurity located at $x=0$. The Hamiltonian describing the system is (in first-quantized language)

$$h=-i\sum_{j=1}^{N^e}\frac{\partial}{\partial x^j}+2J\sum_{j=1}^{N^e}\delta(x_j)\sigma_j\cdot\mathbf{S}\ ,$$

with $\frac{1}{2}\sigma_j$ being the jth electron spin operator and $\mathbf{S}$ the impurity operator. If we set $S=\frac{1}{2}$, we retrieve (2.7b) (here we did not include potential scattering).

We wish to construct a complete set of wave functions which are simultaneous eigenstates of h, $\mathscr{S}$, and $\mathscr{S}_z$ ($\mathscr{S}$ is the total spin operator) and which are antisymmetric under exchange of electrons.

We shall use the formalism developed in Sec. II.G. Thus consider a wave function of the form

$$G_\alpha(x,a)=\sum_P(-1)^P\exp i\sum_{j=1}^{N^e}k_{Pj}x_j\sum_Q\sum_{\nu=0}^{N^e}\theta_\nu(x_Q)\xi^\alpha_{a_{Q,\alpha}}\ , \tag{7.11}$$

where $x=(x_1,\ldots,x_{N^e})$ and $a=(a_1,\ldots,a_{N^e})$ are the coordinates and spin indices of the electrons, and where α denotes the impurity spin index, $-S\le\alpha\le S$.

$$\theta_\nu(x_Q)=\theta(x_{Q1}<\cdots<x_{Q\nu}<0<x_{Q(\nu+1)}<\cdots<x_{QN^e})$$

locates the impurity within a given arrangement of the electrons $Q\in S_{N^e}$, and $a_Q=(a_{Q1},\ldots,a_{QN^e})$. As we have incorporated fermionic bookkeeping, the momenta satisfy $k_1<k_2<k_3<\cdots<k_{N^e}$.

The energy eigenvalue equation places the following jump condition on the ξ coefficients:

$$\xi^{\nu-1}_{a_{Q,\alpha}}-\xi^{\nu}_{a_{Q,\alpha}}+J(\sigma)_{a_{Q\nu},a'_{Q\nu}}\cdot(\mathbf{S})_{\alpha\alpha'}(\xi^{\nu-1}_{a_{Q1}\cdots a'_{Q\nu}\cdots a_{QN^e}\alpha'}+\xi^{\nu}_{a_{Q1}\cdots a'_{Q\nu}\cdots a_{QN^e}\alpha'})=0\ , \tag{7.12}$$

and the periodic boundary condition imposes

$$\xi^{\nu}_{a_j,\underline{a}\alpha}=e^{ik_jL}\xi^{\nu-1}_{\underline{a},a_j\alpha}\ , \tag{7.13}$$

where $\underline{a}$ stands for any permutation of the N^e-1 indices $\{a_k,k\ne j\}$.

Combining (7.12) and (7.13), we obtain an eigenvalue equation for $\xi^0_{a,\alpha}$,

$$Z\xi^0=\lambda\xi^0\ , \tag{7.14}$$

where

$$\lambda=e^{ik_jL}\ , \tag{7.15}$$

$$Z=Y_{0N}\tilde{P}\ . \tag{7.16}$$

Here $\tilde{P}$ is a cyclic permutation operator acting on the electronic indices, $\tilde{P}=2\,3\ldots N^e1$, and

$$Y_{0N}=\hat{r}1+\hat{s}\sigma^z_{N^e}S^z+\tfrac{1}{2}\hat{s}(\sigma^+_{N^e}S^-+\sigma^-_{N^e}S^+)\ , \tag{7.17}$$

where

$$\hat{r}=\frac{J+i[1-S(S+1)J^2]}{J+i[1+S(S+1)J^2]}\ , \tag{7.18a}$$

$$\hat{s}=\frac{2J}{J+i[1+S(S+1)J^2]}\ , \tag{7.18b}$$

so that $|\hat{r}+S\hat{s}|=1$, and we may set $\hat{r}+S\hat{s}=e^{i\eta}$, with η real.

We wish to give $\xi^0_{a,\alpha}$ a more convenient representation. Thus consider a lattice of $N=N^e+1$ sites and a set of M billiard balls, i.e., down spins (here balls and vacancies play the role of the striped and unstriped balls of Sec. II and Appendix B). In contradistinction to the spin-$\frac{1}{2}$ case, however, site 0, the position of the impurity, is special in that it may contain up to $2S$ balls, allowing $2S+1$ impurity spin states with the z component determined by the number of balls α at the site,

$$S_z=(S-\alpha),\quad \alpha=0,\ldots,2S\ .$$

Thus, as we have $2S+N^e$ slots for balls, M of which are occupied (i.e., are down-spin-$\frac{1}{2}$), the total z component is $\mathscr{S}_z=\frac{1}{2}[2S+N^e-2M]$. Denoting by y_j the position of the jth ball, we may represent $\xi^0_{a,\alpha}$ as follows:

$$\xi^0_{a_1,\ldots,a_{N^e},\alpha}=\phi_\alpha(y_1,\ldots,y_q)\ , \tag{7.19}$$

where $S-\alpha$ balls are at the site 0 and $q=M-(S-\alpha)$ of them at sites $y_1,\ldots,y_q$, with $0<y_1<\cdots<y_q\le N^e$. The z-component angular momentum is conserved by the Hamiltonian, which connects only amplitudes which satisfy

$$\mathscr{S}_z=\tfrac{1}{2}[S+N^e-2M]=\tfrac{1}{2}N^e+\alpha-q\ .$$

In terms of this representation the eigenvalue equation becomes

$$\lambda\phi_\alpha(y_1,\ldots,y_{q-1},N^e)=(r-\alpha s)\phi_\alpha(1,y_1+1,\ldots,y_{q-1}+1)+s\sqrt{(S+\alpha)(S-\alpha+1)}\,\phi_{\alpha-1}(y_1+1,\ldots,y_{q-1}+1)\ , \tag{7.20a}$$

$$\lambda\phi_\alpha(y_1,\ldots,y_q<N^e)=(r+\alpha s)\phi_\alpha(y_1+1,\ldots,y_q+1)+s\sqrt{(S-\alpha)(S+\alpha+1)}\,\phi_{\alpha+1}(1,y_1+1,\ldots,y_q+1)\ , \tag{7.20b}$$

where

$$\lambda=\frac{\hat{\lambda}}{\hat{r}+\hat{s}S}\ , \tag{7.21}$$

$$r=\frac{\hat{r}}{\hat{r}+\hat{s}S}=\frac{i+\frac{c}{2}}{i+(S+\frac{1}{2})c}\ , \tag{7.22a}$$

$$s=\frac{\hat{s}}{\hat{r}+\hat{s}S}=\frac{c}{i+(S+\frac{1}{2})c}\ , \tag{7.22b}$$

with $c=2J/[1-S(S+1)J^2]$ generalizing to arbitrary spin the spin-$\frac{1}{2}$ result $c=2J/(1-\frac{3}{4}J^2)$ [obtained from (2.42) by setting $J'=0$, i.e., $J''=J$].

As the Hamiltonian is rotationally invariant, we may require that the wave function have a definite angular momentum $\mathscr{S}$ (it has a definite $\mathscr{S}_z$ by construction). Thus we impose

$$\mathscr{S}_+\xi^0=0\ ,$$

which selects these states with $\mathscr{S}=\mathscr{S}_z$. The other states may be obtained by a repeated application of $\mathscr{S}_-$. In terms of the $\phi_\alpha(y)$ representation, this condition takes the form

$$\sqrt{(S-\alpha)(S+\alpha+1)}\phi_\alpha(y_1,\ldots,y_q)=-\sum_{1\le y\ne y_j}\phi_{\alpha+1}(y_1,\ldots,y,\ldots,y_q)\ , \tag{7.23}$$

where $-S-1\le\alpha\le S$.

This condition may be recognized as generalizing the corresponding condition for spin-$\frac{1}{2}$ as presented in Appendix B.

The discrete eigenvalue problem is again susceptible to a Bethe-ansatz solution. Using the interpretation of the variables $\alpha,y_1,\ldots,y_q$ in the $\phi_\alpha(y)$ representation, and studying the cases of $M=1,2$, we are led to generalize the form of the ansatz, as presented in Appendix A, as follows:

$$\phi_\alpha(y_1,\ldots,y_2)=\nu_\alpha\sum_P A_P\prod_{i=1}^{S-\alpha}\phi(\Lambda_{Pi})\times\prod_{j=1}^{q}f(\Lambda_{P(S-\alpha+j)},y_j)\ , \tag{7.24}$$

where the summation is over all permutations of $M=S-\alpha+q$ symbols, where ϕ and f are single down-spin wave functions, the first being associated with the site 0 and the latter with the rest, and where A_p and ν_α are coefficients to be determined.

By methods analogous to those discussed in Appendix A the following solution to the ansatz is constructed. The single particle wave functions are

$$f(\Lambda_j,y)=\mu_j^{y-1}\ , \tag{7.25a}$$

$$\phi(\Lambda)=\sqrt{2S}\frac{i(1-\Lambda)-\frac{c}{2}}{i(-\Lambda)+Sc}\ , \tag{7.25b}$$

with

$$\mu_j=\frac{i(1-\Lambda_j)+\frac{c}{2}}{i(1-\Lambda_j)-\frac{c}{2}}\ ,$$

and the coefficients are given by

$$\nu_\alpha=\left[(2S)^{-(S-\alpha)}\begin{bmatrix}2S\\S-\alpha\end{bmatrix}\right]^{1/2}\ , \tag{7.26a}$$

$$\frac{A_{p'}}{A_p}=\frac{i(\Lambda_{pj}-\Lambda_{p'j})+c}{i(\Lambda_{pj}-\Lambda_{p'j})-c}\ , \tag{7.26b}$$

where $P'j=P(j+1)$, $P'(j+1)=Pj$, and where the parameters $\{\Lambda_\gamma\}$ satisfy the coupled set of equations

$$\left[\frac{i(1-\Lambda_\gamma)+\frac{c}{2}}{i(1-\Lambda_\gamma)-\frac{c}{2}}\right]^{N^e}\left[\frac{i(-\Lambda_\gamma)+Sc}{i(-\Lambda_\gamma)-Sc}\right]=-\prod_{\delta=1}^{M}\frac{i(\Lambda_\delta-\Lambda_\gamma)+c}{i(\Lambda_\delta-\Lambda_\gamma)-c}\ . \tag{7.27}$$

Setting $S=\frac{1}{2}$ reduces these equations to those presented in Appendix A. In terms of the $\{\Lambda_\gamma\}$ solution of (7.27) the eigenvalue $\hat{\lambda}$ of Z is found to be

$$\hat{\lambda}=e^{i\eta}\prod_{j=1}^{M}\mu_j\ . \tag{7.28}$$

For a detailed proof see Furuya and Lowenstein (1981). Equations (7.27) and (7.28) have been deduced by Fateev and Wiegmann, using transfer-matrix formalism.

B. The spectrum

From the eigenvalues $\hat{\lambda}$ of the Z matrix we find the allowed momenta k_i and construct the energy eigenvalues:

$$E=\sum_{j=1}^{N^e}\frac{2\pi}{L}n_j+D\sum_{\gamma=1}^{M}[\Theta(2\Lambda_\gamma-2)-\pi]\ , \tag{7.29}$$

where again $D=N^e/L$, $\Theta(x)=-2\tan^{-1}(x/c)$, and where we have dropped the term $(N^e/L)\eta$. As before, we introduce the cutoff K by imposing $(2\pi/L)n_j|\le K$, and transfer here the philosophy, discussion, and song and dance presented in the spin-$\frac{1}{2}$ case.

The spin momenta $\{\Lambda_\gamma\}$ are determined from

$$N^e\Theta(2\Lambda-2)+\Theta\left[\frac{1}{S}\Lambda\right]=\sum_{\delta=1}^{M}\Theta(\Lambda_\gamma-\Lambda_\delta)+2\pi I_\gamma\ , \tag{7.30}$$

where I_γ are integers (half integers) depending on (N^e-M)'s being even (odd). The configurations $\{n_j,I_\gamma\}$ are the quantum numbers of the states.

We proceed as in the spin-$\frac{1}{2}$ case:

The ground state. The $\{n_j\}$ quantum numbers are at their minimum, subject to the cutoff K, and the $\{I_\gamma\}$ spin quantum numbers form a consecutive sequence $I_{\gamma+1}=I_\gamma+1$.

We introduce a Λ density, $\sigma(\Lambda)$, by $\Lambda_{\gamma+1}-\Lambda_\gamma = 1/\sigma(\Lambda_\gamma)$, and deduce an integral equation for it (see Sec. III for details),

$$\sigma_{gs}(\Lambda)+\int d\Lambda'\sigma_{gs}(\Lambda')K(\Lambda-\Lambda')=f^s(\Lambda)\ , \tag{7.31}$$

where

$$K(\Lambda)=\frac{1}{\pi}\frac{1}{c^2+\Lambda^2} \tag{7.32a}$$

as before, while f is modified to

$$f^s(\Lambda)=\frac{c}{\pi}\left[\frac{\frac{1}{2}N^e}{(\frac{1}{2}c)^2+(\Lambda-1)^2}+\frac{S}{(Sc)^2+\Lambda^2}\right] . \tag{7.32b}$$

The solution of Eq. (7.31) yields the ground-state Λ distribution (in Fourier space)

$$\tilde{\sigma}_{gs}(p)=\frac{N^e e^{-ip}}{2\cosh\left[\frac{cp}{2}\right]}+\frac{e^{-[S-(1/2)]|c/2|p}}{2\cosh\left[\frac{cp}{2}\right]} . \tag{7.33}$$

Hence we determine the spin of the ground state from $M=\tilde{\sigma}_{gs}(0)=\frac{1}{2}(N^e+1)$, which leads to

$$\mathcal{S}=S+\tfrac{1}{2}N^e-\tfrac{1}{2}(N^e+1)=S-\tfrac{1}{2} . \tag{7.34}$$

The impurity spin thus is only partially screened (Mattis, 1966), and the ground state is $2S$-fold degenerate. When we study the susceptibility, we shall indeed find that the susceptibility diverges accordingly.

While the nature of the ground state is modified, the structure of excitations built upon it is identical to that of the spin-$\frac{1}{2}$ case. That is because the excitations are determined by the kernel $K(\Lambda)$, which is the same in both cases, while the ground state depends also on $f^s(\Lambda)$.

Density excitations. These excitations, associated with the $\{n_j\}$ quantum numbers, are as boring as before. They correspond to massless particle-hole fluctuations which decouple from the rest of the spectrum (in the scaling regime).

Spin excitations. These are induced by variations in the $\{I_\gamma\}$ quantum numbers.

Thus generating "holes" in the ground-state sequence leads (as in Sec. III) to the density equation

$$\sigma(\Lambda)+\sigma^h(\Lambda)+\int K(\Lambda-\Lambda')\sigma(\Lambda')d\Lambda'=f^s(\Lambda)\ , \tag{7.35}$$

with

$$\sigma^h(\Lambda)=\sum_{i=1}^{N^h}\delta(\Lambda-\Lambda_i^h)\ ,$$

where $\{\Lambda_i^h, i=1,\ldots,N^h\}$ are the "hole" locations. The solution is identical to the spin-$\frac{1}{2}$ case,

$$\tilde{\sigma}(p)=\tilde{\sigma}_{gs}(p)-\sum_{j=1}^{N^h}\frac{e^{-i\Lambda_j^h p}e^{(c/2)|p|}}{2\cosh\left[\frac{cp}{2}\right]} , \tag{7.36}$$

and leads to the same excitation energy,

$$\Delta E=E-E_{gs}=2D\sum_{j=1}^{N^h}\tan(e^{(\pi/c)(\Lambda_j^h-1)}) . \tag{7.37}$$

Each hole excitation is a spin-$\frac{1}{2}$ object, as follows from

$$\Delta M=\Delta\tilde{\sigma}(p=0)=-\tfrac{1}{2}N^h . \tag{7.38}$$

These spin-$\frac{1}{2}$ excitations, the dressed electrons, correspond to those in the $S=\frac{1}{2}$ case or to the spin-wave excitations in the Heisenberg model.

If we hold the number of electrons fixed, then N^h must be even, and the simplest excitation is the triplet where the two spins are coupled symmetrically.

The two excitations may also be coupled antisymmetrically to give spin-zero by the addition of a two-string $\{\Lambda^+,\Lambda^-\}$, where $\Lambda^\pm=\frac{1}{2}(\Lambda_1^h+\Lambda_2^h)\pm ic/2$, to the Λ-momenta sea. The same calculation as before now shows that $\Delta M=(-\frac{1}{2}\cdot 2)+1=0$; thus the state is a singlet, and

$$\Delta E=2D[\tan^{-1}(e^{(\pi/c)(\Lambda_1^h-1)})+\tan^{-1}(e^{(\pi/c)(\Lambda_2^h-1)})]$$

and hence degenerate with the triplet. Again the direct contribution from the string is canceled against the induced backflow of the sea.

To obtain the general structure of excitations we adopt (with due caveats) the string hypothesis (see Sec. III) and rewrite Eq. (7.29) in terms of the real parts $\Lambda_\gamma^{(n)}$ of the string

$$\Lambda_\gamma^{(n)}=\Lambda_\gamma^{(n)}+i\frac{c}{2}(n+1-2j),\quad j=1,\ldots,n$$

to yield the generalization of Eq. (3.36) to arbitrary spin

$$N^e\Theta\left[\frac{2\Lambda_\gamma^{(n)}-2}{n}\right]+\sum_{l=1}^{\min(2S,n)}\Theta\left[\frac{2\Lambda_\gamma^{(n)}}{2S+n+1-2l}\right]=\sum_{m=1}^{\infty}\sum_{\delta=1}^{M_m}\Theta_{nm}(\Lambda_\gamma^{(n)}-\Lambda_0^{(m)})+2\pi I_\gamma^{(n)}\ , \tag{7.39}$$

where Θ_{mn} has been defined before and we have M_n n-strings, $n=1,2,\ldots,\infty$.

We convert these algebraic equations to an integral equation by the usual procedure (see Sec. III). Thus having chosen the configurations $\{I_\gamma^{(n)}\}$ and determined the corresponding n-string $\{\Lambda_\gamma^{(n)}\}$, one forms the functions

$$V_n(\Lambda)=\frac{1}{2\pi}\left[N^e\Theta\left[\frac{2\Lambda-2}{n}\right]+\sum_{l=1}^{\min(2S,n)}\Theta\left[\frac{2\Lambda}{2S+n+1-2l}\right]-\sum_{m=1}^{\infty}\sum_{\delta=1}^{M_m}\Theta_{mn}(\Lambda-\Lambda_\delta^{(m)})\right] ,$$

where the $\Lambda_\delta^{(m)}$ have been obtained before. Those values of Λ satisfying $V_n(\Lambda_\gamma^{(n)})=I_\gamma^{(n)}$ are the already known n-strings and those satisfying $V_n(\Lambda_\delta^{(n)h})=J_\delta^{(n)}$, where $J_\delta^{(n)}$

are the integers not included in the $\{I_\gamma^{(n)}\}$ sequence, are the n-string holes. Introducing n-string and n-string hole densities σ_n and σ_n^h, respectively, we have from the definition of V_n

$$\frac{dV_n}{d\Lambda}=\sigma_n^h(\Lambda)+\sigma_n(\Lambda)\ . \tag{7.40}$$

Hence we obtain for σ_n and σ_n^h a set of integral equations

$$f_n=\sigma_n^h+\sum_m A_{nm}\sigma_m\ , \tag{7.41}$$

where

$$f_n=f_n^e+f_n^i\ , \tag{7.42a}$$

$$f_n^e=[n]N^e\delta(\Lambda)\ , \tag{7.42b}$$

$$f_n^i=\sum_{j=1}^{\min(2S,n)}[2S+n+1-2j]\delta(\Lambda) \tag{7.42c}$$

(A_{nm} and $[n]$ have been defined before). This generalizes Eq. (3.40) by modifying the impurity part f_n^i.

Obviously, the above discussion applies to the Heisenberg model, where one spin-$\frac{1}{2}$ has been replaced by a higher spin S.

C. The zero-temperature magnetization

The method used to calculate the zero-temperature magnetization curve has been detailed in Sec. IV, and we shall therefore omit many details here.

In the presence of a magnetic field H the lowest-energy state is characterized by a density of real Λ-momentum which vanishes in the "depletion region" $(-\infty,B)$ and thus satisfies

$$\sigma_B(\Lambda)+\int d\Lambda'\sigma_B(\Lambda')K(\Lambda-\Lambda')=f^s(\Lambda)\ , \tag{7.43}$$

for which f^s is given in (7.32) and K in (7.31). The Fermi level B is determined by the magnetic field. This equation is of the Wiener-Hopf type and is solved by Fourier transforming and factorizing the resulting kernel to obtain for $\rho_\pm(\Lambda)=\theta(\pm\Lambda)\sigma(\Lambda+B)$ (see Sec. IV)

$$K_+\left[\frac{cp}{2}\right]\tilde\rho_+(p)+K_-\left[\frac{cp}{2\pi}\right]\tilde\rho_-(p)=K_-\left[\frac{cp}{2\pi}\right]\tilde f_s(p)e^{ipB}$$
$$=K_+\left[\frac{cp}{2\pi}\right]\tilde g_s(\Lambda)e^{ip\Lambda}\ , \tag{7.44}$$

where we have introduced $\tilde g_s(p)=\tilde f_s(p)/[1+\tilde K(p)]=\tilde\sigma_0(p)$, which is the ground-state Λ density given by (7.33).

The Laplace transforms of $f(\Lambda)$ and $g(\Lambda)$ are given by

$$f(\Lambda)=\frac{1}{\pi}\left[\frac{N^e\frac{1}{2}}{\left[\frac{c}{2}\right]^2+(\Lambda-1)^2}+\frac{S}{(Sc)^2+\Lambda^2}\right]$$
$$=\frac{1}{\pi}\int_0^\infty dt\left[N^e e^{-|\Lambda-1|t}\sin\frac{ct}{2}+e^{-|\Lambda|t}\sin(Sct)\right]\ , \tag{7.45}$$

$$g(\Lambda)=\frac{1}{c}\frac{N^e}{2\cosh\left[\frac{\pi}{c}(\Lambda+1)\right]}+\frac{2}{c}\sum_{j=0}(-1)^j\frac{j+S}{(j+S)^2+\Lambda^2}$$
$$=\frac{N^e}{c}\sum_{j=0}^\infty(-1)^j e^{-(\pi/c)(\Lambda-1)(2j+1)}$$
$$+\frac{2}{c}\sum_{j=0}^\infty(-1)^j\int_0^\infty dt\sin[(j+S)ct]e^{-|\Lambda|t}\ , \tag{7.46}$$

which allows us to solve for $\rho_\pm(p)$:

$$\tilde\rho_\pm(p)=\begin{cases}\displaystyle\sum_{j=0}^\infty(-1)^j\frac{N^e}{c}\tilde\rho_\pm\left[p,t=(2j+1)\frac{\pi}{c};B-1\right]+2\int^\infty dt\sin[(j+S)ct]\tilde\rho_\pm(p,t;B), & B\le 0\\ \displaystyle\frac{1}{\pi}\int_0^\infty dt\left[N^e\sin\left[\frac{ct}{2}\right]\tilde\rho_\pm(p,t;B-1)+\sin(Sct)\tilde\rho_\pm(p,t;B)\right], & 0\le B\ll 1\ ,\end{cases} \tag{7.47}$$

with $\tilde\rho_\pm(p,t;B)$ given by (4.21).

We find therefore the following expression for the magnetization:

$$\mathcal{M}=2\mu\mathcal{S}=\mu[N^e+2S-2\tilde\rho_+(0)]=\mu[2S-1+\tilde\rho_-(0)]=\mathcal{M}^e+\mathcal{M}^i\ . \tag{7.48}$$

The electronic contribution is given by

$$\mathcal{M}^e=\mu\left[\frac{2}{\pi c}\right]^{1/2}LT_0e^{\pi B/c}\ , \tag{7.49}$$

where we have kept only the leading term in $e^{-\pi/c}$ (higher terms lead to departure from scaling). The impurity contribution is

$$\mathcal{M}^i=\begin{cases}\mu\left[2S-1+\sum_{j=1}^{\infty}(-1)^j\int_0^\infty\frac{dt}{t}\sin[(j+S)2\pi t]e^{2\pi Bt/c}e^{t(\ln t-1)}\Gamma(\frac{1}{2}+t)\right], & B\le 0\\ \mu\left[2S-\pi^{-3/2}\int_0^\infty\frac{dt}{t}\sin(2\pi St)e^{-2\pi Bt/c}e^{t(1-\ln t)}\Gamma(\frac{1}{2}+t)\right], & 0\le B\ll 1\ .\end{cases}\tag{7.50}$$

Identifying the electronic part, as before, with the contribution of a free gas of electrons, we find

$$\mathcal{M}^e=\mu\left[\frac{2}{\pi e}\right]^{1/2}LT_0e^{\pi B/c}=\mu\frac{HL}{\pi}\ ,\tag{7.51}$$

so that the dependence of the Fermi level B on the magnetic field is given by $e^{\pi B/c}=(e/2\pi)^{1/2}(H/T_0)\equiv H/T_1$, as before.

Inserting this relation into the expressions for $\mathcal{M}^i$, we find the magnetization curve $\mathcal{M}^i=\mathcal{M}^i(H/T_0)$ exhibiting the scaling phenomenon.

In the present case, where $S>\frac{1}{2}$, the asymptotic behavior for $H\gg T_0$ and $H\ll T_0$ are controlled by the behavior of the integrand near $t=0$. In order to get the expansion of the integrand in (7.50) for small t of the expression valid for $B\le 0$, we first sum over j, using the following identity:

$$\sum_{j=0}^{\infty}(-1)^j\sin[(j+S)2\pi t]=\frac{1}{2}\frac{\sin[(2S-1)\pi t]}{\cos\pi t},\quad t<\tfrac{1}{2},\ S\ge 1\ .$$

We then get

$$\mathcal{M}^i=\begin{cases}\mu(2S-1)\left[1+\int_0^\infty dt\,e^{-2t\ln(H/T_1)}[1-t(1-C-2\ln 2)+t\ln t+O(t^2)]\right], & H\ll T_1\\ \mu 2S\left[1+\int_0^\infty e^{-2t\ln(H/T_1)}[1+t(1-C-2\ln 2)-t\ln t+O(t^2)]\right], & T_1\ll H\ ,\end{cases}\tag{7.52}$$

where C is Euler's constant. Using again Eq. (4.31), we find the asymptotic form of $\mathcal{M}^i$:

$$\mathcal{M}^i=\mu\begin{cases}(2S-1)\left[1+\frac{1}{2\ln(T_H/H)}-\frac{\ln\ln(T_H/H)}{4\ln^2(T_H/H)}+O\left[\frac{1}{\ln^3(H/T_H)^2}\right]\right], & H\ll T_1\\ 2S\left[1-\frac{1}{2\ln(H/T_H)}-\frac{\ln\ln(H/T_H)}{4\ln^2(H/T_H)}+O\left[\frac{1}{\ln^3(H/T_H)^2}\right]\right], & T_1\ll H\ll D\ ,\end{cases}\tag{7.53}$$

where we have redefined the scale in such a way as to absorb the $\ln^{-2}(H/T_0)$ term.

The new scale is given by

$$T_H=\left[\frac{\pi}{e}\right]^{1/2}T_0\ ,\tag{7.54}$$

just as before; however, note that it is the same scale that governs both regions. This need not have been the case, as the $\ln^{-2}(T_1/H)$ term in the one region and the $\ln^{-2}(H/T_1)$ in the other could have been multiplied by different coefficients and thus have led to two different scales. As it turns out, the situation here is simpler than in the spin-$\frac{1}{2}$ case, and we find (Furuya and Lowenstein, 1981),

$$\tilde{W}'=1,\quad S>\tfrac{1}{2}\ .\tag{7.55}$$

We still do observe a crossover behavior. At high fields we have the perturbative region where the magnetization tends to the free value $\mathcal{M}_{\text{free}}=2\mu S$ and has logarithmic corrections typical of asymptotic freedom.

At low temperature, however, the impurity is partially screened (Mattis, 1964), and the magnetization approaches, up to logarithmic corrections, the value $\mu(2S-1)$, as expected from the study of the ground state.

D. The thermodynamics

The procedure is the same as discussed in Sec. V. The partition function factorizes, $Z=Z_1Z_0$, where Z_0 describes, as before, the density fluctuations and is not modified. The spin-fluctuation part Z_Λ is expressed as a functional integral over σ_n,σ_n^h the n-string and n-string hole density, respectively. In terms of the densities we have for the energy and entropy (see Sec. V),

$$E_\Lambda+2HM=\sum_{n=1}^{\infty}\int d\Lambda\,\sigma_n(\Lambda)g_n(\Lambda)\ ,\tag{7.56}$$

$$\mathcal{S}=\sum_{n=1}^{\infty}\int d\Lambda[(\sigma_n+\sigma_n^h)\ln(\sigma_n+\sigma_n^h)-\sigma_n\ln\sigma_n-\sigma_n^h\ln\sigma_n^h]\ ,\tag{7.57}$$

where

$$g_n(\Lambda)=D\left[\Theta\left[\frac{2\Lambda-2}{n}\right]-\pi\right]+2Hn\ .$$

The densities $\{\sigma_n,\sigma_n^h\}$ are constrained by the equations

$$f_n=\sigma_n^h+\sum_{m=1}^{\infty}A_{nm}\sigma_m,\quad n=1,2,\ldots, \tag{7.58}$$

where

$$f_n=[n]N^e\delta(\Lambda-1)+\sum_{j=1}^{\min(2S,n)}[2S+n+1-2S]\delta(\Lambda)\ .$$

Thus the change from the spin-$\frac{1}{2}$ case occurs only in the impurity part of f_n (Fateev and Wiegmann, 1981; see also Furuya and Lowenstein, 1981).

As the dependence of the entropy and energy on the densities is as before, the minimization leads to the same set of thermodynamic equations for $\eta_n=\sigma_n^h/\sigma_n$:

$$\ln[1+\eta_n(\Lambda)]=\frac{g_n(\Lambda)}{T}+\sum_{m=1}^{\infty}A_{nm}\ln[1+\eta_m^{-1}(\Lambda)]\ . \tag{7.59}$$

The free energy, however, does take on a different form. As before, it is given by

$$F_\Lambda=-T\sum_{n=1}^{\infty}\int d\Lambda\, f_n(\Lambda)\ln(1+\eta_n^{-1})\ ,$$

but f_n may now be rewritten, using the identities

$$[n]=GA_{n1} \tag{7.60a}$$

and

$$\sum_{j=1}^{\min(2S,n)}[2S+n+1-2j]=GA_{2S,n}\ , \tag{7.60b}$$

as

$$f_n(\Lambda)=G[N^eA_{n1}\delta(\Lambda-1)+A_{2S,n}\delta(\Lambda)]\ . \tag{7.61}$$

Again, using the GT equations (7.59), for $n=1$ and $n=2S$, F_Λ becomes

$$F_\Lambda=\int d\Lambda\,\sigma_0^e(\Lambda)\{g_1(\Lambda)-T\ln[1+\eta_1(\Lambda)]\}+\int d\Lambda\,\sigma_0^i[g_{2S}-T\ln(1+\eta_{2s})]\ , \tag{7.62}$$

where

$$\sigma_0^e=GN^e\delta(\Lambda-1)=\frac{1}{2c}\frac{N^e}{\cosh\left[\frac{\pi}{c}(\Lambda-1)\right]}$$

and

$$\sigma_0^i=G\delta(\Lambda)=\frac{1}{2c}\frac{1}{\cosh\frac{\pi}{c}\Lambda}\ .$$

Combining now the spin and density contributions, we find

$$F=E_0-\frac{\pi LT^2}{12}-T\int d\Lambda\{\sigma_0^e(\Lambda)\ln[1+\eta_1(\Lambda)]+\sigma_0^i(\Lambda)\ln[1+\eta_{2S}(\Lambda)]\}\ , \tag{7.63}$$

with E_0 being the T- and H-independent ground-state energy (cf. discussion in Sec. V).

The impurity part F^i has thus been modified to

$$F^i=-T\int d\Lambda\,\sigma_0^i(\Lambda)\ln[1+\eta_{2S}(\Lambda)]\ , \tag{7.64}$$

while the electronic contribution is unaltered.

The scaling form of thermodynamics is achieved by taking the proper limit: $D\to\infty$, $c\to0$ with $T_0=De^{-\pi/c}$ fixed. Changing to the $\zeta=\pi\Lambda/c+\ln(T_0/T)$ variable, we find for the GT equations

$$\ln\eta_1=-2e^{\zeta}+G\ln(1+\eta_2)\ , \tag{7.65a}$$

$$\ln\eta_n=G\ln(1+\eta_{n+1})+G\ln(1+\eta_{n-1})\ , \tag{7.65b}$$

$$\lim_{n\to\infty}\{[n+1]\ln(1+\eta_n)-[n]\ln(1+\eta_{n+1})\}=\frac{2H}{T}\ , \tag{7.65c}$$

and for the free energy

$$F=E_0+F^e+F^i+O(e^{-\pi D/T})\ , \tag{7.66}$$

with

$$F^e=-\frac{\pi LT^2}{12}-TLDf_{(1)}\left[\frac{T}{D},\frac{H}{T}\right]\Bigg|_{D\to\infty}\ , \tag{7.67}$$

$$F^i=-Tf_{(2S)}\left[\frac{T}{T_0},\frac{H}{T}\right]\ , \tag{7.68}$$

where

$$f_{(j)}(t,h)=\frac{1}{2\pi}\int d\zeta\frac{1}{\cosh(\zeta+\ln t)}\ln[1+\eta_j(\zeta,h)]\ . \tag{7.69}$$

We proceed to discuss now the low- and high-temperature regions. The functions η_n are smooth, monotonously decreasing in ζ, and tending to finite limits as $\zeta\to\pm\infty$. The most important contribution to F^i comes from the region $\zeta\sim\ln(T_0/T)$, i.e., ζ large and negative (positive) for $T\gg T_0$ ($T\ll T_0$). The case $S=\frac{1}{2}$ is an exception, in that $\eta_1(\zeta=\infty)=0$, so that also the intermediate region in ζ becomes important. This case was discussed in Sec. V, where it was shown that information may still be extracted through the known properties of the free fermion gas. The $S>\frac{1}{2}$ case is more straightforward, and we can construct an asymptotic expansion from the properties of η_n.

Thus

$$F_i\sim-\frac{T}{2}\ln[1+\eta_{2S}(\zeta\to\mp\infty,x_0)]\ , \tag{7.70}$$

$$\frac{T}{T_0}\to\begin{cases}\infty, & S>\frac{1}{2};\\ 0,\end{cases}$$

and using the asymptotic values $\eta_{2S}^{\pm}$ (5.32), we have

$$F^i=\begin{cases}-T\ln\left[\dfrac{\sinh(2S+1)x_0}{\sinh x_0}\right], & \dfrac{T}{T_0}\to\infty\\ -T\ln\left[\dfrac{\sinh 2Sx_0}{\sinh x_0}\right], & \dfrac{T}{T_0}\to 0\,,\end{cases}\tag{7.71}$$

where $x_0=H/T$.

To find corrections to the leading behavior, we have simply to use the asymptotic expansion of $\eta_n(\zeta)$ as studied in Sec. V:

$$\tfrac{1}{2}\ln[1+\eta_n(\zeta,h)]=\ln\left[\frac{\sinh mx_0}{\sinh x_0}\right]+(m\coth mx_0-\coth x_0)x_0\left[\frac{1}{2\zeta}-\frac{1}{4\zeta^2}\ln|\zeta|\right]+O\left[\frac{1}{\zeta^2}\right],\tag{7.72}$$

with

$$m=\begin{cases}n+1 & \text{if } \zeta\to-\infty\\ n & \text{if } \zeta\to+\infty\,.\end{cases}$$

Hence

$$-\frac{F^i}{T}=\begin{cases}\ln\dfrac{\sinh(2S+1)x_0}{\sinh x_0}+[(2S+1)\coth(2S+1)x_0-\coth x_0]x_0\left[\dfrac{1}{2\ln(T_0/T)}-\dfrac{\ln|\ln(T_0/T)|}{4\ln(T_0/T)}\right], & D\gg T\gg T_0\\ \ln\dfrac{\sinh 2Sx_0}{\sinh x_0}+(2s\coth 2Sx_0-\coth x_0)x_0\left[\dfrac{1}{2\ln(T_0/T)}-\dfrac{\ln|\ln(T_0/T)|}{4\ln(T_0/T)}\right], & T\ll T_0\,,\end{cases}\tag{7.73}$$

where the corrections are of order $\ln^{-2}(T/T_0)$ in both the high-temperature and low-temperature regions. These corrections can be used to define the respective scales T_K^h and T_K^l by absorbing them into the $\ln^{-2}$ term. We shall find $T_K^h=T_K^l=T_K=WT_0$, where W is determined in (7.85). This is the spin-S Wilson crossover number.

The two asymptotic expressions differ only in that the factor $(2S+1)$ in the high-temperature expression is replaced by the factor $2S$ in the low-temperature expression, thus indicating a transition from the asymptotically free limit, where the impurity spin is unscreened to the low-temperature region, where its value is effectively reduced to $S-\frac{1}{2}$, in accord with the study of the ground state.

E. The numerical solution

We now proceed to display some thermodynamic functions for different values of the impurity spin S. To do so one needs to solve the GT equations (7.65), which are the same as for the spin-$\frac{1}{2}$ impurity case (Sec. V). Having obtained the set of functions $\{\eta_n\}$, one finds the impurity free energy for the spin-S case from the function η_{2S}, using Eq. (7.64).

The results (Rajan *et al.*, 1982) are displayed below. In Fig. 17 we have plotted $T\chi(T)/\mu^2$ as a function of the temperature for different values of the spins $S=\frac{1}{2},1,\frac{3}{2}$. This quantity is a measure of the effective spin squared. The curves manifest the screening of half a unit of the spin as the temperature is lowered below T_k to the strong coupling regime.

In Fig. 18 we have plotted the specific heat C_v^i as a function of the temperature for spin values $S=\frac{1}{2},1,\frac{3}{2}$. We have also indicated for comparison the results of a renormalization-group calculation (Oliveira and Wilkins, 1981) performed on the GRL (generalized resonance level) model.

We observe that for a higher spin the curve C_v^i,free$=0$ is approached. This is the value of the specific heat (for

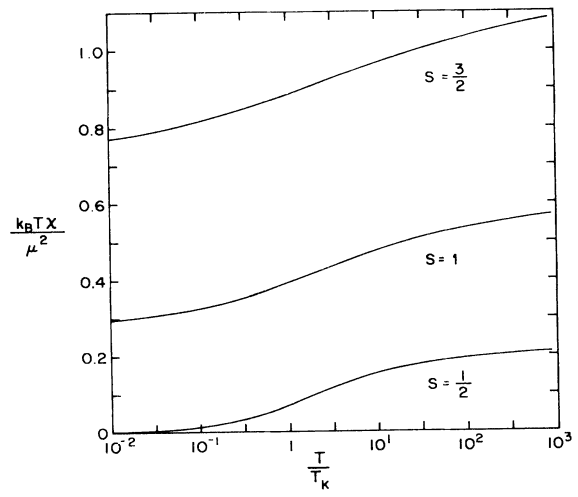

FIG. 17. Effective spin $[\frac{1}{3}S_{\rm eff}(S_{\rm eff}+1)]$ graphs as a function of the temperature for $S=\frac{1}{2},1,\frac{3}{2}$. The curves cross over from $S_{\rm eff}=S$ at high temperatures to $S_{\rm eff}=S-\frac{1}{2}$ at $T=0$.

zero magnetic field) in the absence of interactions. The fact that by increasing the spin value the effective interaction is weakened ("asymptotic freedom phenomenon in the spin") is also manifest in Fig. 17, where again the higher-spin curves tend towards the corresponding noninteracting spin-S curve $(T\chi/\mu^2)\text{free}=\frac{1}{3}S(S+1)$. Of course, this can be deduced more directly from the GT equations by studying η_n for large n.

F. Scales and universal numbers

In this subsection we determine the dimensional scales parametrizing the impurity magnetization $\mathcal{M}^i$ in different asymptotic regions of the H-T plane. We use the behavior of the magnetization to provide the various scales.

Consider thus the zero-temperature magnetization. As we have seen in (7.53), the asymptotic expansion for $\mathcal{M}^i$ is given by

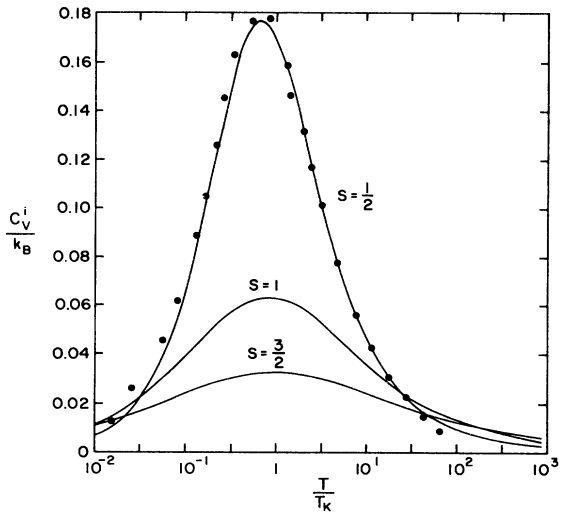

FIG. 18. The specific heat as a function of the temperature for spin values $S=\frac{1}{2},1,\frac{3}{2}$.

$$\mathcal{M}^i=\mu\begin{cases}(2S-1)\{1+(\frac{1}{2})\ln^{-1}(T_H^l/H)-(\frac{1}{4})\ln^{-2}(T_H^l/H)\ln\ln(T_H^l/H)+O[\ln^{-3}(T_H/H)]\}, & H\ll T_H\\ 2S\{1-\frac{1}{2}\ln^{-1}(H/T_H^h)-\frac{1}{4}\ln^{-2}(H/T_H^h)\ln\ln(H/T_H^h)+O[\ln^{-3}(T_H/H)]\}, & T_H\ll H\ ,\end{cases} \tag{7.74}$$

where T_H^h and T_H^l, the scales parametrizing high and low magnetic field regions, are determined by the respective $\ln^{-2}(T/T_0)$ term in the expansion. We saw that

$$T_H^h=T_H^l=\left[\frac{\pi}{e}\right]^{1/2}T_0 \tag{7.75}$$

and hence found Wilson's magnetic number to be 1:

$$\tilde{W}'=\frac{T_H^h}{T_H^l}=1\ . \tag{7.76}$$

A similar analysis may be given for the temperature dependence of the impurity magnetization. From (7.73),

$$\mathcal{M}^i=\mu\left[\frac{H}{T}\right]\frac{4}{3}\begin{cases}S(S+1)\{1-\ln^{-1}(T/T_k^h)-\frac{1}{2}\ln^{-2}(T/T_k^h)\ln\ln(T/T_k^h)+O[\ln^{-3}(T/T_k^h)]\}, & T\gg T_k\\ (S-\frac{1}{2})(S+\frac{1}{2})\{1+\ln^{-1}(T_k^l/T)-\frac{1}{2}\ln^{-2}(T_k^l/T)\ln\ln(T_k^l/T)+O[\ln^{-3}(T_k^l/T)]\}, & T\ll T_k\ ,\end{cases} \tag{7.77}$$

where the temperature scales T_k^l and T_k^h have been defined in the low- and high-temperature regions, respectively, by absorbing the corresponding $\ln^{-2}(T/T_k)$ term.

We were not able to determine, in Sec. V, the numerical relation between T_k^h (or T_k^l) and T_0 but did prove that in the expansion of the function $\eta_n(\zeta)$ in the form (5.55) or (5.56) we obtain $x_2^+=x_2^-$, leading therefore to Wilson's temperature crossover number $\tilde{W}$,

$$\tilde{W}=\frac{T_k^h}{T_k^l}=1\ . \tag{7.78}$$

We still may wish to calculate the ratio

$$W=\frac{T_k}{T_0}\ , \tag{7.79}$$

which is required in Eq. (7.77) and which provides a different generalization of the basic quantity calculated in Sec. VI.

Since we were not able to determine $x_2^{\pm}$ in the expansion (5.58), we cannot directly find W; yet, following the arguments of universality, we can set

$$W=\frac{T_k}{T_H}\frac{T_H}{T_0}\ , \tag{7.80}$$

where $U=T_k/T_H$, being a universal number, can be determined in any scheme—in particular, in the momentum cutoff scheme that is convenient for a perturbative expansion.

The ratio U in its own right may be interpreted as characterizing crossover by writing

$$U=\frac{T_k}{T_H}=\frac{T_k^h}{T_H^l}\ . \tag{7.81}$$

In Appendix C we evaluate the magnetization for large magnetic fields and also for large temperatures in terms of the momentum cutoff $\mathscr{D}$ and the accompanying coupling constant g. Thus

$$\mathscr{M}^i \underset{H\gg\mathscr{T}_H}{\rightarrow} 2\mu S\left[1-\frac{1}{2\ln(H/\mathscr{T}_H)}+\cdots\right], \tag{7.82}$$

where $\mathscr{T}_H=\frac{1}{2}\mathscr{D}e^{-\pi/2g}$ is the perturbative magnetic scale, and

$$\mathscr{M}^i \underset{T\gg\mathscr{T}_k}{\rightarrow} \mu\frac{H}{T}\frac{4}{3}S(S+1)\left[1-\frac{1}{\ln(T/\mathscr{T}_k)}+\cdots\right], \tag{7.83}$$

where

$$\mathscr{T}_k=\beta\gamma e^{-7/4}\exp\{\alpha\tfrac{1}{10}[1-\tfrac{4}{3}S(S+1)]\}\mathscr{D}e^{-\pi/2g}\ .$$

Here

$$\ln\alpha=\int_0^1 dx\, x[\pi^2\csc^2(\pi x)-x^{-2}-(1-x)^{-2}]$$
$$=0.841\,166\ldots,$$
$$\ln\beta=\int_0^1 dx\,(1-x^2)x[\pi^2\csc^2(\pi x)-x^{-2}]$$
$$=0.662\,122\ldots,$$
$$\ln\gamma=0.577\,216\ldots$$

($\ln\gamma$ is Euler's constant). We thus have

$$U=\frac{T_k}{T_H}=\frac{\mathscr{T}_k}{\mathscr{T}_H}$$
$$=2\beta\gamma e^{-7/4}\exp\frac{\alpha}{10}\{1-\tfrac{4}{3}[S(S+1)]\}\ , \tag{7.84}$$

where the second equality follows from universality. We find therefore

$$W=\frac{T_k}{T_0}=2\sqrt{\pi}\beta\gamma e^{-7/4}\exp\frac{\alpha}{10}[1-\tfrac{4}{3}S(S+1)]$$
$$=\mathscr{W}\exp\frac{\alpha}{10}[1-\tfrac{4}{3}S(S+1)]\ , \tag{7.85}$$

where $\mathscr{W}$ is the basic Wilson number for spin-$\frac{1}{2}$, as calculated in Sec. VI.

The crossover numbers, however, are as simple as can be:

$$\tilde{W}=\tilde{W}'=1\ . \tag{7.86}$$

VIII. THE SU(ν) KONDO MODEL (THE COQBLIN-SCHRIEFFER MODEL)

In Sec. VII we discussed the Kondo model of a localized spin-S impurity interacting with a band of spin-$\frac{1}{2}$ electrons. We found, as expected, only partial screening of the spin and thus a susceptibility that is diverging as the temperature tends to zero.

A different approach [Coqblin and Schrieffer (CS), 1969] is provided by expanding the electronic plane waves in total angular momentum eigenstates around the impurity and taking into account the various hoppings between these states via the impurity. This leads us to consider the SU($2S+1$) generalization of the basic SU(2) Kondo problem,

$$\mathscr{H}_{\rm CS}=\sum_{m,k} kc_{km}^*c_{km}-2J\sum_{km}\sum_{k'm'}c_{k'm'}^*c_{km}a_m^*a_{m'}\ , \tag{8.1}$$

where $-S\le m\le S$. The operators c_{km}^* (a_{km}^*) create an electron (impurity) in the fundamental representation of SU(ν), $\nu=2S+1$, with generalized-spin component m. The Hamiltonian is invariant under generalized rotations

$$c_{km}\rightarrow\sum_{m'}U_{mm'}c_{km'},\quad a_m\rightarrow\sum_{m'}U_{mm'}a_{m'}\ ,$$

where $U_{mm'}$ is a ν-dimensional unitary matrix. In contrast, the symmetry group in the last section was SU(2), with the electrons in the fundamental spin-$\frac{1}{2}$ and the impurity in any spin-S representation.

Actually, the symmetry group of the Hamiltonian is U(ν); but, as we saw in Sec. III, the spectrum of the model splits into two noncommunicating sectors. The density fluctuations carry the U(1) charge, while the spin fluctuations carry the SU(ν) quantum numbers (Witten, 1978).

In configuration space (see discussion in Sec. II) the Hamiltonian takes the form

$$\mathscr{H}_{\rm CS}=-i\sum_{\beta=0,1}\int dx\,\psi_{m\beta}^*(x)\beta\partial_x\psi_{m\beta}(x)$$
$$+J\int dx\,\psi_{m,1}^*(x)\psi_{m',0}^*(x)\psi_{m',1}(x)\psi_{m,0}(x)\ , \tag{8.2}$$

with summation over m implied. The operators $\psi_{m,1}^*$ and $\psi_{m,0}^*$ are the Fourier transforms of the electronic and impurity creation operators c_{km}^* and a_m^*, respectively. The purity index β denotes an electron ($\beta=1$) or an impurity ($\beta=0$), just as in Sec. II.

The identical Hamiltonian with $\beta=\pm1$ is, of course, the SU(ν) generalization of the Gross-Neveu (backscattering) model, and we shall discuss them in parallel.

The model is Bethe soluble with the ansatz this time requiring a generalization of the scheme used thus far. The generalization occurs at the discrete eigenvalue problem level, which is solved by ($\nu-1$) applications of a Bethe-Yang technique if the symmetry is SU(ν). This method is due to Sutherland (1967), who generalized the corresponding Yang-Gaudin problem.

We shall find that the ground state is a SU(ν) singlet; thus again a complete screening takes place. This fact was one of the motivations for the introduction of this model, as uncompensated magnetic moments have not

been experimentally observed, even when the impurity spin is larger than half.

The spectrum has some novel features to it. It consists of the fundamental excitations and also of bound states. There were no bound states in the spin-$\frac{1}{2}$ case, while here up to $\nu-1$ fundamental excitations can bind, the bound-state excitation of r particles being the antiparticle of the bound state formed out of $(\nu-r)$ particles. There are also unbound excitations, and their generalized spins can be coupled to various values by the means of n-strings.

The discussion of magnetization and finite temperature can also be formulated along the familiar lines. Explicit results, however, are not yet available for the magnetization for general ν, since the resulting equation is now a generalized Wiener-Hopf equation consisting of several kernels. It is easy, however, to show complete screening at low fields and a crossover to a weakly coupled regime. We shall include only a brief discussion of the thermodynamics.

The model (8.2) was diagonalized (Andrei and Lowenstein, 1980a) for the case $\beta=\pm 1$ (Gross-Neveu model). The spectrum was unravelled and the S-matrix deduced (Andrei and Lowenstein, 1980b). The SU(ν)-Kondo diagonalization follows by setting $\beta=0,1$ and was recently employed (Tsvelick and Wiegmann, 1981) to discuss the thermodynamics.

A. The diagonalization

The Hamiltonian conserves the number of electrons N^e and impurities N^i, so that the most general state in the Hilbert space labeled by the conserved numbers is

$$|\mathcal{F}\rangle=\sum_{\beta_i m_i}\int\prod_{i=1}^{N}dx_i\mathcal{F}(x_i\beta_i m_i)\prod_{i=1}^{N}\psi^{*}_{\beta_i m_i}(x_i)\,|0\rangle\ , \tag{8.3}$$

where $-S\le m_i\le S$, $\beta_i=0,1$ and $N=N^e+N^i$. N^e of the purity indices have the value $\beta=1$ and N^i the value $\beta=0$.

We wish to determine the wave functions $\mathcal{F}$ such that

$$\mathcal{H}|\mathcal{F}\rangle=E|\mathcal{F}\rangle\ . \tag{8.4}$$

Using the canonical anticommutation relations

$$\{\psi_{m\beta}(x),\psi^{*}_{m'\beta'}(x')\}=\delta_{mm'}\delta_{\beta\beta'}\delta(x-x')\ ,$$

$$\{\psi_{m\beta}(x),\psi_{m'\beta'}(x')\}=0\ ,$$

we find that $\mathcal{F}(x\beta m)$ must satisfy a first-quantized Schrödinger equation

$$h\mathcal{F}=E\mathcal{F}\ , \tag{8.5}$$

where

$$h=-i\sum_{i=1}^{N}\beta_i\partial_i+J\sum_{i,j=1}^{N}\delta(x_i-x_j)(\beta_i-\beta_j)^2P^{ij}_{m}\ , \tag{8.6}$$

with $P^{ij}_{m}a$ (generalized) spin exchange operator;

$$P^{ij}_{m}\mathcal{F}(\cdots m_i\cdots m_j\cdots)=\mathcal{F}(\cdots m_j\cdots m_i\cdots)\ .$$

The complete antisymmetry of the wave function

$$\mathcal{F}(\cdots(x\beta m)_i\cdots(x\beta m)_j\cdots)=(-)\mathcal{F}(\cdots(x\beta m)_j\cdots(x\beta m)_i\cdots)$$

allows us to rewrite the Hamiltonian as

$$h=-i\sum_{i=1}^{N}\beta_i\partial_i-J\sum_{i,j=1}^{N}\delta(x_i-x_j)(\beta_i-\beta_j)^2P^{ij}_{\beta}\ , \tag{8.7}$$

where P^{ij}_{β} is a purity exchange operator.

The Hamiltonian is now of the form discussed in Sec. II [Eq. (2.8) with $J'=J$], but acts on wave functions $\mathcal{F}(x,\beta,m)$, where m, the generalized spin index, can take $\nu=2S+1$ values.

The Hamiltonian does not depend explicitly on the spin indices, so the wave function may be factorized

$$\mathcal{F}(x\beta m)=F(x,\beta)t(m)\ , \tag{8.8}$$

with $t(m)=t(m_1,\ldots,m_N)$ the generalized-spin-wave function.

The solution of the eigenvalue problem

$$hF(x,\beta)=EF(x,\beta) \tag{8.9}$$

proceeds as in Sec. II. One writes the ansatz

$$F^{k,\alpha}(x,\beta)=\sum_{P,Q\in S_N}\xi_{Q,P}\theta(x_Q)e^{i\sum_j x_{Qj}k_{Pj}}\prod_{l}\delta^{\alpha_{Pl}}_{\beta_{Ql}}\ , \tag{8.10}$$

which satisfies the Hamiltonian, provided that the columns ξ_P are connected by the consistent set of Y matrices given in Eq. (2.15) (with $J'=J$)

$$Y^{ab}_{ij}=\begin{cases}\dfrac{1-J^2}{1+J^2}\mathcal{P}^{ab}+\dfrac{2i(\alpha_i-\alpha_j)J}{1+J^2}\mathbb{1}, & \alpha_i\neq\alpha_j\\ -\mathbb{1}, & \alpha_i=\alpha_j\ .\end{cases} \tag{8.11}$$

Imposing now periodic boundary conditions leads to the eigenvalue equation studied in Sec. II.E:

$$Z'_j\Phi\equiv X'_{j+1,j}\cdots X'_{Nj}X'_{1j}\cdots X'_{j-1,j}\Phi=\exp(ik_jL)\Phi\ , \tag{8.12}$$

where

$$X'_{ij}=e_{ij}\frac{i(\alpha_i-\alpha_j)+c\mathcal{P}^{ij}}{i(\alpha_i-\alpha_j)+c}\ , \tag{8.13}$$

with

$$c=\frac{2J}{1-J^2},\quad e_{ij}=e^{i(\alpha_i-\alpha_j)\phi} \tag{8.14a}$$

$$e^{i\phi}=\frac{1+2iJ-J^2}{1+J^2}\ , \tag{8.14b}$$

and where Φ is a function of the permutations $Q\in S_N$ defined in terms of the reference column ξ_0,

$$\Phi(Q)=(-1)^{\delta(Q)}\xi_{Q,0}\ . \tag{8.15}$$

Once the discrete eigenvalue problem (8.10) has been solved and the allowed momenta k_i found, the energy eigenvalues can be determined from

$$E=\sum_{i=1}^{N}\alpha_i k_i \ . \tag{8.16}$$

Thus far the procedure has been the same as in Sec. II. In the present case, however, the wave function $F(x,\beta)$ enjoys a permutation symmetry that is larger than in the SU(2) case, allowing for new types of solutions. It is specified by a Young tableau $\tilde{T}$, the conjugate of the spin-wave function tableau, T, which can have up to $\nu=2S+1$ rows, $T=(L_0,L_1,\ldots,L_{\nu-1})$, since that is the number of generalized-spin states a particle can occupy. The symmetry of a wave function expressed in a Bethe form (8.10) is determined by the symmetry of the reference column ξ_0, which therefore must be in the $\tilde{T}$ representation.

This, then, is the problem: We wish to solve the discrete eigenvalue equation (8.12), where the symmetry of the function $\Phi(Q)$, the conjugate of $\xi_0(Q)$, is specified by $T=(L_0,L_1,\ldots,L_{\nu-1})$, generalizing the results of Sec. II from $\nu=2$ to arbitrary ν.

The first step is to find a convenient way of presenting $\Phi(Q)$. We shall adopt our favorite billiard ball picture and regard Q as describing the order on a line of N balls numbered $1,2,\ldots,N$, the jth position being occupied by ball number Qj. We wish to construct Φ to be symmetric with respect to permutations of those balls labeled with the numbers g_{0l}, $l=1,\ldots,L_0$, corresponding to the first row of the tableau T (see Fig. 19); hence we may write

$$\Phi(Q)=\phi(y_1^1,\ldots,y_{M^1}^1) \ , \tag{8.17}$$

where $M^1=L_1+\cdots+L_{\nu-1}=N-L_0$ is the number of the balls in the remaining $\nu-1$ rows, and where the discrete variables $\{y_\gamma^1,\gamma=1,\ldots,M^1\}$ denote their positions, so that ball γ is located at $y_\gamma^1=Q^{-1}\gamma$. The function ϕ is symmetric in those y_γ^1 variables corresponding to entries in the same row of T.

The eigenvalue problem (8.12) with $\Phi(Q)$ of the form (8.15) as specified by a tableau T is again Bethe soluble (Sutherland, 1967). The function ϕ is expanded in terms of the single-particle wave function $f(\Lambda,y)$ discussed in the SU(2) case, Eq. (2.44),

$$\phi(y_1^1,\ldots,y_{M^1}^1)=\sum_{Q,P\in S_{M^1}}\zeta_{Q,P}\theta(y_Q^1)\prod_{\gamma=1}^{M^1}f(\Lambda_{P\gamma}^1,y_{Q\gamma}^1) \ , \tag{8.18}$$

and is a solution with eigenvalue

$$\lambda_j=e^{ik_jL}=\prod_{i\neq j}e_{ij}\prod_{\gamma=1}^{M^1}\frac{i(\alpha_j-\Lambda_\gamma^1)+c/2}{i(\alpha_j-\Lambda_\gamma^1)-c/2} \ , \tag{8.19}$$

provided that

(1) The neighboring columns of the $M^1!\times M^1!$ coefficient matrix are connected by the set of consistent Y matrices

$$Y_{\gamma\delta}^{1a,b}=\frac{(-ic)+(\Lambda_\gamma^1-\Lambda_\delta^1)\mathscr{P}^{ab}}{ic+(\Lambda_\gamma^1-\Lambda_\delta^1)} \ , \tag{8.20}$$

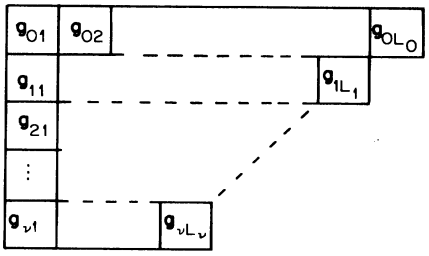

FIG. 19. The most general Young tableau for SU(ν) symmetry.

where $\mathscr{P}^{ab}\in S_{M^1}$.

(2) The reference column $\Phi^1=\zeta_0$ satisfies the eigenvalue equation

$$Z_\gamma^1\Phi^1=\lambda_\gamma^1\Phi^1 \ , \tag{8.21}$$

where

$$\lambda_\gamma^1=-\prod_{\delta=1}^{M^1}\frac{i(\Lambda_\gamma^1-\Lambda_\delta^1)+c}{i(\Lambda_\gamma^1-\Lambda_\delta^1)-c}\prod_{j=1}^{N}\frac{i(\Lambda_\gamma^1-\alpha_j)-c/2}{i(\Lambda_\gamma^1-\alpha_j)+c/2} \tag{8.22}$$

and

$$Z_\gamma^1=X_{\gamma+1,\gamma}^1\cdots X_{M^1,\gamma}^1X_{1,\gamma}^1\cdots X_{\gamma-1,\gamma}^1 \ , \tag{8.23}$$

with

$$X_{\gamma,\delta}^1=P_{\gamma\delta}Y_{\gamma\delta}^{1\gamma\delta}=\frac{i(\Lambda_\gamma^1-\Lambda_\delta^1)+c\mathscr{P}^{ab}}{i(\Lambda_\gamma^1-\Lambda_\delta^1)-c} \ . \tag{8.24}$$

One now repeats the process, writing a Bethe ansatz for Φ^1 considered as a function of $M^2=L_2+\cdots+L_{\nu-1}=M^1-L_1$ discrete variables,

$$\Phi_Q^1=\phi^1(y_1^2,\ldots,y_{M^2}^2)=\sum_{Q,P\in S_{M_2}}\zeta_{Q,P}^1\theta(y_Q^2)\prod_{\gamma=1}^{M^2}f(\Lambda_{P\gamma}^2,y_Q^2) \ . \tag{8.25}$$

Such a function will satisfy (8.19), provided (1) that

$$\lambda_\gamma^1=\prod_{\delta=1}^{M^2}\frac{i(\Lambda_\gamma^1-\Lambda_\delta^2)+c/2}{i(\Lambda_\gamma^1-\Lambda_\delta^2)-c/2} \tag{8.26}$$

and (2) that the neighboring columns of the $M^2!\times M^2!$ coefficient matrix ζ^1 are connected by Y matrices

$$Y_{\gamma\delta}^{2ab}=\frac{(-ic)+(\Lambda_\gamma^2-\Lambda_\delta^2)\mathscr{P}^{ab}}{ic+(\Lambda_\gamma^2-\Lambda_\delta^2)} \ , \tag{8.27}$$

with $\mathscr{P}^{ab}\in S_{M^2}$, and (3) that the column vector $\Phi^2=\zeta_0^1$ satisfies

$$Z_\gamma^2\Phi^2=\lambda_\gamma^2\Phi^2 \ , \tag{8.28}$$

where

$$\lambda_\gamma^2=-\prod_{\delta=1}^{M^2}\frac{i(\Lambda_\gamma^2-\Lambda_\delta^2)+c}{i(\Lambda_\gamma^2-\Lambda_\delta^2)-c}\prod_{\epsilon=1}^{M^1}\frac{i(\Lambda_\gamma^2-\Lambda_\epsilon^1)-c/2}{i(\Lambda_\gamma^2-\Lambda_\epsilon^1)+c/2} \ .$$

Clearly, one may iterate the above procedure until, eventually, one exhausts the rows of the Young tableau. At that point the eigenvalue condition becomes purely algebraic:

$$Z_\gamma^{(\nu-1)}\Phi^{(\nu-1)}=1=\lambda_\gamma^{(\nu-1)} = \prod_{\delta=1}^{M^{(\nu-1)}} \frac{[i(\Lambda_\gamma^{\nu-1}-\Lambda_\delta^{\nu-1})+c]}{[i(\Lambda_\gamma^{\nu-1}-\Lambda_\delta^{\nu-1})-c]} \times \prod_{\epsilon=1}^{(\nu-2)} \frac{[i(\Lambda_\gamma^{\nu-1}-\Lambda_\epsilon^{\nu-2})-c/2]}{[i(\Lambda_\gamma^{\nu-1}-\Lambda_\epsilon^{\nu-2})+c/2]} . \tag{8.29}$$

The original eigenvalue problem for h thus reduces to the task of solving the coupled equations

$$\prod_{\delta=1}^{M^r} \frac{i(\Lambda_\gamma^r-\Lambda_\delta^r)+c}{i(\Lambda_\gamma^r-\Lambda_\delta^r)-c} = -\prod_{k=1}^{M^{r-1}} \frac{i(\Lambda_\gamma^r-\Lambda_k^{r-1})+c/2}{i(\Lambda_\gamma^r-\Lambda_k^{r-1})-c/2} \times \prod_{\lambda=1}^{M^{r+1}} \frac{i(\Lambda_\gamma^r-\Lambda_\lambda^{r+1})+c/2}{i(\Lambda_\gamma^r-\Lambda_\lambda^{r+1})-c/2} , \tag{8.30}$$

with $r=1,2,\ldots,\nu-1$. Here $M^\nu=0$ and by convention $M^0=N$ and $\Lambda_j^0=\alpha_j$.

Once these equations have been solved for $\{\Lambda_\gamma^r\}$, the energy is given in terms of the rank-one variables $\{\Lambda_\gamma^1,\gamma=1,\ldots,M^1\}$ by Eqs. (8.16) and (8.19), providing a solution for the SU(ν) Gross-Neveu model (Andrei and Lowenstein, 1980) if we choose $\alpha=\pm 1$, and for the Kondo model if we choose $\alpha=0,1$.

B. The spectrum

The eigenvalues of the Kondo model, in terms of the rank-one spin momenta $\{\Lambda_\gamma^1\}$, are (dropping inessential constants)

$$E=\sum_{j=1}^{N^e}\frac{2\pi}{L}n_j+D\sum_{\gamma=1}^{M^1}[\Theta(2\Lambda_\gamma^1-2)-\pi] , \tag{8.31}$$

with $D=N^e/L$ and the density quantum numbers, n_j cut off as before, $|(2\pi/L)n_j|<K$.

The spin momenta are determined from the following set of coupled equations:

$$\sum_{\kappa=1}^{M^{r-1}}\Theta(2\Lambda_\gamma^r-2\Lambda_\kappa^{r-1})+\sum_{\lambda=1}^{M^{r+1}}\Theta(2\Lambda_\gamma^r-2\Lambda_\lambda^{r+1}) = \sum_{\delta=1}^{M^r}\Theta(\Lambda_\gamma^r-\Lambda_\delta^r)+2\pi I_\gamma^r , \tag{8.32}$$

where I_γ^r are integers or half integers coming from the logarithm. The configurations $\{n_j,I_\gamma^r\}$ are the quantum numbers of the states.

As usual, we convert the algebraic equations to integral relations among the various spin momenta densities. Thus we introduce, in the thermodynamic limit, the densities $\sigma^r(\Lambda)$ to describe the distribution of the rank-r spin momenta $\{\Lambda_\gamma^r\}$, with $N^{r,h}$ isolated holes and $N^{r,s}$ two-strings of various ranks located at $\{\Lambda_j^{r,h},\ r=1,\ldots,\nu-1,\ j=1,\ldots,N^{r,h}\}$ and $\{\Lambda_l^{r,s},\ r=1,\ldots,\nu-1,\ l=1,\ldots,N^{r,s}\}$, respectively. Equation (8.30) then becomes

$$\int d\Lambda\,\sigma^{r+1}(\Lambda)\Theta(2\Lambda^r-2\Lambda)+\sum_{l=1}^{N^{r+1,s}}\Theta(\Lambda^r-\Lambda_l^{r+1,s})+\int d\Lambda\,\sigma^{r-1}(\Lambda)\Theta(2\Lambda^r-2\Lambda)+\sum_{l}^{N^{r-1,s}}\Theta(\Lambda^r-\Lambda_l^{r-1,s}) = \int d\Lambda\,\sigma^r(\Lambda)\Theta(\Lambda^r-\Lambda)+\sum_{l=1}^{N^{r,s}}\{\Theta[\tfrac{2}{3}(\Lambda^r-\Lambda_l^{r,s})]+\Theta[2(\Lambda^r-\Lambda^{r,s})]\}+2\pi I^r(\Lambda) . \tag{8.33}$$

Differentiating with respect to Λ^r and taking the Fourier transform lead to a simple recursive relation (Andrei and Lowenstein, 1980a)

$$2ch\frac{c}{2}p\tilde{\sigma}^{*r}(p)=\tilde{\sigma}^{*r-1}(p)+\tilde{\sigma}^{*r+1}(p)-h^r(p) , \tag{8.34}$$

where

$$\tilde{\sigma}^{*r}(p)=\tilde{\sigma}_r(p)+\sum_{l=1}^{N^{r,s}}\exp(-i\Lambda_l^{r,s}p)\exp\left[-\frac{c}{2}|p|\right] , \tag{8.35a}$$

$$\tilde{h}^r(p)=\exp\frac{c}{2}|p|\sum_{j=1}^{N^{r,h}}\exp(-i\Lambda_j^{r,h}p) , \tag{8.35b}$$

and with boundary conditions on the densities

$$\tilde{\sigma}^\nu(p)=\tilde{\sigma}^{*\nu}(p)=0 , \tag{8.36a}$$

by construction,

$$\tilde{\sigma}^0(p)=\tilde{\sigma}^{*0}(p)=N^e\exp(-ip)+N^i , \tag{8.36b}$$

by convention.

The system may be readily solved to yield

$$\tilde{\sigma}^r(p)=\frac{\sinh[(\nu-r)cp/2]}{\sinh(\nu cp/2)}\tilde{\sigma}^0(p)-\sum_{l=1}^{N^{r,s}}\exp(-i\Lambda^{r,s}p)\exp(-c\,|p|/2)$$
$$-\left[\sum_{q=1}^{r}\tilde{h}^q(p)\frac{\sinh(qcp/2)\sinh[(\nu-r)cp/2)]}{\sinh(\nu cp/2)\sinh(cp/2)}+\sum_{q=r+1}^{\nu-1}\tilde{h}^q(p)\frac{\sinh[(\nu-q)cp/2]\sinh(rcp/2)}{\sinh(\nu cp/2)\sinh(cp/2)}\right]. \tag{8.37}$$

There are, we observe, contributions to σ^r from holes of all ranks, whereas the explicit string contributions are limited to rank r.

Setting $p=0$ in (8.37), we learn about the symmetry of the state,

$$M^r=\frac{\nu-r}{\nu}N-\sum_{q=1}^{r}\frac{q(\nu-r)}{\nu}N^{q,h}$$
$$-\sum_{q=r+1}^{\nu-1}\frac{r(\nu-q)}{\nu}N^{q,h}-N^{r,s}, \tag{8.38}$$

so that the length of the rth row in the tableau T is

$$L_r=N^r-M^{r+1}=\frac{N}{\nu}-\sum_{q=1}^{r-1}\frac{qN^{q,h}}{\nu}$$
$$-\sum_{q=r+1}^{\nu-1}N^{q,h}-N^{r,s}+N^{r+1,s}. \tag{8.39}$$

This means that T, in the absence of strings, must have $N^{1,h}$ columns of length 1, $N^{2,h}$ columns of length 2, etc., and $(N-\sum_q qN^{q,h})/\nu$ columns of length ν. A rank-r two-string removes a box from the last of the columns of length r and adds it to the first one to form a column of length $r+1$.

One may thus view, for example, a rank-one hole as carrying generalized-spin $(1/\nu)$!

In terms of the various distributions the energy is given by

$$E=\sum_{j=1}^{N^e}\frac{2\pi}{L}n_j+D\int d\Lambda^1\sigma^1(\Lambda^1)[\Theta(2\Lambda^1-2)-\pi]$$
$$+D\sum_{l=1,i=\pm1}^{N^{1,s}}[\Theta(2\Lambda_{l,i}^{1,s}-1)-\pi], \tag{8.40}$$

where $\sigma^1(\Lambda^1)$ is the rank-one density of real Λ momenta (one-strings) and the third term is the direct contribution of the rank-one two-strings $\{\Lambda_{l,\pm}^{1,s}=\Lambda_l^{1,s}\pm ic/2,\ l=1,\ldots,N^{1,s}\}$.

We proceed now to discuss the ground-state and some low-lying excitations.

The ground state is given by consecutive configurations of $\{I_\gamma^r\}$, without holes of any rank and therefore without n-strings of any rank. From Eq. (8.39) it follows that the symmetry is specified by a tableau whose rows are of equal length $L_i=N/\nu$ (see Fig. 20). The state is an SU(ν) singlet; hence the impurity spin is completely screened. This is a rather satisfying feature, as experimental observations tend to reveal compensated spins even in the case of higher-spin impurities, dissolved in metals.

The ground-state distribution of the rank-one spin momenta, in Fourier space, is

$$\sigma_{\rm gs}^1(p)=\frac{\sinh(\nu-1)cp/2}{\sinh\nu cp/2}[N^e e^{-ip}+N^i], \tag{8.41}$$

and the ground-state energy is

$$E_{\rm gs}=\sum_{j=1}^{N^e}\frac{2\pi}{L}n_j+D\int d\Lambda^1\sigma_{\rm gs}^1(\Lambda^1)[\Theta(2\Lambda-2)-\pi], \tag{8.42}$$

but we need no explicit form.

Density excitations are obtained by raising one of the $\{n_j\}$ quantum numbers from its ground state to a higher level, without changing the spin momenta. This produces a massless fermion spectrum carrying the U(1) charges and decoupling from the rest of the spectrum.

Spin excitations are generated by introducing holes and strings of various ranks. The change induced in $\sigma^1(\Lambda^1)$ by a hole of rank r at $\Lambda^{r,h}$ is, from (8.37),

$$\Delta^r\tilde{\sigma}^1(p)=-\frac{\sinh[(\nu-r)cp/2]}{\sinh(\nu cp/2)}$$
$$\times\exp(c\,|p|/2)\exp(-i\Lambda^{r,h}p) \tag{8.43}$$

and the energy associated with it

$$\Delta^r E(\Lambda^{r,h})=D\left[2\tan^{-1}\{\tan[(\nu-r)\pi/2\nu]\right.$$
$$\left.\times\tanh[(\Lambda-1)\pi/\nu c]\}+\frac{\nu-r}{\nu}\pi\right], \tag{8.44}$$

so that for a distribution of holes at $\{\Lambda_{i_r}^{r,h},\ r=1,\ldots,\nu-1,\ i_r=1,\ldots,N^{r,h}\}$ we find

$$\Delta E=\sum_{r=1}^{\nu}\sum_{i_r=1}^{N^{r,h}}\Delta^r E(\Lambda_{i_r}^{r,h}). \tag{8.45}$$

The number of holes $\{N^{r,h}\}$ must be such that they lead to integer M^r so as to produce a correct tableau. For example, a rank-one hole leads to a change

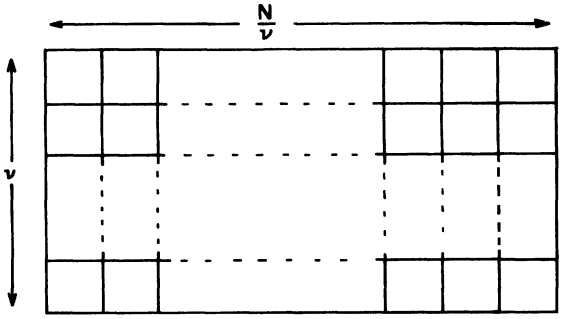

FIG. 20. Ground-state tableau. It corresponds to a SU(ν) singlet.

$$\Delta^1 M^r = -\frac{\nu - r}{\nu} ,$$

and we therefore need ν of them. Instead, we may add a hole of rank $\nu - 1$, which induces

$$\Delta^{\nu-1} M^r = -\frac{r}{\nu} ,$$

so that we now have an allowed two-hole state with

$$\Delta M^r = -1, \quad r \geq 1$$

see (Fig. 21), yielding a state in the adjoint representation which is the SU(ν) generalization of the triplet case discussed earlier.

The rank-r excitation may therefore be viewed as carrying a fractional spin r/ν, and the two excitations in the adjoint representation are coupled symmetrically.

One may couple the spins antisymmetrically to form an excited singlet by adding a "ladder" of two-strings of all ranks $r = 1, \ldots, \nu - 1$ equally spaced between $\Lambda^{1,h}$ and $\Lambda^{\nu-1,h}$ (see Appendix D).

The energy of the excitation is determined by the position of the holes only. As before, the change induced in the Λ sea by the ladder of strings exactly cancels their direct contribution, and we find

$$\Delta E^{\text{adj}} = \Delta E^{\text{sing}} = D \left[2\tan^{-1}\{a(1) th[(\Lambda^{1,h} - 1)\pi/\nu c]\} + 2\tan^{-1}\left[a(\nu - 1) th(\Lambda^{\nu-1,h} - 1)\frac{\pi}{\nu c} \right] + \pi \right] , \tag{8.46}$$

where $a(r) = \tan[1 - (r/\nu)](\pi/2)$.

Thus far, the discussion has concerned a fixed number of particles, therefore precluding, in particular, one-hole states. Still, they may be generated by adding electrons to the system. This may be done in various ways. Consider the case in which we add q electrons in such a way as to generate q holes of rank one. The state will transform as a symmetrized q tensor and will have energy

$$\Delta E^{\text{sym}} = \sum_{i=1}^{q} \Delta E^{\text{fund}}(\Lambda_i^{1,h}) + \sum_{i=1}^{q} \frac{2\pi}{L} n^i , \tag{8.47}$$

where n^i are the levels into which the electrons have been inserted. The addition of an electron to the system excites both a spin excitation and a density excitation. This expression is the sum of energy terms corresponding to the fundamental rank-one one-hole excitation

$$\Delta E^{\text{fund}}(\Lambda^{1,h}) = 2D \left[\tan^{-1}\{a(1)\tanh[(\Lambda^{1,h} - 1)\pi/\nu c]\} + \left[1 - \frac{1}{\nu} \right] \pi \right] \tag{8.48}$$

and also a sum of decoupled density excitations (which will not be further mentioned). We thus observe a situation of q unbound excitations in the system (see Fig. 22). On the other hand, they may be added antisymmetrically (see Fig. 23) in which case a single rank-q hole is generated with the energy

$$\Delta E^{\text{antisym}} = D \left[2\tanh^{-1}[a(q)\tanh(\Lambda^{q,h} - 1)\pi/\nu c] + \frac{\nu - q}{\nu} \pi \right] , \tag{8.49}$$

corresponding now to a *bound* state of q fundamental excitations. Thus in particular, the adjoint excitation is the case where a fundamental excitation interacts symmetrically with a bound state of $(\nu - 1)$ excitations, and the singlet is the case where they interact antisymmetrically. The S matrix describing this interaction was constructed for the backscattering model in Kurak and Swieca (1979) and Berg and Weisz (1979), using factorizability and in Andrei and Lowenstein (1980b), using the exact solution.

As we saw in Sec. V, the scaling regime singles out the region of Λ momenta large and negative. In this region the energy of the q-bound state is

$$\Delta E(\Lambda^{q,h}) \to 2T_0 q e^{(\pi/c)\Lambda^{q,h}} , \tag{8.50}$$

where the qth fluctuation scale is

$$T_0^q = T_0 \sin\left[\frac{q}{\nu} \pi \right] , \tag{8.51}$$

where the fundamental scale is

$$T_0 = D e^{-2\pi/\nu c} . \tag{8.52}$$

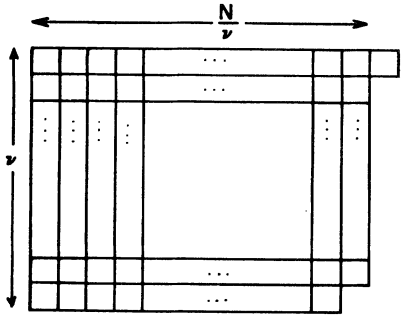

FIG. 21. Excited state described by a rank-1 hole and a rank-$\nu - 1$ hole. It is in the adjoint representation.

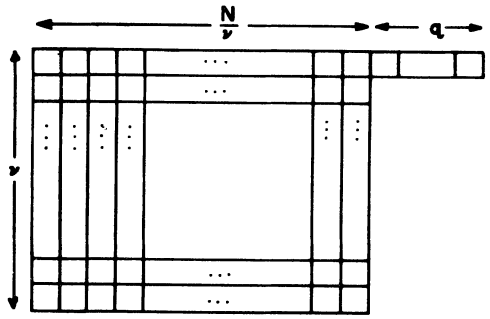

FIG. 22. Excited state consisting of q added electrons in a symmetrized spin configuration. They are unbound.

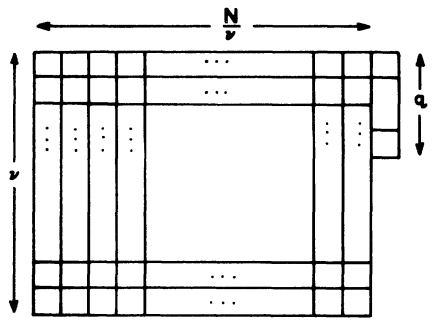

FIG. 23. Excited state consisting of q added electrons in an antisymmetrized spin configuration. They form a bound state.

The scale ratio $T_0^q/T_0=\sin(q/\nu)\pi$ is typical of soluble models and also characterizes, for example, the sine-Gordon spectrum.

We have enumerated only a few excitations. There are a host of others, all of which can be classified as various combinations of bound states of the fundamental excitations. The spin of these combinations depends on the presence of the appropriate ladders of n-strings.

To obtain the general structure of excitations we employ again, with due caution, the strong hypothesis, now assumed for each rank. In terms of the real parts $\Lambda_\gamma^{r(n)}$ of a rank-r n-string ($r=1,\ldots,\nu-1$, $n=1,2,\ldots,\infty$),

$$\Lambda_{\gamma,j}^{r(n)}=\Lambda_\gamma^{r(n)}=i\frac{c}{2}(n+1-2j),\quad j=1,\ldots,n\ ,$$

and equation (8.32) becomes

$$\sum_{p,\lambda}\hat{\Theta}_{np}[2(\Lambda_\gamma^{r(n)}-\Lambda_\lambda^{r-1(p)})]+\sum_{q,\epsilon}\hat{\Theta}_{n,q}[2(\Lambda_\gamma^{r(n)}-\Lambda_\epsilon^{r+1q})]=\sum_{(n,\delta)}\Theta_{nm}(\Lambda_\gamma^{r(n)}-\Lambda_\delta^{r(m)})+2\pi I_\gamma^{r(n)}\ , \tag{8.53}$$

where

$$\hat{\Theta}_{nm}(x)=\Theta[x/(|n-m|+1)]+\Theta[x/(|n-m|+3)]+\cdots+\Theta[x/(|n+m|+1)]\ , \tag{8.54}$$

and Θ_{mn} has been defined before.

We employ the usual procedure to convert Eq. (8.53) to integral equations determining the densities of the rank-r n-strings $\sigma_n^r(\Lambda)$ and their hole densities $\sigma_n^{r,h}(\Lambda)$. Having specified the configurations $\{I_\gamma^{r(n)}\}$ and found the corresponding Λ momenta $\{\Lambda_\gamma^{r(n)}\}$, one forms the functions

$$V_n^r(\Lambda)=\frac{1}{2\pi}\left[\sum_{k,\lambda}\hat{\Theta}_{nk}[2(\Lambda-\Lambda_\lambda^{r-1(k)})]+\sum_{l,\epsilon}\hat{\Theta}_{nl}[2(\Lambda-\Lambda_\epsilon^{r+1(l)})]-\sum_{m,\delta}\Theta_{nm}(\Lambda-\Lambda_\delta^{r(m)})\right]\ ,$$

where $\{\Lambda_\delta^{r(n)},\ r=1,\ldots,\nu-1,\ n=1,\ldots,\infty\}$ have been determined before. The string densities describe the solutions of

$$V_n^r(\Lambda_r^{r(n)})=I_\gamma^{r(n)}$$

and the hole densities the solutions of

$$V_n^r(\Lambda_\delta^{r(n)h})=J_\delta^{r(n)}\ ,$$

where the $J_\delta^{r(n)}$ are those integers omitted from the specified configuration $\{I_\gamma^{r,(n)}\}$, so that

$$\frac{dV_n^r}{d\Lambda}=\sigma_n^{r,h}+\sigma_n^r\ .$$

The densities thus satisfy the following set of integral equations

$$-\sum_{m,s}A_{nm}^{rs}\sigma_m^s=\sigma_n^{r,h} \tag{8.55}$$

where

$$A_{nm}^{rs}=A_{nm}\delta^{r,s}-B_{nm}(\delta^{r,s+1}+\delta^{r,s-1})\ . \tag{8.56a}$$

The operators A_{nm} have been defined before and

$$B_{nm}=[\,|n-m|+1]+[\,|n-m|+3]+\cdots+[n+m-1]\ . \tag{8.56b}$$

In the SU(2) case, $\sigma^2=0$, $\sigma^0=N^e\delta(\Lambda-1)+N^i\delta(\Lambda)$, thus reproducing Eq. (3.30).

The energy of the state $\{I_\gamma^{r(n)}\}$ is given by the rank-one string densities $\{\sigma_n^1\}$ and the particle-hole fluctuations $\{n_j\}$:

$$E=\sum_j\frac{2\pi}{L}n_j+D\sum_{n=1}^{\infty}\int d\Lambda\,\sigma_n^1(\Lambda)\left[\Theta\left[\frac{2\Lambda-2}{n}\right]-\pi\right]\ . \tag{8.57}$$

Obviously, the above discussion applies just as well to the SU(ν) Gross-Neveu model and the SU(ν) Heisenberg model.

C. Magnetization and finite-temperature thermodynamics

We consider now the application of a magnetic field H to our system. The role of $S_z=\left(\begin{smallmatrix}1/2 & 0\\ 0 & -1/2\end{smallmatrix}\right)$ is played by the obvious generalization

$$\Sigma_z=\begin{bmatrix}S & & & \\ & S-1 & & \\ & & \ddots & \\ & & & -S\end{bmatrix}\ ,$$

counting how many electrons are in each spin projection. Thus

$$\mathscr{H}_{\rm mag}=-2H\sum_{k=0}^{2S}L_k(S-k)=-2H\left[SN-\sum_{r=1}^{2S}M^r\right]\ , \tag{8.58}$$

where $L_k=M^k-M^{k+1}$ is the number of boxes in the kth

row $k=0,\ldots,\nu-1$.

The energy in the presence of a magnetic field thus becomes

$$E=-2HSN+D\sum_n\int d\Lambda\,\sigma_n^1(\Lambda)\left[\Theta\left[\frac{2\Lambda-2}{n}\right]-\pi\right]+H\sum_{n=1}^{\infty}\sum_{r=1}^{\nu-1}n\int d\Lambda\,\sigma_n^r(\Lambda)$$
$$=-2HSN+\sum_{n,r}\int d\Lambda\,\sigma_n^r(\Lambda)g_n^r(\Lambda)\,, \tag{8.59}$$

where

$$g_n^r=D\left[\Theta\left[\frac{2\Lambda-2}{n}\right]-\pi\right]\delta^{r,1}+2Hn\,. \tag{8.60}$$

The presence of a magnetic field leads to a macroscopic excitation of holes on the negative Λ axis, the rank-r Λ momenta being depleted over the interval $(-\infty,B^r)$, where B^r is determined by the magnetic field. Again, no strings are excited at $T=0$.

Taking a derivative of Eq. (8.53) with respect to Λ^r, we find the magnetization equation

$$\int_{B^{r+1}}^{\infty}d\Lambda'\hat{K}(\Lambda-\Lambda')\sigma^{r+1}(\Lambda')+\int_{B^{r-1}}^{\infty}d\Lambda'\hat{K}(\Lambda-\Lambda')\sigma^{r-1}(\Lambda')-\int_{B^r}^{\infty}K(\Lambda-\Lambda')\sigma(\Lambda')d\Lambda'=\sigma^r(\Lambda),\quad r=1,\ldots,\nu-1\,, \tag{8.61}$$

where $K(x)=(1/\pi)[c/(c^2+x^2)]$ as before and

$$\hat{K}(x)=\frac{1}{\pi}\frac{(\frac{1}{2}c)}{(\frac{1}{2}c)^2+x^2}\,.$$

In the SU(2) case this reduces to the magnetization equation (4.1), when we insert $\sigma^2=0$, $\sigma^0=N^e\delta(\Lambda-1)+N^i\delta(\Lambda)$ and of course $B^0=-\infty$.

Equation (8.61) is a generalized Wiener-Hopf equation, and its solution will not be attempted here.

We turn now to formulate the thermodynamic equations. For a given set $\{\sigma_n^r,\sigma_n^{r,h};n=1,\ldots,\infty,\ r=1,2,\ldots,\nu-1\}$ the entropy $\mathcal{S}$ (namely, the logarithm of the number of configurations $\{I_\gamma^r\}$ consistent with the above set of densities) is

$$\mathcal{S}=\sum_{n,r}\int d\Lambda[(\sigma_n^r+\sigma_n^{r,h})\ln(\sigma_n^r+\sigma_n^{r,h})-\sigma_n^r\ln\sigma_n^r-\sigma_n^{r,h}\ln\sigma_n^{r,h}]\,. \tag{8.62}$$

The equations are derived by minimizing the free energy

$$F=E-T\mathcal{S}$$

with respect to the densities, subject to the constraint following from Eq. (8.55)

$$\delta\sigma_n^{r,h}=-\sum_{m,s}A_{nm}^{rs}\delta\sigma_m^s,\quad \delta\sigma_n^0=\delta\sigma_n^\nu=0\,. \tag{8.63}$$

We find that the densities must satisfy

$$\ln(1+\eta_n^r)=\frac{g_n^r}{T}+\sum_{m,s}A_{nm}^{r,s}\ln[1+(\eta_m^s)^{-1}]\,, \tag{8.64}$$

where

$$\eta_b^r\equiv\frac{\sigma_n^{r,h}}{\sigma_n^r}\,, \tag{8.65}$$

and we have defined $(\eta_n^\nu)^{-1}=(\eta_n^0)^{-1}=0$.

Equation (8.64) may be simplified using the identities

$$A_{nm}^{r,s}-G(A_{n-1,m}^{r,s}+A_{n+1,m}^{r,s})=\delta_{nm}\delta^{rs}-G\delta_{nm}(\delta^{r,s+1}+\delta^{r,s-1}), \tag{8.66a}$$

$$A_{l,m}^{r,s}-GA_{2,m}^{r,s}=\delta_{1m}\delta^{rs}-G\delta_{1m}(\delta^{r,s+1}+\delta^{r,s-1})\,, \tag{8.66b}$$

to yield

$$\ln\eta_1^r=-\frac{2D}{T}\tan^{-1}e^{(\pi/c)(\Lambda-1)}\delta^{r,1}+G\ln(1+\eta_2^r)-G\{\ln[1+(\eta_1^{r+1})^{-1}]+\ln[1+(\eta_1^{r-1})^{-1}]\}\,, \tag{8.67a}$$

$$\ln\eta_n^r=G[\ln(1+\eta_{n+1}^r)+\ln(1+\eta_{n-1}^r)]-G\{\ln[1+(\eta_n^{r+1})^{-1}]+\ln[1+(\eta_n^{r-1})^{-1}]\}\,, \tag{8.67b}$$

with the boundary condition

$$\lim_{n\to\infty}([n+1]\ln(1+\eta_n^r)-[n]\ln(1+\eta_{n+1}^r))=-\frac{2H}{T}\,, \tag{8.68}$$

which follows from

$$[n+1]A_{nm}^{rs}-[n]A_{n+1,m}^{rs}=\begin{cases}0, & m<n\\ -([m+1]+[m-1])\delta^{rs}+[m](\delta^{r,s+1}+\delta^{r,s-1}), & m>n\,.\end{cases} \tag{8.69}$$

We may further transform the thermodynamic equations. If we denote

$$Q_n^r=\ln[1+(\eta_n^r)^{-1}]\,,$$

they take the form

$$G^{-1}Q_n^r=Q_n^{r+1}+Q_n^{r-1}-h_n^r\,, \tag{8.70}$$

where

$$h_n^r = G^{-1}\ln(1+\eta_n^r) - \ln(1+\eta_{n+1}^r) - \ln(1+\eta_{n-1}^r) ,$$

with

$$\ln(1+\eta_0^r) = \left[-\frac{2D}{T} \right] \delta^{r,1} G^{-1} \tan^{-1} e^{(\pi/c)(\Lambda-1)} . \tag{8.71}$$

The boundary conditions are

$$Q^\nu = Q^0 = 0 .$$

This recursion relation has been studied in Eq. (8.34) and we can immediately write the solution in momentum space

$$Q_n^r = \ln[1+(\eta_n^r)^{-1}] = -\left[\sum_{q=1}^{r} \tilde{h}_n^q(p) \frac{\sinh(qcp/2)\sinh[(\nu-r)cp/2]}{\sinh(\nu cp/2)\sinh(cp/2)} + \sum_{q=r+1}^{\nu-1} \tilde{h}_n^q(q) \frac{\sinh(rcp/2)\sinh[(\nu-q)cp/2]}{\sinh(\nu cp/2)\sinh(cp/2)} \right] , \tag{8.72}$$

so that the thermodynamic equations become

$$-\ln[1+(\eta_1^r)^{-1}] = -\frac{\Delta E^{\text{fund}}(\Lambda)}{T} + \sum_{q=1}^{\nu-1} G_\nu^{r,q}[\ln(1+\eta_2^q) - G^{-1}\ln(1+\eta_1^q)] , \tag{8.73a}$$

$$-\ln[1+(\eta_n^r)^{-1}] = \sum_{q=1}^{\nu-1} G_\nu^{r,q}[\ln(1+\eta_{n+1}^q) + \ln(1+\eta_{n-1}^q) - G^{-1}\ln(1+\eta_n^q)] , \tag{8.73b}$$

where $G_\nu^{r,q}$ is an integral operator whose kernel, in momentum space, is

$$G_\nu^{r,q}(p) = \frac{\sinh[\min(r,q)cp/2]\sinh\{[\nu-\max(r,q)]cp/2\}}{\sinh(\nu cp/2)\sinh(cp/2)} , \tag{8.74}$$

and the driving term is

$$\Delta E^{\text{fund}}(\Lambda) = D(2\tan^{-1}\{\tan[(\nu-r)\pi/2\nu]\tanh[(\Lambda-1)\pi/\nu c]\} + (\nu-r)\pi/\nu) . \tag{8.75}$$

Note, that just as in the SU(2) case, the driving term is the energy of the fundamental excitation [Eq. (8.41)]. The operator $G_\nu^{r,q}$ generalizes to arbitrary ν the operator G, which is given by $G = G_2^{1,1}$.

The corresponding equations for the SU(ν) backscattering model are obtained by replacing the Kondo driving term by the Gross-Neveu driving term, namely, the fundamental excitation energy in that model (Andrei and Lowenstein, 1980a):

$$\Delta E_{\text{GN}}^{\text{fund}}(\Lambda) = D(\tanh^{-1}\{\tan[(\nu-r)\pi/2\nu]\tanh[(\Lambda-1)\pi/\nu c]\} + [(\Lambda-1)\to(\Lambda+1)](\nu-r)\pi/\nu) .$$

Once we have determined the functions $\{\eta_n^r\}$, free energy is given by

$$F = F_0 + \sum_{n,r} \int d\Lambda \{\sigma_n^r g_n^r - T\sigma_n^r \ln(1+\eta_n^r) - T\sigma_n^{r,h}\ln[1+(\eta_n^r)^{-1}]\} , \tag{8.76}$$

where F_0 is the free energy of the free spinless fermion gas. Using Eqs. (8.55) and (8.64), and inserting the form of σ_n^0, we find

$$F = F_0 - T\sum_n \int d\Lambda [N^e\delta(\Lambda-1) + N^i\delta(\Lambda)][n]\ln(1+(\eta_n^1)^{-1}] .$$

This expression may be rewritten by means of (8.64) for $n=1$

$$\ln(1+\eta_1^r) = \frac{g_1^r}{T} \sum_{n=1}^{\infty} ([n+1]+[n-1])\ln[1+(\eta_n^r)^{-1}] + \sum_{n=1}^{\infty} [n]\{\ln[1+(\eta_n^{r+1})^{-1}] + \ln[1+(\eta_n^{r-1})^{-1}]\} ,$$

which again may be cast into the recursion relation

$$G^{-1}H^r = H^{r+1} + H^{r-1} - h^r ,$$

where

$$H^r = \sum_n [n]\ln[1+(\eta_n^r)^{-1}] ,$$

$$h^r = \frac{g_1^r}{T} - \ln(1+\eta_1^r) ,$$

with boundary conditions $H^0 = H^\nu = 0$. The solution again is, in momentum space,

$$H^1(p) = -\sum_{q=1}^{\nu-1} G_\nu^{q,1} \left[\frac{g_1^q}{T} - \ln(1+\eta_1^q) \right] ,$$

so that the free energy is

$$F=F_0-T\int d\Lambda[N^e\delta(\Lambda-1)+N^i\delta(\Lambda)]H^1(\Lambda)$$
$$=F_0-T\int d\Lambda[N^e\delta(\Lambda-1)+N^i\delta(\Lambda)]$$
$$\times\int d\Lambda'\sum_{q=1}^{\nu-1}G_\nu^{q,1}(\Lambda-\Lambda')\left[\frac{g_1^q}{T}-\ln(1+\eta_1^q)\right], \tag{8.77}$$

where the kernel of the operator $G_\nu^{q,1}$ is

$$G_\nu^{q,1}(\Lambda)=\frac{\sin\pi(\nu-q)/\nu}{\cosh 2\pi\Lambda/\nu c+\cos(\nu-q)\pi/\nu}. \tag{8.78}$$

Thus

$$F=E_{\rm gs}+F^e+F^i, \tag{8.79a}$$

where

$$E_{\rm gs}=\int d\Lambda[N^e\delta(\Lambda-1)+N^i\delta(\Lambda)]$$
$$\times G_\nu^{1,1}(\Lambda-\Lambda')[\Theta(2\Lambda-2)-\pi] \tag{8.79b}$$

is the ground-state energy given in (8.41) and (8.42), F^e is the electronic free energy

$$F^e=F^0-N^eT\int d\Lambda\sum_{q=1}^{\nu}G_\nu^{q,1}(\Lambda-1)\ln(1+\eta_1^q), \tag{8.79c}$$

and F^i is the impurity free energy

$$F^i=-N^iT\int d\Lambda\sum_{q=1}^{\nu}G_\nu^{q,1}(\Lambda)\ln(1+\eta_1^q). \tag{8.79d}$$

We are interested in the thermodynamics in the scaling regime $H,T\ll D$, where the model may be expected to approximate the physical situation and where the cutoff effects are negligible. The dynamic scale we hold fixed in the limit is

$$T_0=De^{-2\pi/\nu c}, \tag{8.80}$$

which is the fluctuation scale (8.52) characterizing the spin excitations at low momenta. When expressed in terms of T_0 the thermodynamic functions become universal and independent of the defining scheme.

The equations then take the form

$$-\ln[1+(\eta_1^r)^{-1}]=-2e^{\zeta}\sin(r\pi/\nu)$$
$$+\sum_{q=1}^{\nu-1}G_\nu^{r,q}[\ln(1+\eta_2^q)$$
$$-G^{-1}\ln(1+\eta_1^q)], \tag{8.81a}$$

$$-\ln[1+(\eta_n^r)^{-1}]=\sum_{q=1}^{\nu-1}G_\nu^{r,q}[\ln(1+\eta_{n+1}^q)+\ln(1+\eta_{n-1}^q)$$
$$-G^{-1}\ln(1+\eta_n^q)], \tag{8.81b}$$

where $\zeta=(2\pi/\nu c)\Lambda+\ln(T_0/T)$ and now the operators $G_\nu^{r,q}$ are given by Eq. (8.74), with c replaced by π. The free energy is given by

$$F^e=F^0-TLD\int d\zeta\sum_{q=1}^{\nu}\frac{\sin\pi(\nu-q)/\nu}{\cosh[\zeta-\ln(D/T)]+\cos[(\nu-q)\pi/\nu]}\ln(1+\eta_1^q), \tag{8.82a}$$

and

$$F^i=-\frac{N^iT}{2\pi}\int d\zeta\sum_{q=1}^{\nu}\frac{\sin\pi(\nu-q)/\nu}{\cosh[\zeta-\ln(T_0/T)]+\cos[(\nu-q)\pi/\nu]}\ln(1+\eta_1^q), \tag{8.82b}$$

where $\eta_n^q=\eta_n^q(\zeta,H/T)$.

In the electronic free energy we must take the limit $D\to\infty$ to obtain the universal result. We observe that the same functions, $\ln(1+\eta_1^q)$, appear in both the impurity and the electron free energy. This allows us to relate them in the low-energy region just as in the SU(2) case (Sec. V) and obtain a Fermi-liquid picture.

In the high-temperature regime, again, the methods developed in Sec. V yield the asymptotic behavior which tends to the free field behavior with increasing temperature.

The actual thermodynamic curves resulting from Eqs. (8.81) and (8.82b) are now under consideration. The present method extends techniques employed earlier to solve the SU(2) thermodynamic equations (Rajan *et al.*, 1982).

IX. THE MAGNETORESISTANCE

In the preceding sections we have explored the equilibrium properties of various versions of the Kondo model. These properties are expressed by means of the thermodynamic functions which can be deduced from the free energy.

Transport properties, on the other hand, concern nonequilibrium, though stationary, situations, such as conduction. Their formulation and calculation is much more involved and subtle. In general, they must be expressed in terms of time-dependent correlation function, and careful limiting procedures must be taken to assure irreversibility. One of the more fundamental transport quantities is the conductivity, the response function of the system to an external electric field.

In the present section we shall calculate this quantity at $T=0$ for arbitrary magnetic field. Our methods, though, are generalizable to arbitrary temperatures. The finite temperature conductivity will be discussed in a future publication.

The magnetoresistance will be calculable by S-matrix techniques which allow us to evade the hard problem of calculating time-dependent correlation functions. This is

possible due to the simple nature of the localized exchange interaction between the electrons and the impurities.

The physical situation one tries to model is of a metal in which magnetic impurities have been dissolved at low concentration. Their position is considered random so as to produce irreversibility. Still, their concentration is sufficiently small that a low-density expansion is valid and that one may concentrate on one impurity at a time. Only the s-wave component of the conduction electrons is assumed to be affected, which again allows the calculation of a three-dimensional quantity in terms of the effective one-dimensional model (1.1).

The magnetoresistance was calculated in Andrei (1982), and the finite-temperature extension is currently under investigation.

A. The T-matrix formulation of the Kondo resistivity

The presence of the impurities in the metal leads to a finite conductivity σ_{ij} given by Kubo's formula [see, for example, Doniach and Sondheimer (1974)]

$$\sigma^{ij}=\lim_{\omega\to 0}\lim_{q\to 0}\frac{\Pi^{ij}(q,\omega)-ne\delta^{ij}}{i\omega}\ , \tag{9.1}$$

where n is the electronic density (in three dimensions), i,j are the spatial components, and e is the electric charge. The quantity

$$\Pi^{ij}(\mathbf{x}-\mathbf{x}',t-t')=i\theta(t-t')\overline{\langle[J_i(\mathbf{x},t),J_j(\mathbf{x}',t')]\rangle} \tag{9.2}$$

is the retarded two-point function of the current density operator

$$J_i=\frac{i}{2m}e\left[\psi_a^*(\mathbf{x})\left[\frac{\partial}{\partial x^i}\psi_a(\mathbf{x})\right]-\left[\frac{\partial}{\partial x^i}\psi_a^*(\mathbf{x})\right]\psi_a(\mathbf{x})\right]\ ,$$

where ψ_a is the electronic field, and where the bar indicates averaging over the positions of the impurities, so that Π^{ij} becomes translationally invariant.

In the case of an electron gas in which the electrons interact only with the impurities and not with each other, the two-particle Green's function Π^{ij} factorizes and can be expressed in terms of one-particle Green's function G.

The factorization also takes place in our case, although the impurity spin induces correlations in the motions of the electrons. Still, all vertex corrections vanish upon integration over the angular variables, so long as we restrict ourselves to s-wave scattering. We thus have [see Doniach and Sondheimer (1974) for details]

$$\sigma=\tfrac{1}{3}\sigma^{ii}=-\tfrac{2}{3}e^2\rho_0\tau(\varepsilon=0)\ , \tag{9.3}$$

where ρ_0 is the density of states, and where the transport time τ, which is also the one-particle lifetime, is given through the use of the optical theorem by Nagaoka (1965), and Hamann (1967):

$$\tau(\varepsilon)=\frac{1}{2c\ \mathrm{Im}T(\varepsilon)} \tag{9.4}$$

where c is the impurity concentration and T is the one-particle nonflip T matrix defined by

$$G_{kk'}(\omega)=G^0_{kk'}(\omega)+G^0_{kk'}(\omega)T_{kk'}(\omega)G^0_{kk'}(\omega)\ . \tag{9.5}$$

Here $G_{kk'}(\omega)$ is the Fourier transform of the retarded one-particle Green's function

$$G_{kk'}(t)=-i\theta(t)\langle\Omega\,|\,[c^*_{k\tau}(t),c_{k'\tau}(0)]\,|\,\Omega\rangle\ , \tag{9.6}$$

where c_{ka} is the Fourier transform of $\psi_a(x)$, the electronic field. The T matrix is actually independent of the momenta

$$T=T(\omega)$$

as a result of the independence of the exchange interaction in our model.

The T-matrix elements can be expressed (Lippman and Schwinger, 1950) in terms of ingoing and outgoing scattering eigenstates of the Hamiltonian. To define these states, we consider an additional electron with momentum p incident on the system far away from the impurity. This incident state is given by the action of the creation operator on the ground state

$$c^*_{pa}\,|\,\Omega\rangle$$

and is approached in the far past by the incoming state $|\,p,a:\Omega\,\rangle^+$ and in the far future by the outgoing state $|\,p,a:\Omega\,\rangle^-$. Thus

$$|\,p,a;\Omega\,\rangle^{\pm}=c^*_{pa}\,|\,\Omega\,\rangle+|\,\chi\,\rangle^{\pm}_a\ , \tag{9.7}$$

with $|\,\chi\,\rangle^{\pm}$ the appropriate scattered wave (Suhl, 1965).

The nonflip S matrix is then given by

$$S_{pp'}={}^-\langle p\tau,\Omega\,|\,p'\tau\Omega\,\rangle\ , \tag{9.8}$$

from which the related T matrix is obtained through

$$S=1-iT\ . \tag{9.9}$$

B. Calculation of the T matrix

Having formulated the problem in terms of scattering eigenstates of the Hamiltonian, we now proceed to calculate the nonflip S-matrix element from the exact solution.

As we have a complete classification of all eigenstates of the Hamiltonian, we can easily identify the appropriate scattering eigenstates $|\,pa;\Omega\,\rangle^{\pm}$.

These states have been only briefly considered in Sec. III, where we concentrated on the study of excitations with a fixed and even number N. Thus consider an eigenstate with $N^e=N^e+1$ electrons. This leads to a doublet (spin-$\frac{1}{2}$) state characterized by a hole at Λ^h, say. (Thus even in the ground state of an odd number of particles there is a kink.) The energy of this "dressed electron" state is (see Sec. III)

$$\Delta E^d=2D\tan^{-1}(e^{(\pi/c)(\Lambda^h-1)})+\frac{2\pi}{L}n\ , \tag{9.10}$$

where $(2\pi/L)n$ is the energy of an associated massless density excitation carrying the charge. The quantum

number n indicates the level into which the additional electron has been inserted [this discussion parallels the discussion of chiral excitations in the chiral Gross-Neveu model (Andrei and Lowenstein, 1979)].

Having identified the appropriate state, we now proceed to extract the S-matrix element. The direct way, via Eq. (9.8), is not practicable due to the complexity of the wave function. We may, however, deduce the S-matrix element from the spectrum, since we were able to quantize the system on a *finite* line segment L with periodic boundary conditions.

To illustrate the idea [which is well known—see, for instance, Landau and Lifshitz (1960b)], let us consider the following model:

$$h=-i\partial_x+2J\delta(x)\ , \tag{9.11}$$

whose eigenfunctions are

$$F(x)=A[\theta(-x)+e^{i\delta}\theta(x)]e^{ikx}\ , \tag{9.12}$$

where $e^{i\delta}=(i+J)/(i-J)$ is the phase shift incurred by the particle crossing the potential. Incidentally, this is a Bethe model, since the same momentum k appears in both regions and since no reflected wave e^{-ikx} is produced. The correctly normalized scattering states are

$$F^{+}(x)=\frac{1}{\sqrt{2\pi}}[\theta(-x)+e^{i\delta}\theta(x)]e^{ikx}\ , \tag{9.13a}$$

and

$$F^{--}(x)=\frac{1}{\sqrt{2\pi}}[\theta(-x)e^{-i\delta}+\theta(x)]e^{ikx}\ , \tag{9.13b}$$

and we find that the S-matrix element is given by

$$S=\ ^{-}\langle F\,|\,F\rangle^{+}=e^{i\delta}\ . \tag{9.14}$$

We may instead deduce the phase from the spectrum of the model. Thus, imposing periodic boundary conditions, we find

$$E=K=\frac{2\pi}{L}n-\frac{\delta}{L}\ , \tag{9.15}$$

and the phase shift makes its appearance as a shift of the allowed momentum from its free value $(2\pi/L)n$.

By studying the spectrum of the model, we can deduce the S matrix, provided we were able to solve it on a finite-length ring [this method was used to abstract the S matrix of the Gross-Neveu model, for example (Andrei and Lowenstein, 1980b)].

We shall consider the scattering in the presence of a finite magnetic field H in whose presence the Fermi level in the Λ space is shifted from $(-\infty)$ to $B=(e/2\pi)^{1/2}H/T_0$ (see Sec. IV). The ground state in the presence of the field, $|\Omega_H\rangle$, is characterized by a Λ distribution obtained by solving Eq. (4.1). The spin of the system, S, is determined by the magnetic field H, and is given by Eq. (4.4), while the energy is found from Eq. (4.3).

The state $|p,a;\Omega_H\rangle$ is a one-hole state, as discussed earlier, and its spin-momentum distribution, σ_{B,Λ^h}, is found from

$$\sigma_{B,\Lambda^h}(\Lambda)=f(\Lambda)-\int_B^{\infty}K(\Lambda-\Lambda')\sigma_{B,\Lambda^h}(\Lambda')d\Lambda' -\delta(\Lambda-\Lambda^h)\ , \tag{9.16}$$

with the function f and the kernel K defined as before. The solution of Eq. (9.16) can be obtained by Wiener-Hopf techniques (Furuya, 1981), but this is not necessary if we consider zero-temperature resistivity. In this case the hole Λ^h is placed at the Fermi level B, so that its sole effect is to change the spin S, as determined in the ground state by the magnetic field, to $S\pm\frac{1}{2}$, where the sign is determined by the symmetry of the final state, namely, whether the additional box in the Young tableau was added to the upper or lower row (see Fig. 24). This corresponds to states where the dressed electron scatters with its spin parallel or antiparallel to the impurity spin. If we

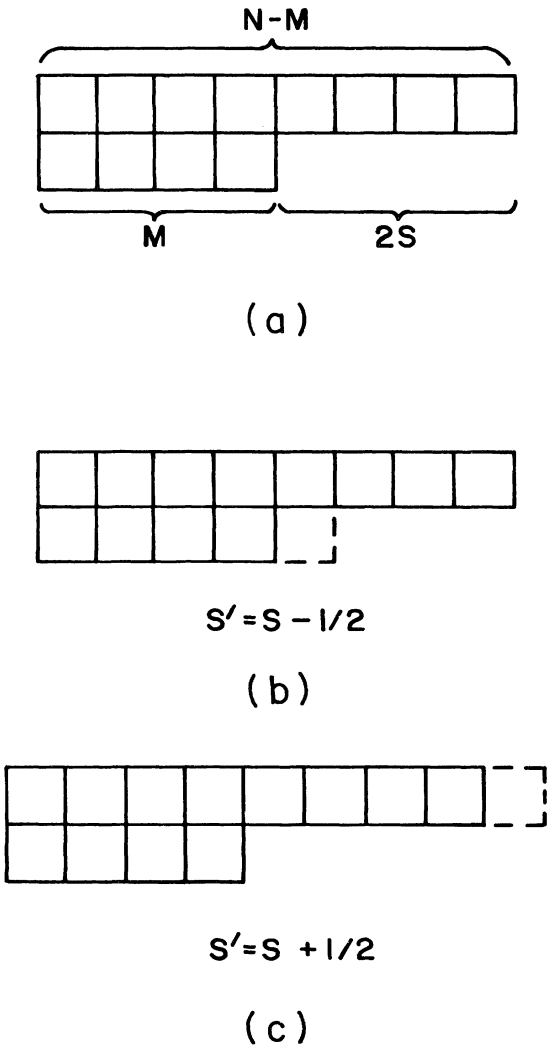

FIG. 24. (a) The ground-state configuration (spin-$\frac{1}{2}$ case) in the presence of a magnetic field H, which determines the value of S. (b) If an electron is now added to the system it can decrease the spin $S'=S\frac{1}{2}$, or (c) increase it, $S'=S+\frac{1}{2}$.

denote by $E(N^e,S)$ the lowest-energy state for a system with N^e electrons and with net spin S, we find that the energy, $E(p,\Omega_H)$, of the state $|pa;\Omega_H\rangle$, with p at the Fermi level, $p=p_{\text{Fermi}}(H)$, is

$$E(p,\Omega_H)|_{p=p_F(H)}=E(N+1,S\pm\tfrac{1}{2}) .$$

On the other hand, since we may add the electron far from the impurity, we find

$$E(p,\Omega_H)=p+E(\Omega_H) , \tag{9.17}$$

so that only those values of p are allowed which satisfy

$$\begin{aligned} p&=E(p,\Omega_H)-E(\Omega_H)\\ &=E(N^e+1,S\pm\tfrac{1}{2})-E(N^e,S) \end{aligned} \tag{9.18}$$

(p is on the Fermi level). By studying the deviation of the allowed incident momentum p from the free value $(2\pi/L)n$, we can determine the scattering phase shift $\delta=\delta[p=p_F(H)]$

$$\delta=Lp(\text{mod}2\pi n) , \tag{9.19}$$

and hence the S-matrix element $S=e^{i\delta}$. This, however, describes the one-dimensional scattering process. If we want to interpret it as the s-wave scattering in three dimensions, we have

$$S=e^{2i\delta_0} , \tag{9.20}$$

so that

$$\delta_0=\tfrac{1}{2}\delta . \tag{9.21}$$

The expression for $E(N,S)$ is given in (4.3):

$$\begin{aligned} E(N,S)&=D\int_B^\infty \sigma_B(\Lambda)[\Theta(2\Lambda-2)-\pi]\\ &=\int_{-\infty}^B d\Lambda\,\sigma_B(\Lambda)2D\tan^{-1}(e^{\pi(\Lambda-1)/c}) , \end{aligned} \tag{9.22}$$

where the second form expresses the energy as an integral of distribution of holes excited by the magnetic field from $-\infty$ to B. Thus $2D\tan^{-1}(e^{\pi(\Lambda-1)/c})$ is the hole excitation energy [Eq. (9.10)], and $\sigma_B(\Lambda)$ is the hole density for $\Lambda<B$.

By studying the scaling limit, we obtain

$$\begin{aligned} E(N,S)&=2T_0\int_{-\infty}^B d\Lambda\,\sigma_B(\Lambda)e^{(\pi/C)\Lambda}\\ &=2T_0\int d\Lambda\,\rho_-(\Lambda)e^{\rho\Lambda/c}e^{\pi B/c}\\ &=2T_0e^{\pi B/c}\tilde{\rho}_-\left[\frac{i\pi}{c}\right] , \end{aligned} \tag{9.23}$$

where $\rho_-(\Lambda)=\theta(\Lambda)\sigma(\Lambda+B)$ was studied extensively in Sec. IV, and where an explicit expression was obtained for its Fourier transform $\tilde{\rho}_-(p)$ [see Eq. (4.22)].

Combining Eqs. (9.18), (9.19), and (9.23), we find that the phase shift in the scaling limit is given by

$$\delta_0(H)=\pm\frac{\pi}{2}[1-M^i(H)] , \tag{9.24}$$

where M^i is related to the magnetization [Eq. (4.29c)], now expressed in terms of the magnetic scale T_H:

$$M^i=\frac{\mathcal{M}^i}{\mu}=\begin{cases}\dfrac{1}{\sqrt{2\pi}}\displaystyle\sum_{k=1}^{\infty}(-\tfrac{1}{2})^k(k!)^{-1}(k+\tfrac{1}{2})^{k-(1/2)}e^{-[k+(1/2)]}\left[\dfrac{H}{T_H}\right]^{2k+1}, & H\le\sqrt{2}T_H\\ 1-\pi^{(-3/2)}\displaystyle\int\frac{dt}{t}\sin(\pi t)e^{-t\ln(t/2e)}\left[\dfrac{T_H}{H}\right]^{2t}\Gamma(t+\tfrac{1}{2}), & \sqrt{2}T_H\le H\ll D.\end{cases} \tag{9.25}$$

The $(\pm)$ sign corresponds to parallel and antiparallel scattering, respectively.

The phase shift attains its unitarity limit at $H=0$,

$$\delta_0(H=0)=\pm\frac{\pi}{2} , \tag{9.26}$$

and crosses over to perturbative behavior when the magnetic field is sufficiently high:

$$\delta_0(H)\underset{H\gg T_H}{\to}\frac{\pi}{4}\left[\frac{1}{\ln(H/T_H)}+\frac{1}{2}\frac{\ln\ln(H/T_H)}{\ln^2(H/T_H)}+O\ln^{-3}(H/T_H)\right] , \tag{9.27}$$

and there are no $\ln^{-2}(H/T_H)$ terms by our choice of T_H as the scale (the phase shift is shown in Fig. 25).

The inverse transport time is now given by

$$\frac{1}{2\tau}=c\,\text{Im}\frac{1-e^{2i\delta_0}}{i}=-c\sin^2\delta_0 , \tag{9.28}$$

and hence the magnetoresistance r is

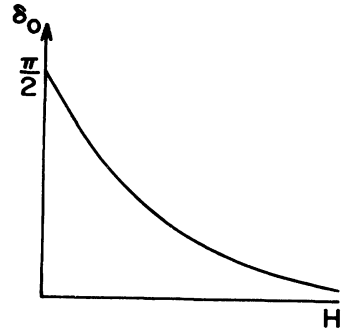

FIG. 25. The s-wave phase shift as a function of the magnetic field.

$$r=r_0\cos^2\frac{\pi}{2}M^i(H)\ , \tag{9.29}$$

where r_0 is the zero-field resistance.

It turns out that the relation between resistivity and magnetization can be derived via Friedel's sum rule or Anderson's theorem (Yosida and Yoshimori, 1973). These methods, however, require that the incident particle be on the Fermi surface. Our approach is more general, and the T matrix for scattering at arbitrary momentum as well as the finite energy resistivity can be thus obtained. This will be the subject of a forthcoming work.

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The work reported here was started in discussions with E. Rezayi. A year later, when the connection with the Gross-Neveu model was realized, it was J. Sak who eloquently and patiently presented to us the condensed matter physicist's point of view. We are grateful to him for his questioning attitude, which drove us to determine W. We would like to thank M. Croft for clarifications about the experimental background, and P. W. Anderson, P. Bak, M. Fowler, E. Fradkin, C. Krishna-Murthy, and J. Wilkins for useful comments, suggestions, and criticisms. In particular, we are indebted to E. Witten and L. Yaffe for many illuminating discussions and helpful remarks.

APPENDIX A: DETAILED SOLUTION OF THE DISCRETE EIGENVALUE PROBLEM

In Sec. II, we were faced with the following discrete eigenvalue problem:

$$Z'_N\Phi=\lambda_N\Phi\ , \tag{A1}$$

where Φ is a complex-valued function of M distinct integer-valued arguments $y_1,\ldots,y_M$, with $0<y_1<y_2<\cdots<y_M\le M$ and

$$Z'_N=X'_{1N}X'_{2N}\cdots X'_{N-1,N}\ , \tag{A2}$$

$$X'_{ij}=(a_{ij}+b_{ij}\mathscr{P}_{ij})\equiv e_{ij}\frac{i(\alpha_i-\alpha_j)+c\mathscr{P}_{ij}}{i(\alpha_i-\alpha_j)+c} \tag{A3}$$

(notation of Sec. II). In the Kondo model, α_i, $i=1,\ldots,N$ are the respective purities ($=1,0$) of the N electrons and impurities. In other models, the α_i have other interpretations and ranges of values (not always discrete). The treatment which follows is valid for all such cases.

According to Yang (1967), the discrete eigenvalue problem (A1) has a modified Bethe-ansatz solution

$$\Phi(y_1,\ldots,y_M)=\sum_{P\in S_M}A_P\prod_{\gamma=1}^{M}f(\Lambda_{P\gamma},y_\gamma)\ ,$$

where

$$f(\Lambda,y)=\begin{cases}\displaystyle\prod_{j=1}^{y-1}\frac{i(\alpha_j-\Lambda)+\frac{c}{2}}{i(\alpha_{j+1}-\Lambda)-\frac{c}{2}}, & y>1\\ 1, & y=1\ .\end{cases}$$

The coefficients A_P are determined, up to a common factor, by Eq. (2.46), and the parameters Λ_γ must satisfy the coupled equations (2.47). We now wish to demonstrate, for arbitrary M and N, the validity of this claim.

Let us begin by studying the case $M=1$, where

$$\Phi(y)=f(\Lambda,y)\ . \tag{A4}$$

Observing that

$$\frac{i(\alpha_i-\Lambda)+\frac{c}{2}}{i(\alpha_j-\Lambda)-\frac{c}{2}}=\frac{\mu_j(\Lambda)-a_{ij}}{b_{ij}}\ , \tag{A5}$$

where

$$\mu_j(\Lambda)=\frac{i(\alpha_j-\Lambda)+\frac{c}{2}}{i(\alpha_j-\Lambda)-\frac{c}{2}}\ , \tag{A6}$$

one can easily show

$$a_{N-1,N}\Phi(N-1)+b_{N-1,N}\Phi(N)=\mu_N\Phi(N-1)\ ,$$
$$b_{N-1,N}\Phi(N-1)+a_{N-1,N}\Phi(N-1)=\mu_{N-1}^{-1}\Phi(N)\ , \tag{A7}$$

so that, writing Φ as an N-dimensional column vector,

$$X'_{N-1,N}\Phi=e_{N-1,N}\begin{pmatrix}\Phi(1)\\ \vdots\\ \Phi(N-2)\\ \mu_N\Phi(N-1)\\ \mu_{N-1}^{-1}\Phi(N)\end{pmatrix}\ . \tag{A8}$$

Repeating the process with $X'_{N-2,N}$, $X'_{N-3,N}$, etc., one obtains

$$Z'_N\Phi=\left[\prod_{i<N}e_{iN}\right]\mu_N\begin{pmatrix}\Phi(1)\\ \vdots\\ \Phi(N-1)\\ \displaystyle\prod_{i=1}^{N}\mu_i^{-1}(\Lambda)\Phi(N)\end{pmatrix}\ . \tag{A9}$$

If now we require that Λ be a solution of

$$\prod_{i=1}^{N}\mu_i(\Lambda)=1\ , \tag{A10}$$

which is equivalent to imposing the periodic boundary conditions

$$\Phi(N+1)=\Phi(1)\ , \tag{A11}$$

we obtain, finally, the desired eigenvalue relation (A1), with

$$\lambda_N=\left[\prod_{i<N}e_{iN}\right]\mu_N\ . \tag{A12}$$

Our strategy for general M, $1\le M\le\frac{1}{2}N$, is motivated by the above discussion of the case $M=1$. We shall define Φ_k, $k=0,1,\ldots,N$, in such a way that

$$X'_{kN}\Phi_k = e_{kN}\Phi_{k-1} ,$$

and, moreover, that Φ_0 ends up being proportional to Φ with proportionality constant $\prod_{\gamma=1}^{M}\mu_N(\Lambda_\gamma)$. The correct definition of Φ_k is, we claim,

$$\Phi_k = \sum_P A_P \prod_\gamma f_k(\Lambda_{P\gamma}, y_{Q_k(y)\gamma}) , \tag{A13}$$

with

$$f_k(\Lambda,y) = \begin{cases} f(\Lambda,y), & y \le k \\ \mu_N(\Lambda) f(\Lambda,y), & k+1 \le y \le N-1 , \\ \mu_{k+1}^{-1}(\Lambda)\cdots\mu_{N-1}^{-1}(\Lambda) f(\Lambda,N), & y = N \end{cases} \tag{A14}$$

and, writing permutations as in Sec. II,

$$Q_k(y) = \begin{cases} 12\ldots M & \text{if } y_\gamma < N \text{ for all } \gamma \\ 12\ldots(\beta-1)M\beta(\beta+1)\ldots M-1 & \text{if } y_M = N . \end{cases} \tag{A15}$$

Here y_β is the smallest $y_\gamma > k$, and the coefficients are to be determined from (2.38).

Our first goal is to show that Φ_k defined by (A13) satisfies $X'_{kN}\Phi_k = e_{kN}\Phi_{k-1}$. From our treatment of the case $M=1$, this is immediately seen to be true if either k or N is not among the y_γ, $\gamma = 1,\ldots,M$. If, on the other hand, $y_M = N$ and $y_{\beta-1} = k$ (definition of β), one checks to see that X'_{kN} maps the function

$$f_k(\Lambda_{P1},y_1)\cdots f_k(\Lambda_{P(\beta-1)},k) f_k(\lambda_{P\beta},N) f_k(\Lambda_{P(\beta+1)},y_\beta)\cdots f_k(\Lambda_{PM},Y_{M-1})$$

into

$$e_{kN} f_{k-1}(\Lambda_{P1},y_1)\cdots f_{k-1}(\Lambda_{P(\beta-2)},y_{\beta-2})(\mu_N^{-1}(\Lambda_{P(\beta-1)}) f_{k-1}(\Lambda_{P(\beta-1)},k)$$
$$\times[\mu_k(\Lambda_{P\beta}) f_{k-1}(\Lambda_{P\beta,N})] f_{k-1}(\Lambda_{P(\beta+1)},y_\beta)\cdots f_{k-1}(\Lambda_{PM},y_M) .$$

But since $y_{\beta-1}=k$, $y_M = N$, we have $Q_k(y) = Q_{k-1}(y)P_{\beta,\beta-1}$, where $P_{\beta,\beta-1}$ interchanges β and $\beta-1$ and

$$Q_{k-1}(y) = 12\ldots M(\beta-1)\beta\ldots M-1 ,$$

so that

$$X'_{kN}\Phi_k = e_{kN}\sum_P A_P\mu_k(\Lambda_{P\beta})\mu_N^{-1}(\Lambda_{P(\beta-1)})\prod_{\gamma=1}^{M} f_{k-1}(\Lambda_{PP_{\beta,\beta-1}},y_{Q_{k-1}(y)\gamma})$$
$$= e_{kN}\sum_P A_{PP_{\beta-1,\beta}}\mu_k(\Lambda_{P(\beta-1)})\mu_N^{-1}(\Lambda_{P\beta})\prod_{\gamma=1}^{M} f_{k-1}(\Lambda_{P\gamma},y_{Q_{k-1}(y)\gamma}) . \tag{A16}$$

The rhs of (A6) will be equal to $e_{kN}\Phi_{k-1}$, provided that the coefficients A_P satisfy the following identity:

$$A_P f_{k-1}(\Lambda_{P(\beta-1)},N) f_{k-1}(\Lambda_{P\beta},k) + A_{P'} f_{k-1}(\Lambda_{P(\beta-1)},k) f_{k-1}(\Lambda_{P\beta},N)$$
$$= A_{P'}\mu_k(\Lambda_{P(\beta-1)})\mu_N^{-1}(\Lambda_{P\beta}) f_{k-1}(\Lambda_{P(\beta-1)},N) f_{k-1}(\Lambda_{P\beta},k) + A_P\mu_k(\Lambda_{P\beta})\mu_N^{-1}(\Lambda_{P(\beta-1)}) f_{k-1}(\Lambda_{P\beta},N) f_{k-1}(\Lambda_{P(\beta-1)},k) , \tag{A17}$$

where $A_{P'} \equiv A_{PP_{\beta-1,\beta}}$. Equation (A17) reduces, after some algebra to

$$\frac{A_{P'}}{A_P} = \frac{i(\Lambda_{P(\beta-1)} - \Lambda_{P\beta}) + c}{i(\Lambda_{P(\beta-1)} - \Lambda_{P\beta}) - c} , \tag{A18}$$

which must hold for arbitrary P and β. This determines A_P uniquely, apart from a common factor.

Carrying the iteration to its conclusion, we obtain

$$Z'_N\Phi = \left[\prod_{k=1}^{N-1} e_{kN}\right]\Phi_0 ,$$
$$\Phi_0 = \sum_P A_P \prod_\gamma f_0(\Lambda_{P\gamma}, Y_{Q_0\gamma}) \tag{A19}$$
$$= \sum_P A_P \prod_\gamma f_0(\Lambda_{PQ_0-1_\gamma}, y_\gamma)$$
$$= \sum_P A_{PQ_0} \prod_\gamma f_0(\Lambda_{P\gamma}, y_\gamma) ,$$

where

$$f_0(\Lambda,y) = \begin{cases} \mu_N(\Lambda) f(\Lambda,y), & y < N \\ \prod_{j=1}^{N-1}\mu_j^{-1}(\Lambda) f(\Lambda,N), & y = N \end{cases}$$

and

$$Q_0 = M\,12\ldots(M-2)(M-1) .$$

We see that

$$\Phi_0 = \left[\prod_{\gamma=1}^{M}\mu_N(\Lambda_\gamma)\right]\Phi , \tag{A20}$$

provided that

$$A_{PQ_0} f_0(\Lambda_{PM},N) = A_P f(\Lambda_{PM},N)\mu_N(\Lambda_{PM}) , \tag{A21}$$

i.e.,

$$A_{PQ_0}/A_P = \prod_{j=1}^{N}\mu_j(\Lambda_{PM}) ,$$

which once again is equivalent to demanding periodic boundary conditions for Φ on the one-dimensional chain. But

$$A_{PQ_0}=A_{(PM)(P1)\cdots[P(M-1)]}$$
$$=\left[\frac{i(\Lambda_{P1}-\Lambda_{PM})+c}{i(\Lambda_{P1}-\Lambda_{PM})-c}\right]A_{(P1)(PM)\cdots[P(M-1)]}$$
$$=\cdots=\prod_{\gamma=1}^{M-1}\left[\frac{i(\Lambda_{P\gamma}-\Lambda_{PM})+c}{i(\Lambda_{P\gamma}-\Lambda_{PM})-c}\right]A_P \qquad \text{(A22)}$$

by repeated application of (A18). Equation (A21) then becomes

$$\prod_{\gamma\neq M}\left[\frac{i(\Lambda_{P\gamma}-\Lambda_{PM})+c}{i(\Lambda_{P\gamma}-\Lambda_{PM})-c}\right]=\prod_j \mu_j(\Lambda_{PM}) .$$

Since this must hold for arbitrary P, we require

$$-\prod_{\gamma=1}^{M}\left[\frac{i(\Lambda_\beta-\Lambda_\gamma)+c}{i(\Lambda_\beta-\Lambda_\gamma)-c}\right]=\prod_{j=1}^{N}\mu_j(\Lambda_\gamma) , \qquad \text{(A23)}$$

which is just (2.43). Combining (A20) with (A19) now yields the desired result.

An alternate way to derive these results is provided by the Baxter-type scheme of Faddeev *et al.* (1979). For a detailed account see Ho (1981).

We also note that partial derivation ($M=2$) for Yang's results were recently provided [Fung (1981), Bahder (1981), and probably more]. There exists also an account by Gaudin (1969, 1973).

In Appendix B we discuss the transformation properties of the solution.

APPENDIX B: THE PERMUTATION SYMMETRY OF ϕ_Q

We wish to show that the solution ϕ_Q constructed in Appendix A has indeed the required symmetry, $[N-M,M]$, of a Young tableau with upper and lower rows of lengths $N-M$ and M, respectively. We remind the reader that this is the symmetry required in order that $\xi_{Q,0}$ be of conjugate symmetry to the spin-wave function $t(a)$.

A convenient picture of the permutation $Q\in S_N$ is provided by considering a one-dimensional lattice of N sites occupied by N numbered balls. Ball number Qj occupies site j in the arrangement Q. If the function ϕ_Q, whose symmetry we study, is to be described by a Young tableau with the integers $g_1,\ldots,g_{N-M}$ in the upper row and integers $h_1,\ldots,h_M$ in the lower one (see Fig. 8), then paint a stripe on balls $h_1,\ldots,h_M$ and leave the other balls unstriped. We denote by $y_1,\ldots,y_M$ the sites occupied by the striped balls and by $z_1,\ldots,z_{N-M}$ those occupied by the unstriped balls.

With this representation we proceed to consider (using the notation of Sec. II and Appendix A)

$$\phi_Q=(-1)^{\delta_Q}\xi_{Q,0}=\hat{\phi}(y_1,\ldots,y_M,z_1,\ldots,z_{N-M}) , \qquad \text{(B1)}$$

where

$$y_j=h_j \quad \text{and} \quad Qz_l=g_l .$$

The requirement that ϕ_Q be of the symmetry $[N-M,M]$ means, in the language of the billiard ball picture, that

(1) $\hat{\phi}$ must be a symmetric function of the y_j and separately of the z_e. This permits us to write

$$\hat{\phi}=\phi(y_1,\ldots,y_M) \quad \text{with } y_1<y_2<\cdots<y_M .$$

(2) $\hat{\phi}(y_1,\ldots,y_M;z_1,\ldots,z_{N-M})$ must vanish if symmetrized with respect to $z_1,\ldots,z_{N-M}$ and (say) y_M. Expressed in terms of ϕ, this means

$$\sum_{\substack{y=1\\ y\neq y_1}}^{M}\phi(y_1,\ldots,y,\ldots,y_{M-1})=0 .$$

We shall show now that the wave functions we constructed do indeed satisfy this condition. We summarize results of Appendix A:

$$\phi(y_1,\ldots,y_M)=\sum_P A_P\prod_{\gamma=1}^{M}f(\Lambda_{P\gamma},y_\gamma) , \qquad \text{(B2)}$$

where the single-particle wave functions are

$$f(\Lambda,y)=\prod_{j=1}^{y-1}\left[\frac{i(\alpha_j-\Lambda)+\frac{c}{2}}{i(\alpha_{j+1}-\Lambda)-\frac{c}{2}}\right]$$
$$=\frac{1}{1-\mu_1(\Lambda)}\left[\prod_{j=1}^{y-1}\mu_j(\Lambda)-\prod_{j=1}^{y}\mu_j(\Lambda)\right] , \qquad \text{(B3)}$$

and where the coefficients are determined from

$$\frac{A_{P'}}{A_P}=\frac{i(\Lambda_{P\gamma}-\Lambda_{P(\gamma+1)})+c}{i(\Lambda_{P\gamma}-\Lambda_{P(\gamma+1)})-c} , \qquad \text{(B4)}$$

where

$$P'\gamma=P(\gamma+1), \quad P'(\gamma+1)=P\gamma ,$$

and

$$P'\beta=P\beta \text{ for } \beta\neq\gamma,\gamma+1$$

and

$$\mu_j(\Lambda)=\frac{i(\alpha_j-\Lambda)+\frac{c}{2}}{i(\alpha_j-\Lambda)-\frac{c}{2}} , \qquad \text{(B5)}$$

and where finally, $\Lambda_1,\ldots,\Lambda_M$ are determined from

$$\prod_{j=1}^{N}\mu_j(\Lambda_\gamma)=-\prod_\delta\frac{i(\Lambda_\delta-\Lambda_\gamma)+c}{i(\Lambda_\delta-\Lambda_\gamma)-c} . \qquad \text{(B6)}$$

For the Kondo problem $\alpha_j=0$ or 1 [for the chiral Gross-Neveu (backscattering) model, $\alpha_i=\pm1$; for the Heisenberg model $\alpha=0$ and for the Yang-Gaudin model $\alpha_j=k_j$].

Let us proceed to the proof:

$$\sum_{y\notin\{y_j\}}^{N}\phi(y_1\cdots y\cdots y_{M-1})=\sum_P A_P\left[\left[\sum_{y=1}^{y_1-1}f(\Lambda_{P1},y)\right]\prod_{\gamma=1}^{M}f(\Lambda_{P(\gamma+1)},Y_\gamma)\right.$$
$$+\sum_{\beta=1}^{M-2}\prod_{\gamma=1}^{\beta}f(\Lambda_{P\gamma},Y_\gamma)\left[\sum_{y_\beta<y<y_{\beta+1}}f(\Lambda_{P(\beta+1)},y)\right]\prod_{\delta=\beta+1}^{M-1}f(\Lambda_{P(\delta+1)},y_\delta)$$
$$\left.+\prod_{\gamma=1}^{M-1}f(\Lambda_{P\gamma},y_\gamma)\left[\sum_{y_{M-1}<y\le N}f(\Lambda_{PM},y)\right]\right]. \tag{B7}$$

We wish to simplify (B7). Using

$$\sum_{y=a}^{b}f(\Lambda,y)=\frac{\Lambda}{1-\mu_1(\Lambda)}\left[\sum_{j=1}^{a+1}\mu_j(\Lambda)-\prod_{j=1}^{b}\mu_j(\Lambda)\right], \tag{B8}$$

we deduce

$$\sum_{y=1}^{y_1-1}f(\Lambda_{P1},y)=\frac{1}{1-\mu_1(\Lambda_{P1})}\left[1-\prod_{j=1}^{y_1-1}\mu_j(\Lambda_{P1})\right], \tag{B9a}$$

$$\sum_{y_\beta<y<y_{\beta+1}}f(\Lambda_{P(\beta+1)},y)=\frac{1}{1-\mu_1(\Lambda_{P(\beta+1)})}\left[\prod_{j=1}^{y_\beta}\mu_j(\Lambda_{P(\beta+1)}-\prod_{j=1}^{y_{\beta+1}-1}\mu_j(\Lambda_{P(\beta+1)})\right], \tag{B9b}$$

$$\sum_{y_{M-1}\le y\le N}f(\Lambda_{PM},y)=\frac{1}{1-\mu_1(\Lambda_{PM})}\left[\prod_{j=1}^{y_M-1}\mu_j(\Lambda_{PM})-\prod_{j=1}^{N}\mu_j(\Lambda_{PM})\right]. \tag{B9c}$$

Furthermore, let us show that

$$\sum_P A_P\left[\prod_{\gamma=1}^{\beta}f(\Lambda_{P\gamma},y_\gamma)\left[\frac{\prod_{j=1}^{y_\beta}j(\Lambda_{P(\beta+1)})}{1-\mu_1(\Lambda_{P(\beta+1)})}\right]\prod_{\delta=\beta+1}^{M-1}f(\Lambda_{P(\delta+1)},y_\delta)\right.$$
$$\left.+\prod_{\gamma=1}^{\beta-1}f(\Lambda_{P\gamma},y_\gamma)\left[\frac{-\prod_{j=1}^{y_{\beta-1}}\mu_j(\Lambda_{P\beta})}{1-\mu_1(\Lambda_{P_\beta})}\right]\prod_{\delta=\beta}^{M-1}f(\Lambda_{P(\delta+1)},y_\delta)\right]=0. \tag{B10}$$

This follows from

$$A_P\left[f(\Lambda_{P\beta},y_\beta)\frac{\prod_{j=1}^{y_\beta}\mu_j(\Lambda_{P(\beta+1)})}{1-\mu_1(\Lambda_{P(\beta+1)})}-f(\Lambda_{P(\beta+1)},y_\beta)\frac{\prod_{j=1}^{y_\beta-1}\mu_j(\Lambda_{P\beta})}{1-\mu_1(\Lambda_{P\beta})}\right]$$
$$+A_{P'}\left[f(\Lambda_{P(\beta+1)},y_\beta)\frac{\prod_{j=1}^{y_\beta}\mu_j(\Lambda_{P\beta})}{1-\mu_1(\Lambda_{P\beta})}-f(\Lambda_{P\beta},y_\beta)\frac{\prod_{j=1}^{}\mu_j(\Lambda_{P(\beta+1)})}{1-\mu_1(\Lambda_{P(\beta+1)})}\right]=0,$$

which can be rewritten as

$$A_P[2\mu_{y_\beta}(\Lambda_{P(\beta+1)})-\mu_{y_\beta}(\Lambda_{P(\beta+1)})\mu_{y_\beta}(\Lambda_{P\beta})-1]+A_{P'}[2\mu_{y_\beta}(\Lambda_{P\beta})-\mu_{y_\beta}(\Lambda_{P(\beta+1)})\mu_{y_\beta}(\Lambda_{P\beta})-1]=0.$$

However, this follows immediately from the conditions on the coefficients—(B4).

Using (B9) and (B10), we can now reduce the Young condition (B7) to give

$$\sum_{y\notin\{y_j\}}^{N}\phi(y_1\cdots y\cdots y_{M-1})=\sum_P A_P\left[\frac{1}{1-\mu_1(\Lambda_{P1})}\prod_{\gamma=1}^{M-1}f(\Lambda_{P(\gamma+1)},y_\gamma)-\frac{\prod_{j=1}^{N}\mu_j(\Lambda_{PM})}{1-\mu_1(\Lambda_{PM})}\prod_{\gamma=1}^{M-1}f(\Lambda_{P\gamma},y_\gamma)\right]. \tag{B11}$$

But $A_P\prod_{j=1}^{N}\mu_j(\Lambda_{PM})=A_{\tilde{P}}$, where $\tilde{P}=PM\,P1\,P2\ldots P(M-1)$ as a consequence of (B4) and (B6) [see (A21)]. Thus

$$\sum_{y\in\{y_j\}}^{N}\phi(y_1\cdots y\cdots y_{M-1})=\sum_P\left[\frac{A_P}{1-\mu_1(\Lambda_{P1})}\sum_{\gamma=1}^{M-1}f(\Lambda_{P(\gamma+1)},y_\gamma)-\frac{A_P}{1-\mu_1(\Lambda_{\tilde{P}1})}\prod_{\gamma=1}^{M-1}f(\Lambda_{\tilde{P}(\gamma+1)},y_\gamma)\right]=0\ ,$$

which is the required condition.

APPENDIX C: PERTURBATIVE CALCULATION OF THE FREE ENERGY

In order to calculate the free energy for high magnetic fields or high temperature, it is convenient to construct the model using the momentum cutoff scheme ($\mathscr{D}$ scheme) where the Hamiltonian is given by

$$\mathscr{H}=\mathscr{H}_0+V\ , \tag{C1}$$

$$\mathscr{H}_0=\sum_{\substack{|k|\le\mathscr{D}\\ s=\pm1}}(k-sH)a_{ks}^*a_{ks}-2HS_z\ , \tag{C2a}$$

$$V=\frac{2g}{L}\sum_{|k|,|k'|\le\mathscr{D}}\sum[a_{k'+}^*a_{k-}S^-+a_{k'-}^*a_{k+}S^+ +(a_{k'+}^*a_{k+}-a_{k'-}^*a_{k'-})S_z]\ . \tag{C2b}$$

Here a_{ks}^* and a_{ks} are creation and annihilation for free electrons with spin $\frac{1}{2}s$ and momentum k, satisfying the usual anticommutation relations

$$\{a_{ks}^*,a_{k's'}\}=\delta_{kk'}\delta_{ss'}\ ,\quad \{a_{ks}^*,a_{k's'}^*\}=\{a_{ks},a_{k's'}\}=0\ , \tag{C3}$$

from which it follows that

$$\{\mathscr{H}_0,a_k^*\}=ka_{k\pm}^*,\quad \{\mathscr{H}_0,a_{k\pm}\}=-ka_{k\pm}\ , \tag{C4a}$$

$$e^{\beta\mathscr{H}_0}a_{k\pm}^*e^{-\beta\mathscr{H}_0}=e^{\beta k}a_{k\pm}^*\ , \tag{C4b}$$

and H is the magnetic field ($H>0$).

We wish to calculate the partition function

$$Z=\mathrm{tr}e^{-\beta\mathscr{H}}$$

by expanding it in the usual way in V considered as a perturbation:

$$Z=\mathrm{tr}\left[e^{-\beta\mathscr{H}_0}\left[1-\int_0^\beta d\lambda\,V(\lambda)+\int_0^\beta d\lambda_1\int_0^{\lambda_1}d\lambda_2V(\lambda_1)V(\lambda_2)-\cdots\right]\right]\equiv Z_0+Z_1+Z_2+\cdots\ , \tag{C5}$$

where

$$V(\lambda)=e^{\lambda\mathscr{H}_0}Ve^{-\lambda\mathscr{H}_0}=\frac{2g}{L}\sum_{k,k'}e^{\lambda(k'-k)}[a_{k'+}^*a_{k-}S^-+a_{k'-}^*a_{k+}S^+ +(a_{k'+}^*a_{k+}-a_{k'-}^*a_{k-})S_z]\ . \tag{C6}$$

The free energy can then be written as follows:

$$\mathscr{F}=-T\ln Z=-T\left\{\ln Z_0+\frac{Z_1}{Z_0}+\left[\frac{Z_2}{Z_0}-\frac{1}{2}\left[\frac{Z_1}{Z_0}\right]^2\right]+O(g^3)\right\}\ . \tag{C7}$$

The noninteracting ground state is a state with N^e electrons filling the positive spin "sea" from $k=-\mathscr{D}$ to $k=H$ and the negative spin "sea" from $k=-\mathscr{D}$ to $k=-H$, plus one impurity. The Fermi distribution for the electrons is given by

$$\langle n_{ks}\rangle_0=\frac{\mathrm{tr}e^{-\beta\mathscr{H}_0}n_{ks}}{\mathrm{tr}e^{-\beta\mathscr{H}_0}}=\frac{1}{1+e^{\beta(k-sH)}}\equiv f(k-sH)\ , \tag{C8}$$

and the total number of electrons in the ($\pm$) sea given by

$$\langle N_\pm\rangle_0=\sum_{|k|\le\mathscr{D}}\langle n_{k\pm}\rangle_0=\frac{L}{2\pi}(\mathscr{D}\pm H)\ . \tag{C9}$$

Since $\langle N_+\rangle_0+\langle N_-\rangle_0=\mathscr{D}L/\pi=N$,

$$\mathscr{D}=\pi\frac{N}{L}\ . \tag{C10}$$

Note that in order to be consistent with the D-scheme notation one should actually set $|k|\le\mathscr{K}$ and then relate $\mathscr{K}$ and $\mathscr{D}=N/L$ via $\mathscr{K}=\pi\mathscr{D}$. We shall not bother.

The zeroth-order partition function Z_0 is given by

$$Z_0=\mathrm{tr}e^{-\beta\mathscr{H}_0}=Z_0^eZ_0^i\ ,$$

where the electron partition function is

$$Z_0^e=\prod_{\substack{|k|\le\mathscr{D}\\ s=\pm1}}(1-e^{-\beta(k-sH)})$$

and the impurity partition function

$$Z_0^i=\begin{cases}1+2\sum_{j=1}^{S}\cosh(2j\beta H), & S\text{ integer}\\ 2\sum_{j=0}^{S-(1/2)}\cosh[(2j+1)\beta H], & S\text{ half-integer}\ .\end{cases} \tag{C11}$$

Hence, the zeroth-order contribution to the impurity free energy is

$$\mathscr{F}_i^{(0)}=-T\ln Z_0^i\ . \tag{C12}$$

First order. The first-order correction is given [see (C7) and (C5)] by

$$\frac{Z_1}{Z_0}=-2g\frac{\int_0^\beta d\lambda\,\mathrm{tr}\left[e^{-\beta\mathcal{H}_0}\sum_{k,k'}\frac{e^{\lambda(k-k')}}{L}[a^*_{k'+}a_{k-}S^-+a^*_{k'-}a_{k+}S^+ +(a^*_{k'+}a_{k+}-a^*_{k'-}a_{k-})S_z]\right]}{\int_0^\beta d\lambda\,\mathrm{tr}\,e^{-\beta\mathcal{H}_0}} \tag{C13}$$

in this expression, the terms involving S^- and S^+ do not contribute to the trace, and in the last two terms only $k=k'$ contributes. Thus we have

$$\mathcal{F}_i^{(1)}=-T\frac{Z_1}{Z_0}=gF_S\left[\frac{H}{T}\right]\frac{\langle N_+-N_-\rangle_0}{L}=g\frac{2H}{\pi}F_S\left[\frac{H}{T}\right], \tag{C14}$$

where $F_S(H/T)$ is the impurity trace, given by

$$F_S\left[\frac{H}{T}\right]=\frac{\mathrm{tr}e^{2\beta HS_z}S_z}{\mathrm{tr}e^{2\beta HS_z}}=\begin{cases}\dfrac{\sum_{j=1}^{S}2j\sinh(2j\beta H)}{1+2\sum_{j=1}^{S}\cosh(2j\beta H)}, & S\text{ integer}\\[2ex] \dfrac{\sum_{j=0}^{S-(1/2)}(2j+1)\sinh[(2j+1)\beta H]}{\sum_{j=0}^{S-(1/2)}2\cosh[(2j+1)\beta H]}, & S\text{ half-integer}.\end{cases} \tag{C15}$$

Second order. The second-order contribution consists of two terms

$$\mathcal{F}_i^{(2)}=-T\left[\frac{Z_2}{Z_0}-\frac{1}{2}\left[\frac{Z_1}{Z_0}\right]^2\right], \tag{C16}$$

where the second term can be obtained from (C14). The first term is given by

$$-T\frac{Z_2}{Z_0}=-\frac{T}{Z_0}\left[\frac{2g}{L}\right]^2\int_0^\beta d\lambda_1\int_0^{\lambda_1}d\lambda_2\,\mathrm{tr}\left[e^{-\beta\mathcal{H}_0}\sum_{k_1,k_2;k'_1,k'_2}e^{\lambda_1(k_1-k_2)+\lambda_2(k'_1-k'_2)}\right.$$
$$\times[a^*_{k_1+}a_{k_2-}a^*_{k'_1-}a_{k'_2+}S^-S^+ +a^*_{k_1-}a_{k_2+}a^*_{k'_1+}a_{k'_2-}S^+S^-$$
$$\left.+(a^*_{k_1+}a_{k_2+}-a^*_{k_1-}a_{k_2-})(a^*_{k'_1+}a_{k'_2+}-a^*_{k'_1-}a_{k'_2-})S_z^2]\right].$$

Now, there are contributions to the trace from terms with $k_1=k_2, k'_1=k'_2$ and $k_1=k'_2, k'_1=k_2$, which, using the commutation relations (C3), we can express in terms of the averages of quantities involving $n_{k\pm}$ and $n_{k'\pm}$ as follows:

$$-T\frac{Z_2}{Z_0}=-T\left[\frac{2g}{L}\right]^2\int_0^\beta d\lambda_1\int_0^{\lambda_1}d\lambda_2\left\{\sum_{k,k'}e^{(\lambda_1-\lambda_2)(k-k')}\left[W_S^+\left[\frac{H}{T}\right]\langle n_{k+}(1-n_{k'-})\rangle_0+W_S^-\left[\frac{H}{T}\right]\langle n_{k-}(-n_{k'+}+1)\rangle_0\right.\right.$$
$$\left.+W_S\left[\frac{H}{T}\right](\langle n_{k+}(1-n_{k'+})\rangle_0+\langle n_{k-}(1-n_{k'-})\rangle_0)\right]$$
$$\left.+W_S\left[\frac{H}{T}\right]\sum_{k,k'}[\langle n_{k+}n_{k'+}\rangle_0+\langle n_{k-}n_{k'-}\rangle_0-2\langle n_{k+}n_{k'}\rangle_0]\right\}, \tag{C17}$$

where the impurity traces W_S^+, W_S^-, and W_S are given by

$$W_S^+\left[\frac{H}{T}\right]=\frac{\mathrm{tr}(e^{2\beta HS_z}S^-S^+)}{\mathrm{tr}e^{2\beta HS_z}}=\frac{\sum_{j=-S}^{(S-1)}[S(S+1)-j(j+1)]e^{2j\beta H}}{Z_0^i},$$

$$W_S^-\left[\frac{H}{T}\right]=\frac{\mathrm{tr}(e^{2\beta HS_z}S^+S^-)}{\mathrm{tr}e^{2\beta HS_z}}=\frac{\sum_{j=-S+1}^{S}[S(S+1)-j(j-1)]e^{2j\beta H}}{Z_0^i}, \tag{C18}$$

$$W_S\left[\frac{H}{T}\right]=\frac{\mathrm{tr}(e^{2\beta HS_z}S_z^2)}{\mathrm{tr}e^{2\beta HS_z}}=\begin{cases}\dfrac{\sum_{j=1}^{S} j^2 2\cosh(2j\beta H)}{Z_0^i} & S \text{ integer}\\ \dfrac{\sum_{j=0}^{S-(1/2)} (j+\frac{1}{2})^2 2\cosh[(2j+1)\beta H]}{Z_0^i} & Sh \text{ integer}\,.\end{cases}$$

The second summation over k and k' in (C17) can be expressed in terms of a simple average, namely,

$$\sum_{k,k'}[\langle n_{k+}n_{k'+}\rangle_0+\langle n_{k-}n_{k'-}\rangle_0-2\langle n_{k+}n_{k'-}\rangle_0]\equiv\langle(N_+-N_-)^2\rangle_0=\frac{L^2H^2}{\pi^2}\,, \tag{C19}$$

and the λ_1,λ_2 integrals are trivial. The first summation of (C17) is converted into an integral and the averages are expressed in terms of the Fermi distribution (C8); also, we perform a change of variables from λ_1,λ_2 to $\lambda'=(\lambda_1+\lambda_2)$ and $(\lambda_1-\lambda_2)$. Since the integrand only depends on λ, one can integrate λ' to get the following:

$$-T\frac{Z_2}{Z_0}=-\left[\frac{g}{\pi}\right]^2\left\{T\int_0^\beta d\lambda(\beta-\lambda)\left[W_S^+\left[\frac{H}{T}\right]I_+^2(\lambda)+W_S^-\left[\frac{H}{T}\right]I_-^2(\lambda)+W_S\left[\frac{H}{T}\right]2I_+(\lambda)I_-(\lambda)\right]\frac{2H^2}{T}W_S\left[\frac{H}{T}\right]\right\}, \tag{C20}$$

where

$$I_\pm(\lambda)=\int_{-(\mathscr{D}\pm H)}^{(\mathscr{D}\mp H)}dk\frac{e^{k\lambda}}{1+e^{\beta k}}\,. \tag{C21}$$

Hence the second-order contribution (C16) is given by

$$\mathscr{F}_i^{(2)}=\frac{g^2}{\pi^2}\left\{\frac{2H^2}{T}\left[F_S^2\left[\frac{H}{T}\right]-W_S\left[\frac{H}{T}\right]\right]-T\int_0^1 dx(1-x)\left[W_S^+\left[\frac{H}{T}\right]I_+^2(x)+W_S^-\left[\frac{H}{T}\right]I_-^2(x)+W_S\left[\frac{H}{T}\right]2I_+(x)I_-(x)\right]\right\}, \tag{C22}$$

where x is a dimensionless variable, $x=T\lambda$, and $I_\pm(\lambda)=TI_\pm(x)$.

Now we exploit the fact that the momentum cutoff is very large, namely, $\mathscr{D}\gg H$ and $\mathscr{D}\gg T$, and approximate $I_\pm(x)$ by

$$I_\pm(x)=e^{\pm(H/T)x}\int_{-(\mathscr{D}\pm H)/T}^{(\mathscr{D}\mp H)/T}dy\frac{e^{yx}}{1+e^y}\approx e^{\pm H/Tx}\left[\frac{\pi}{\sin\pi x}-\frac{1}{x}e^{-(\mathscr{D}\pm H)/T}-\frac{e^{-(\mathscr{D}\mp H)/T}(1-x)}{(1-x)}\right] \tag{C23}$$

and after some algebra rewrite the integral involved in (C22) as follows:

$$\begin{aligned}I\left[\frac{H}{T},\frac{\mathscr{D}}{T}\right]&\equiv\int_0^1 dx(1-x)\{W_S^+I_+^2(x)+W_S^-I_-^2(x)+2W_SI_+(x)I_-(x)\}\\&=\left[W_S^-\left[\frac{H}{T}\right]\int_0^1 dx\,e^{-2xH/T}F^2\left[x,-\frac{H}{T},\frac{\mathscr{D}}{T}\right]W_S\left[\frac{H}{T}\right]\int_0^1 F\left[x,\frac{H}{T},\frac{\mathscr{D}}{T}\right]F\left[x,-\frac{H}{T},\frac{\mathscr{D}}{T}\right]dx\right],\end{aligned} \tag{C24}$$

where

$$F\left[x,\frac{H}{T},\frac{\mathscr{D}}{T}\right]\equiv[\pi\csc\pi x-x^{-1}e^{-[(\mathscr{D}+H/T)]x}-(1-x)^{-1}e^{-[(\mathscr{D}-H/T)](1-x)}]\,.$$

We use the following identity:

$$W_S^\pm\left[\frac{H}{T}\right]e^{\pm2H/T}-W_S^\mp\left[\frac{H}{T}\right]=0\,, \tag{C25}$$

as well as the invariance of the product $F(x,H/T,\mathscr{D}/T)F(x,-H/T,\mathscr{D}/T)$ under the change of variable x into $(1-x)$.

If we neglect terms of order $H/\mathscr{D}$ or $T/\mathscr{D}$, the integrals appearing in (C24) are given by

$$\int_0^1 dx \left[F\left[x, \frac{H}{T}, \frac{\mathscr{D}}{T}\right] F\left[x, -\frac{H}{T}, \frac{\mathscr{D}}{T}\right] \right] \sim 2\phi_0(0) - 2 + \frac{4\mathscr{D}}{T}\ln 2 + O\left(\frac{T}{\mathscr{D}}, \frac{H}{\mathscr{D}}\right), \tag{C26a}$$

$$\int_0^1 dx\, e^{-(2H/T)x} F^2\left[x, -\frac{H}{T}, \frac{\mathscr{D}}{T}\right] \sim \left[\phi_0\left[\frac{H}{T}\right] + \left\{ e^{-2H/T} + \frac{2H}{T}\left[Ei\left[\frac{2H}{T}\right] + \ln\left|\frac{\mathscr{D}e}{2H}\right| \right] + 2\frac{\mathscr{D}}{T}\ln 2 \right\} + \left\{ -e^{-2H/T} - \frac{2H}{T}\left[Ei\left[-\frac{2H}{T}\right] + \ln\left|\frac{\mathscr{D}e}{2H}\right| \right] + 2\frac{\mathscr{D}}{T}\ln 2 \right\} \right] + O\left(\frac{T}{\mathscr{D}}, \frac{H}{\mathscr{D}}\right), \tag{C26b}$$

where

$$\phi_0\left[\frac{H}{T}\right] \equiv \int_0^1 dx\, e^{-2xH/T}[\pi^2 \csc^2 \pi x - x^{-2}(1-x)^{-2}], \qquad Ei(a) \equiv \int_{-\infty}^{a} \frac{e^x}{x} dx . \tag{C27}$$

From Eqs. (C22), (C24), and (C26), we find the second-order contribution $\mathscr{F}_i^{(2)}$ for $\mathscr{D} \gg H$ and $\mathscr{D} \gg T$. Then, adding $\mathscr{F}_i^{(1)}$ [Eq. (C13)] and $\mathscr{F}_i^{(0)}$ [Eq. (C12)] to it leads to the impurity free energy up to second order in g for $\mathscr{D} \gg H, T$:

$$\begin{aligned}
\mathscr{F}_i = & -T \ln Z_0^i + g\frac{2H}{\pi} F_S\left[\frac{H}{T}\right] \\
& + \left[\frac{g}{\pi}\right]^2 \left[2\frac{H^2}{T}\left[F_S^2\left[\frac{H}{T}\right] - W_S\left[\frac{H}{T}\right] \right] \right. \\
& \quad - T\left\{ \left[W_S^-\left[\frac{H}{T}\right] + W_S^+\left[\frac{H}{T}\right] \right] \left[2\frac{\mathscr{D}}{T}\ln 2 - 1 \right] + 2\frac{H}{T}\left[W_S^+\left[\frac{H}{T}\right] Ei\left[\frac{2H}{T}\right] - W_S^-\left[\frac{H}{T}\right] Ei\left[-\frac{2H}{T}\right] \right] \right. \\
& \quad + 2\frac{H}{T}\left[W_S^+\left[\frac{H}{T}\right] \right] \left[1 + \ln\left|\frac{\mathscr{D}}{2H}\right| \right] + \left[W_S^-\left[\frac{H}{T}\right] \phi_1\left[\frac{H}{T}\right] + W_S^+\left[\frac{H}{T}\right] \phi_1\left[-\frac{H}{T}\right] \right] \\
& \quad \left. \left. + W_S\left[\frac{H}{T}\right] \left[2\phi_0(0) - 2 + 4\frac{\mathscr{D}}{T}\ln 2 \right] \right\} \right] + O\left(\frac{H}{\mathscr{D}}, \frac{T}{\mathscr{D}}\right) + O(g^3),
\end{aligned} \tag{C28}$$

where Z_0^i, F_S, and $(W_S^{\pm}, W_S)$ are given, respectively, by Eqs. (C11), (C15), and (C18); Ei, ϕ_0 by Eq. (C27), and

$$\phi_1\left[\frac{H}{T}\right] = \int_0^1 dx (1-x) e^{-2xH/T}[\pi^2 \csc^2 \pi x - x^{-2} - (1-x)^{-2}] . \tag{C29}$$

Our aim is to find the asymptotic behavior of the free energy $\mathscr{F}_i$ in two regions: (i) $T/H \ll 1$, $H \ll \mathscr{D}$ and (ii) $H/T \ll 1$, $T \ll \mathscr{D}$.

(i) Low-temperature and high magnetic field region ($T/H \ll 1, H \leq \mathscr{D}$). From the corresponding definitions we can obtain the following asymptotic behaviors of the functions of H/T appearing in (C28):

(a) $-T \ln Z_0^i = -2SH + O(e^{-2H/T})$, (C11′)

(b) $F_S\left[\frac{H}{T}\right] = S[1 + O(e^{-2H/T})]$, (C15′)

(c) $W_S\left[\frac{H}{T}\right] = S^2[1 + O(e^{-2H/T})]$, (C18′)

(d) $W_S^+\left[\frac{H}{T}\right] = 2S[e^{-2H/T} + O(e^{-4H/T})]$, (C18′)

(e) $W_S^-\left[\frac{H}{T}\right] = 2S[1 + O(e^{-2H/T})]$, (C18′)

(f) $Ei\left[\pm\frac{2H}{T}\right] = \frac{e^{\pm 2H/T}}{2\frac{H}{T}}\left[1 + O\left[\frac{T^2}{H^2}\right]\right]$, (C27′)

(g) $\phi_i\left[\frac{H}{T}\right] \sim 0, \quad e^{-2H/T}\phi_i\left[-\frac{H}{T}\right] \sim 0, \quad i = 0, 1$. (C27′),(C29′)

Hence

$$\mathcal{F}_i \underset{T\ll H\ll\mathcal{D}}{\sim} -2SH\left[1-\frac{g}{\pi}-2\left[\frac{g}{\pi}\right]^2\ln\left|\frac{\mathcal{D}_e}{2H}\right|\right]+O(g^3) +(H\text{-independent terms}) . \quad \text{(C30)}$$

(ii) The high-temperature, low magnetic field region, $(T/H\gg 1, T\ll\mathcal{D})$.

For this region, the following asymptotic behaviors are obtained:

(a) $$-T\ln Z_0^i=-T\ln(2S+1)-T\left[\tfrac{2}{3}S(S+1)\frac{H^2}{T^2}+O\left[\frac{H^4}{T^4}\right]\right],$$

(b) $$\frac{2H}{T}F_S\left[\frac{H}{T}\right]=\frac{4H^2}{T^2}\frac{S(S+1)}{3}+O\left[\frac{H^4}{T^4}\right],$$

(c) $$W_S\left[\frac{H}{T}\right]=\frac{2S(S+1)}{3}\left[1-\frac{H^2}{T^2}\frac{2}{5}[1-\tfrac{4}{3}S(S+1)]+O\left[\frac{H^4}{T^4}\right]\right],$$

(d) $$\left[W^-\left[\frac{H}{T}\right]+W_S^+\left[\frac{H}{T}\right]\right]=\tfrac{2}{3}S(S+1)\left[2+\tfrac{2}{5}[1-\tfrac{4}{3}S(S+1)]\frac{H^2}{T^2}+O\left[\frac{H^4}{T^4}\right]\right],$$

(e) $$\frac{2H}{T}\left[W_S^+\left[\frac{H}{T}\right]Ei\left[\frac{2H}{T}\right]-W_S^-\left[\frac{H}{T}\right]Ei\left[-\frac{2H}{T}\right]\right]=\left[\frac{4H^2}{T^2}\tfrac{2}{3}S(S+1)\left[2-C-\ln\left|\frac{2H}{T}\right|\right]+O\left[\frac{H^4}{T^4}\right]\right],$$

(f) $$W_S^-\left[\frac{H}{T}\right]\phi_1\left[\frac{H}{T}\right]+W_S^+\left[\frac{H}{T}\right]\phi_1\left[-\frac{H}{T}\right]=\tfrac{2}{3}S(S+1)\left[2\phi_1(0)+\{\tfrac{2}{5}[1-\tfrac{4}{3}S(S+1)]\phi_1(0)-4\ln\beta+2\}\frac{H^2}{T^2}+O\left[\frac{H^4}{T^4}\right]\right],$$

(g) $$\frac{2H}{T}\left[W_S^+\left[\frac{H}{T}\right]-W_S^-\left[\frac{H}{T}\right]\right]=-\frac{4H^2}{T^2}\tfrac{2}{3}S(S+1)+O\left[\frac{H^4}{T^4}\right],$$

where C is Euler's constant and

$$\ln\beta=\int_0^1 dx(1-x)^2x\left[\frac{\pi^2}{\sin^2\pi x}-\frac{1}{x^2}\right].$$

The free energy then follows:

$$\mathcal{F}_i \underset{H\ll T\ll\mathcal{D}}{\sim} -T\ln(2S+1)-\frac{H^2}{T}\frac{2S(S+1)}{3}\left[1-2\frac{g}{\pi}+\left[\frac{2g}{\pi}\right]^2\tfrac{7}{4}-\ln\left|\frac{\gamma\beta\mathcal{D}}{T}\right| +\tfrac{1}{5}[1-\tfrac{4}{3}S(S+1)]\frac{\phi_1(0)-\phi_0(0)}{2}\right]+O(g^3) . \quad \text{(C31)}$$

The remaining integrals can be done numerically:

$$\ln\beta=0.662\,122\ldots,$$

$$\phi_0(0)-\phi_1(0)=\int_0^1 dx(1-x)\left[\frac{\pi^2}{\sin^2\pi x}-\frac{1}{x^2}-\frac{1}{(1-x)^2}\right]=0.841\,166\ldots=\ln\alpha ,$$

$$C=0.577\,216\ldots=\ln\gamma .$$

Hence

$$\mathcal{F}_i \underset{H\ll T\ll\mathcal{D}}{\sim} -T\ln(2S+1)-\frac{H^2}{T}\frac{2S(S+1)}{3}\left[1-\frac{2g}{\pi}\left[\frac{2g}{\pi}\right]^2\ln\left|\frac{T}{\mathcal{D}e^{7/4}\beta\gamma\alpha\frac{1}{10}[1-\frac{4}{3}S(S+1)]}\right|+O(g^3)\right]. \quad \text{(C32)}$$

Equations (C30) and (C32) determine, to all orders in perturbation theory, the ratio

$$U=\frac{\mathcal{T}_k}{\mathcal{T}_H}=2\beta\gamma e^{-7/4} . \quad \text{(C33)}$$

APPENDIX D: DETERMINATION OF STRING POSITIONS

In Sec. II we considered the (simplest) singlet excitation which consists of two holes at Λ_1^h and Λ_2^h and a two-string $\bar{\Lambda}\pm ic/2$. We wish to show here that in order to

satisfy Eq. (2.43) [or (3.2)] the two-string must be placed at $\bar{\Lambda}=\frac{1}{2}(\Lambda_1^h+\Lambda_2^h)$.

In the generalization of the model to SU(ν) symmetry with $\nu>2$ (Sec. VIII), the above statement generalizes as follows: Consider a state consisting of two holes, a rank-R_1 hole at $\Lambda^{R_1,h}$ and rank-R_2 hole at $\Lambda^{R_2,h}$, $R_2\geq R_1$. This state describes a bound state of R_1 fundamental excitation interacting symmetrically with a bound state of R_2 fundamental excitations. If we wish to construct a state where the bound-state excitations interact in an antisymmetric representation we have to add a "ladder" of two-strings $\{\Lambda^{r,s}\}$ of ranks $r=R_1,R_1+1,\ldots R_2$ placed equidistantly between $\Lambda^{R_1,h}$ and $\Lambda^{R_2,h}$,

$$\Lambda^{r,s}=\left[\frac{R_2+1-r}{R_2-R_1+2}\right]\Lambda^{R_1,h}+\left[\frac{r-R_1+1}{R_2-R_1+2}\right]\Lambda^{R_2,h}. \tag{D1}$$

In the basic SU(2) problem this reduces to $\Lambda^s=\frac{1}{2}(\Lambda_1^h+\Lambda_2^h)$, as all Λ momenta are of rank 1.

We proceed now to show that this configuration does indeed solve Eq. (8.33), which for a rank-r two-string $\Lambda_{\pm}^{r,s}=\Lambda^{r,s}\pm ic/2$ takes the form

$$\int d\Lambda\,\sigma^r(\Lambda)\{\Theta[\tfrac{2}{3}(\Lambda-\Lambda^{r,s})]+\Theta[2(\Lambda-\Lambda^{r,s})]\}-\int d\Lambda[\sigma^{r+1}(\Lambda)+\sigma^{r-1}(\Lambda)]\Theta(\Lambda-\Lambda^{r,s})$$
$$=\Theta[\tfrac{2}{3}(\Lambda^{r+1,s}-\Lambda^{r,s})]+\Theta[2(\Lambda^{r+1,s}-\Lambda^{r,s})]+\Theta[\tfrac{2}{3}(\Lambda^{r-1,s}-\Lambda^{r,s})]+\Theta[2(\Lambda^{r-1,s}-\Lambda^{rs})]+2\pi I^r, \tag{D2}$$

where the (Fourier transforms of) one-string rank-r densities, σ^r, are given by

$$\tilde{\sigma}^r(p)=\frac{\sinh(\nu-r)cp/2}{\sinh\nu cp/2}\sigma^0-\frac{\sinh(R_1cp/2)\sinh[(\nu-r)cp/2]}{\sinh(\nu cp/2)\sinh(cp/2)}e^{c|p|/2}e^{-i\Lambda^{R_1,h}p}$$
$$-\frac{\sinh[(\nu-R_2)cp/2]\sinh(rcp/2)}{\sinh(\nu cp/2)\sinh(cp/2)}e^{c|p|/2}e^{-i\Lambda^{R_2,h}p}-e^{-i\Lambda^{r,s}p}e^{-c|p|/2} \tag{D3a}$$

for $r=R_1,\ldots,R_2$, with $\sigma^0=N^e e^{-ip}+N^i$, and

$$\tilde{\sigma}^{R_1-1}(p)=\frac{\sinh[(\nu-R_1+1)cp/2]}{\sinh(\nu cp/2)}\sigma^0-\frac{\sinh[(\nu-R_1)cp/2]\sinh[(R_1-1)cp/2]}{\sinh(\nu cp/2)\sinh(cp/2)}e^{c|p|/2}e^{-i\Lambda^{R_1,h}p}$$
$$-\frac{\sinh[(\nu-R_2)cp/2]\sinh[(R_1-1)cp/2]}{\sinh(\nu cp/2)\sinh(cp/2)}e^{(c/2)|p|}e^{-i\Lambda^{R_2,h}p}, \tag{D3b}$$

$$\tilde{\sigma}^{R_2+1}(p)=\frac{\sinh[(\nu-R_2+1)cp/2]}{\sinh(\nu cp/2)}\sigma^0-\frac{\sinh(R_1cp/2)\sinh[(\nu-R_2-1)cp/2]}{\sinh(\nu cp/2)\sinh(cp/2)}e^{(c/2)|p|}e^{i\Lambda^{R_1,h}p}$$
$$-\frac{\sinh(R_2cp/2)\sinh[(\nu-R_2-1)cp/2]}{\sinh(\nu cp/2)\sinh(cp/2)}e^{(c/2)|p|}e^{-i\Lambda^{R_2,h}p}. \tag{D3c}$$

From (D3a) and (D2) we find

$$\tan^{-1}\frac{\Lambda^{r+1,s}-\Lambda^{r,s}}{c/2}-\tan^{-1}\frac{\Lambda^{r,s}-\Lambda^{r-1,s}}{c/2}=0,$$
$$r=R_1+1,\ldots,R_2-1, \tag{D4}$$

while from (D3a)–(D3c) and (D2) we find

$$\tan^{-1}\frac{\Lambda^{R_1+1,s}-\Lambda^{R_1,s}}{c/2}-\tan^{-1}\frac{\Lambda^{R_1,s}-\Lambda^{R_1,h}}{c/2}=0, \tag{D5a}$$

$$\tan^{-1}\frac{\Lambda^{R_2,h}-\Lambda^{R_2,s}}{c/2}-\tan^{-1}\frac{\Lambda^{R_2,s}-\Lambda^{R_2-1,s}}{c/2}=0. \tag{D5b}$$

Hence

$$\Lambda^{r,s}=\left[\frac{R_2+1-r}{R_2-R_1+2}\right]\Lambda^{R_1,h}+\left[\frac{r-R_1+1}{R_2-R_1+2}\right]\Lambda^{R_2,h}. \tag{D6}$$

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